Space Charge Tune Shift

JAI lectures - Hilary Term 2025

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Table of contents

Goal of this course

- Brief introduction to a complex topic.
- Understand the concept of Space Charge.
- Distinguish different types of effects.
- Understand the limitations imposed to the accelerator operation and performance.

- We will derive some basic parameters related to space charge.
- For more detailed derivations and more realistic cases please go to references.

References

Specialized courses

1. S. Sheehy, "Space charge tune shift", JAI lectures 2019 https://indico.cern.ch/event/774280/contributions/3217261/.
2. K.Schindl "Space charge", CAS lectures https://cds.cern.ch/record/941316?.
3. M.Migliorati, "Space Charge Effects and Instabilities", https://indico.cern.ch/event/779575/contributions/3244564/.

Books

1. I. Hofmann "Space Charge Physics for Particle Accelerators", Springer 2017.
2. H. Wiedemann "Particle Accelerator Physics", Springer 2015.

Introduction

The beam is a distribution of charged particles i.e. they create en EM fields that affect their dynamics.

We can distinguish three main contributions:

- Direct fields.
- Image fields.
- Wakefields (this will covered in Instabilities lectures).

Space Charge Forces

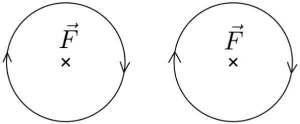
Electric field generated by a point-like charge q:

$$F_{\text{elec}} = \frac{e^2}{4\pi\epsilon r^2} \tag{1}$$

Repulsive!

Since the particle is moving with some speed v, this is equivalent to a current carrying wire with $I = qv$.

$$F_{\text{wire}} = \frac{\mu_0 I}{4\pi r^2} = \frac{v^2}{c^2} F_{\text{elec}} \tag{2}$$



Attractive!

Space Charge Forces

The overall force is repulsive:

$$F_{\text{total}} = (1 - v^2/c^2)F_{\text{elec}} \tag{3}$$

we see that for $v \to c$ the force F_{total} vanishes.

What does this mean?

Space Charge Forces

Two main regimes exist to describe the effects of Coulomb interactions in a system with many particles.

Which regime are we? Debye length

$$\lambda_D = \sqrt{\frac{\epsilon_0 \gamma^2 k_B T}{q^2 n}} \tag{4}$$

Collisional regime

Dominated by particle-on-particle collisions and described by single particle dynamics.

$$\lambda_D \gg a$$

Space Charge regime

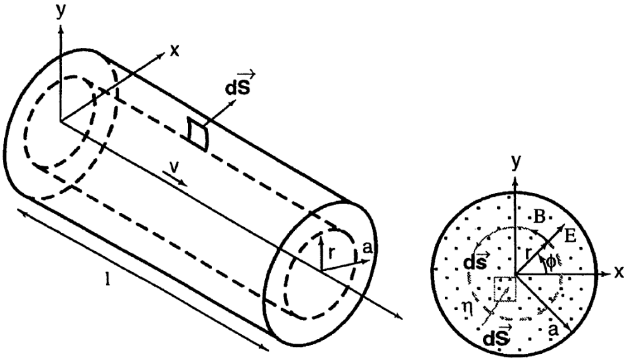
Dominated by the self fields of the particle distribution and it is described by collective effects.

$$\lambda_D \ll a$$

Space Charge Forces

Simple model: beam as a continuous cylinder of charge q, length l and radius a.

$$\rho(r) = qn(r) = \frac{I_{\text{beam}}}{\pi a^2 v} \tag{5}$$



Space Charge Forces

Electric field

$$\nabla \cdot \vec{E} = \frac{\eta}{\epsilon_0} \tag{6}$$

Gauss' law:

$$\int_V \nabla \cdot \vec{E} dV = \int_S \vec{E} d\vec{S} \tag{7}$$

cylinder of radius r and length l:

$$\pi r^2 l \frac{\eta}{\epsilon_0} = E_R 2\pi r l \tag{8}$$

$$E_r = \frac{I}{2\pi\epsilon_0\beta c} \frac{r}{a^2} \tag{9}$$

Magnetic field

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{10}$$

Stoke's law:

$$\oint \vec{B} d\vec{S} = \int_S \nabla \times \vec{B} d\vec{S} \tag{11}$$

$$B_\phi 2\pi r = \mu_0 \pi r^2 \beta c \eta \tag{12}$$

$$B_\phi = \frac{I}{2\pi\epsilon_0 c^2} \frac{r}{a^2} \tag{13}$$

Space Charge Forces

The force acting on a test particle is given by the Lorentz equation:

$$F_r = q(E_r - v_s B_\phi) \tag{14}$$

where:

$$F_r = \frac{eI}{2\pi\epsilon_0 \beta c \gamma^2} \frac{r}{a^2} \tag{15}$$

and in transverse coordinates:

$$F_x = \frac{eI}{2\pi\epsilon_0 \beta c \gamma^2 a^2} x, \quad F_y = \frac{eI}{2\pi\epsilon_0 \beta c \gamma^2 a^2} y \tag{16}$$

Space Charge Forces: circular vs. Gaussian beam

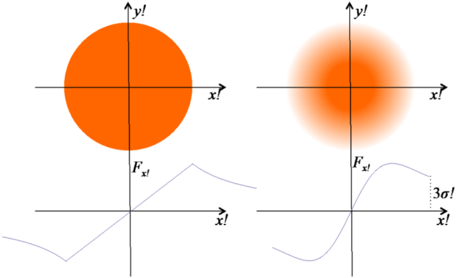


Figure: Space charge force for a homogeneous circular beam (left) and a Gaussian-shaped beam (right).

Self field tune shift

SC produces an extra defocusing. Let's include it in the Hill's equation:

$$x'' + (K(s) + K_{SC}(s))x = 0 \tag{17}$$

$$x'' + \left(K(s) - \frac{2r_0 I}{ea^2\beta^3\gamma^3 c}\right) x = 0 \tag{18}$$

Tune shift due to an error in focusing strength ΔK:

$$\Delta Q_{x,y} = \frac{1}{4\pi}\int \Delta K(s)\beta_{x,y}(s)ds \tag{19}$$

In our case $\Delta K = K_{SC}$:

$$\Delta Q_{x,y} = \frac{1}{4\pi}\int K_{SC}(s)\beta_{x,y}(s)ds \tag{20}$$

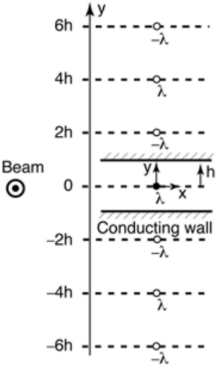
$$\Delta Q_{x,y} = -\frac{r_0 RI}{e\beta^3\gamma^3 c}\left\langle \frac{\beta_{x,y}(s)}{a^2(s)} \right\rangle \tag{21}$$

Finally:

$$\Delta Q_{x,y} = -\frac{r_0 N}{2\pi\epsilon_{x,y}\beta^2\gamma^3} \tag{22}$$

Image Effects

A second effect is coming from image currents due to conducting walls.



Electric field produced by a charge λ at a distance $2n \cdot d$:

$$E_y = \frac{\lambda}{2\pi\epsilon_0} \frac{1}{d} \tag{23}$$

$$E_{2h} = \frac{\lambda}{2\pi\epsilon_0} \left(\frac{1}{2h - y} - \frac{1}{2h + y} \right) \tag{24}$$

$$E_{4h} = \frac{\lambda}{2\pi\epsilon_0} \left(\frac{1}{4h - y} - \frac{1}{4h + y} \right) \tag{25}$$

Image Effects

Let's do some algebra:

$$E_{inh} = \tag{26}$$

$$= (-1)^n \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{2nh+y} - \frac{1}{2nh-y}\right) = \tag{27}$$

$$= (-1)^n \frac{\lambda}{4\pi\epsilon_0}\frac{y}{n^2h^2} \tag{28}$$

$$E_{iy} = \sum_{n=1}^{\infty} E_{iny} = \frac{\lambda}{4\pi\epsilon_0 h^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}y = \tag{29}$$

$$= \frac{\lambda}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}y \tag{30}$$

We obtain the corresponding fields and forces:

$$E_{ix} = -\frac{\lambda}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}x \tag{31}$$

$$F_{iy} = \frac{e\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48}y \tag{32}$$

$$F_{ix} = -\frac{e\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48}x \tag{33}$$

Incoherent Tune Shift

The total contribution to the incoherent tune shift can be summarized:

$$\Delta Q_x = -\frac{2r_0 I_b R\langle\beta_x\rangle}{qc\beta^3\gamma}\left(\frac{1}{2\langle a^2\rangle\gamma^2} - \frac{\pi^2}{48^2}\right) \tag{34}$$

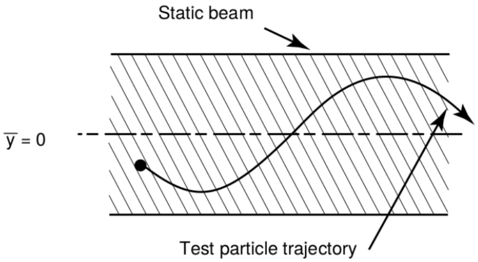
$$\Delta Q_y = -\frac{2r_0 I_b R\langle\beta_y\rangle}{qc\beta^3\gamma}\left(\frac{1}{2\langle a^2\rangle\gamma^2} + \frac{\pi^2}{48^2}\right) \tag{35}$$

- Direct field.
- Image field.

Coherent vs Incoherent effects

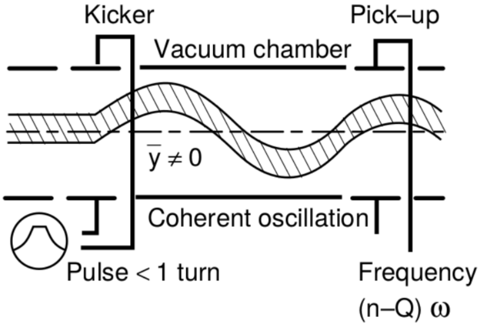
Incoherent

Each particle is independent (has its own betatron oscillation, phase and tune). Impossible to observe any betatron motion. The beam "does not move".



Coherent

The kick gets a fast deflection that affects the full distribution and starts to perform betatron oscillations as a whole. The source of space charge is now moving.



Coherent Tune Shift

Taking ρ the beam pipe radius and $\bar{x}$ the center o mass position.
Image charge at $b = \rho^2/\bar{x}$.

$$E_{ix} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{b-\bar{x}} \approx \frac{\lambda}{2\pi\epsilon_0}\frac{1}{b} = \frac{\lambda}{2\pi\epsilon_0}\frac{\bar{x}}{\rho^2} \tag{36}$$

$$F_{ix} = \frac{e\lambda}{2\pi\epsilon_0}\frac{\bar{x}}{\rho^2} \tag{37}$$

$$\Delta Q_{x,y} = -\frac{r_0 R\langle\beta_{x,y}\rangle I}{ec\beta^3\gamma\rho^2} = -\frac{r_0\langle\beta_{x,y}\rangle}{2\pi\beta^2}\frac{N}{\gamma\rho^2} \tag{38}$$

- The force is linear in $\bar{x}$.
- $1/\gamma$ dependence.
- The coherent tune shift is never positive.
- Perfectly conducting beampipe assumed. Realistic effects are delicate.

Laslett coefficients

A more realistic scenario is when we consider elliptic, unbunched uniformly distirbuted beams travelling at a speed βc through an elliptic vacuum chamber. For these geometries the tune shift can be expressed in terms of the "laslett coefficients".

Incoherent: $\epsilon_{0,1,2}$, Coherent: $\xi_{1,2}$

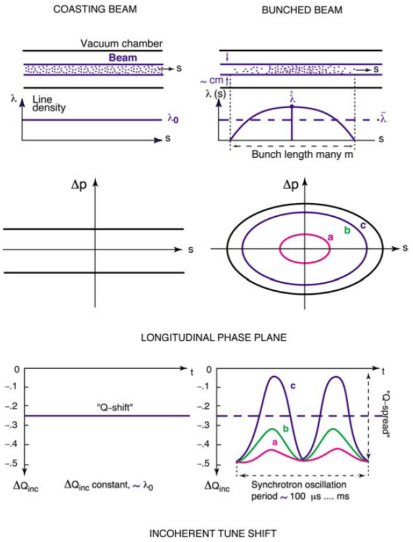
$$\Delta Q_{y,inc.} = \frac{-Nr_0\langle\beta_y\rangle}{\pi\beta^2\gamma}\left(\frac{\epsilon_0^y}{b^2\gamma^2} + \frac{\epsilon_1^y}{h^2} + \beta^2\frac{\epsilon_2^y}{g^2}\right) \quad (39)$$

$$\Delta Q_{y,coh.} = \frac{-Nr_0\langle\beta_y\rangle}{\pi\beta^2\gamma}\left(\frac{\xi_1^y}{h^2} + \beta^2\frac{\xi_2^y}{g^2}\right) \quad (40)$$

Laslett coefficients	Circular $(a = b,\ w = h)$	Elliptical (e.g. $w = 2h$)	Parallel plates $(h/w = 0)$
ε_0^x	1/2	$\frac{b^2}{a(a+b)}$	
ε_0^y	1/2	$\frac{b}{a+b}$	
ε_1^x	0	-0.172	-0.206
ε_1^y	0	0.172	0.206
ξ_1^x	1/2	0.083	0
ξ_1^y	1/2	0.55	$0.617(\pi^2/16)$
ε_2^x	$-0.411(-\pi^2/24)$	-0.411	-0.411
ε_2^y	$0.411(\pi^2/24)$	0.411	0.411
ξ_2^x	0	0	0
ξ_2^y	$0.617(\pi^2/16)$	0.617	0.617

Unbunched vs. bunched beams

- So far we just considered unbunched homogeneous beams. Create a constant tune shift. Easy to solve.
- When bunched beams are considered, the space charge effects are more notorious.
- In bunched beams, each "slice" of the beam feels a different space charge.
- Synchrotron oscillations modulate the space charge force felt by a single particle.
- This generates a tune spread.



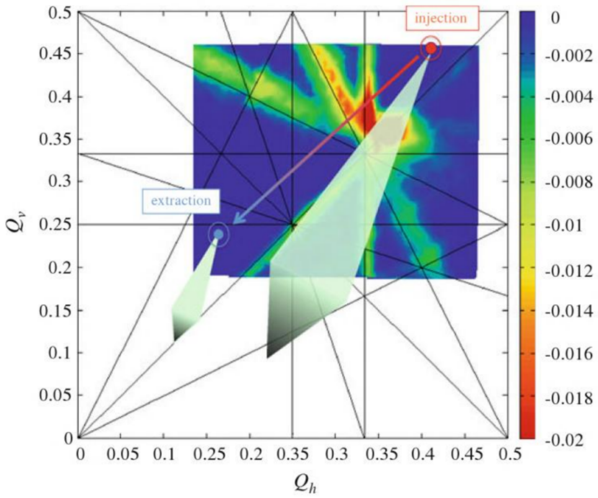
Space Charge Limit

Space Charge may limit the operation if the tune shift is too large and important resonances are crossed.

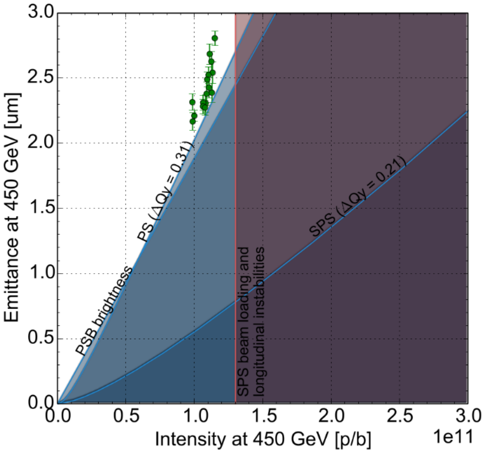
$$\Delta Q \sim \frac{N}{\beta^2\gamma^2} \tag{41}$$

What can we do?

Space Charge Limit

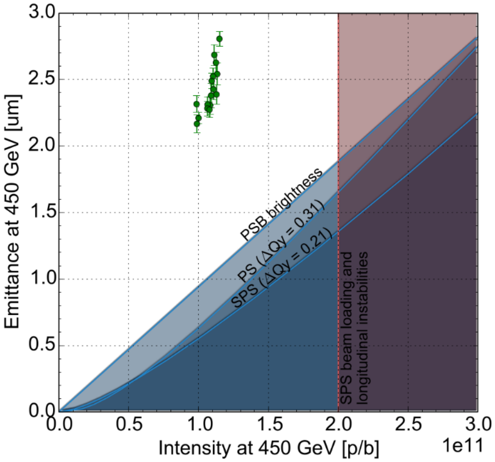


Space Charge Limit: How to mitigate it



- At CERN accelerator chain, previous injector configuration limited the bunch intensity.
- To overcome this limitation, a major upgrade of the injectors was required to achieve HL-LHC desired performance.
- In the example, we can see that the PSB, the PS and the SPS needed to be upgraded.

Space Charge Limit: How to mitigate it



Linac

- Linac4 (H^-) replaces Linac2 (H^+).

PSB

- Energy upgrade.
- Injection: 160 MeV (50 MeV).
- Extraction: 2 GeV (1.4 GeV).

PS

- Replace 43 dipoles.

SPS

- Cabling and Acceleration system.

Summary

- Space Charge limits the performance of particle accelerators.
- Particular impact on low-energy hadron machines.
- We mainly focused on the induced tune shift and tune spread.
- Two main effects: incoherent and coherent.
- There are ways to mitigate the impact of space charge.
- Many other effects not considered here.

Thank you!