Quantum Tomography in Flavor Oscillations

Kun Cheng @ Lepton-Photon
Based on arXiv:2507.12513
with Tao Han, Matthew Low and Arthur Wu

Quantum correlations in flavor space

- Particles can be entangled in flavor space, e.g., meson pair decayed from $\Upsilon(J^{PC} = 1^{--})$

$$\frac{1}{\sqrt{2}}\left(\left|B^0\bar{B}^0\right\rangle - \left|\bar{B}^0 B^0\right\rangle\right)$$

  - Comparing quantum and classicsal prediction [J.Six PLB (1982); CPLEAR, PLB (1998);…]

  - Bell inequality   [Belle, J.Mod.Opt (2004); Belle PRL (2007);…]

  - Quantum decoherence   [KLOE, PLB (2006), KLOE-2, JHEP(2022);…]

  - CPT invariance   [Dunietz, Hauser and Rosner, PRD (1987); KLOE, PLB (2006), KLOE-2, JHEP(2022);…],

  - ……

- Reconstruct the complete quantum information of initial flavor state

  - Meson flavor can correlate in different ways, such as $\frac{|M\bar{M}\rangle - |\bar{M}M\rangle}{\sqrt{2}}, \frac{|M\bar{M}\rangle + |\bar{M}M\rangle}{\sqrt{2}}, \cdots$

  - Most general parametrization:

$$\rho_{MM} = \frac{\mathbb{1}_4 + b_i^{\mathcal{A}}\sigma_i \otimes \mathbb{1}_2 + b_i^{\mathcal{B}}\mathbb{1}_2 \otimes \sigma_i + C_{ij}\sigma_i \otimes \sigma_j}{4}$$

Introduction: Quantum tomography

- One qubit: 3 parameters

$$\rho = \frac{\mathbb{1}_2 + b_i \sigma_i}{2}, \quad b_i = \langle \sigma_i \rangle$$

- Two qubits: 3+3+9 parameters

$$\rho = \frac{\mathbb{1}_4 + b_i^{\mathcal{A}} \sigma_i \otimes \mathbb{1}_2 + b_i^{\mathcal{B}} \mathbb{1}_2 \otimes \sigma_i + C_{ij} \sigma_i \otimes \sigma_j}{4}$$

$$b_i^{\mathcal{A}} = \langle \sigma_i \otimes \mathbb{1}_2 \rangle$$
$$b_i^{\mathcal{B}} = \langle \mathbb{1}_2 \otimes \sigma_i \rangle$$
$$C_{ij} = \langle \sigma_i \otimes \sigma_j \rangle$$

- A complementary set of measurements is needed

- **Qubit in spin space:**

  - $\vec{\sigma}$ is embeded in 3d spatial space.

  - Different direction of $\vec{\sigma} \implies$ complementary.

- **Qubit in flavor space**?

  - e.g. flavor tagging, only σ_z: $\sigma_z |B^0\rangle = |B^0\rangle, \quad \sigma_z |\bar{B}^0\rangle = -|\bar{B}^0\rangle$

  - cannot choose the "*direction*" of measurement freely

Introduction: Flavor/Mass eigenstate

- Hamitonian in flavor eigenstate: $|M\rangle, |\bar{M}\rangle$:
$$H = \mathbf{M} - i\mathbf{\Gamma}/2 = \begin{pmatrix} m - i\frac{\Gamma}{2} & P^2 \\ Q^2 & m - i\frac{\Gamma}{2} \end{pmatrix},$$

- Mass eigenstates are
$$|M_1\rangle = p|M\rangle + q|\bar{M}\rangle, \qquad (m_1, \Gamma_1)$$
$$|M_2\rangle = p|M\rangle - q|\bar{M}\rangle, \qquad (m_2, \Gamma_2)$$

- CP conserving limit: $|M_{1/2}\rangle \approx (|M\rangle \pm |\bar{M}\rangle)/\sqrt{2}$

- $\rho_M = \dfrac{\mathbb{1}_2 + b_i \sigma_i}{2}$, b_i still have physical meaning. Our conversions:

  ▸ z-direction $\Longrightarrow$ flavor eigenstate,
    (oscillates with y)
    $$|M\rangle, b_z = 1$$
    $$|\bar{M}\rangle, b_z = -1$$

  ▸ x-direction $\Longrightarrow$ mass eigenstates
    (and CP eigenstates)
    $$|M_1\rangle, b_x = 1$$
    $$|M_2\rangle, b_x = -1$$

> difference between flavor and mass eigenstate leads to flavor oscillation

State evolution in Bloch-vector space

$$|M(t)\rangle \propto U(t)|M(0)\rangle$$
$$\rho(t) \propto U(t)\rho(0)U(t)^\dagger$$

$$U = \begin{pmatrix} \frac{1}{2}(e^{-\Gamma_1 t/2 - im_1 t} + e^{-\Gamma_2 t/2 - im_2 t}) & \frac{q}{2p}(e^{-\Gamma_1 t/2 - im_1 t} - e^{-\Gamma_2 t/2 - im_2 t}) \\ \frac{p}{2q}(e^{-\Gamma_1 t/2 - im_1 t} - e^{-\Gamma_2 t/2 - im_2 t}) & \frac{1}{2}(e^{-\Gamma_1 t/2 - im_1 t} + e^{-\Gamma_2 t/2 - im_2 t}) \end{pmatrix}$$

- In Bloch-vector space: [Kerbikov, NPA (2018)]
  [Karamitros, McKelvey and Pilaftsis, PRD (2023)]

  ▸ Neglect CPV and decay ($\Gamma \ll \Delta m$):

  Schrödinger picture

  density matrix $\rho = \frac{I_2 + \vec{b} \cdot \vec{\sigma}}{2}$

  $$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] = (\vec{X} \times \vec{b}(t)) \cdot \vec{\sigma},$$
  $$\frac{d\vec{b}(t)}{dt} = \vec{X} \times \vec{b}(t).$$

  Heisenberg picture

  operator $A = \vec{a} \cdot \vec{\sigma}$

  $$\frac{dA(t)}{dt} = i[H, A(t)] = -\left(\vec{X} \times \vec{a}(t)\right) \cdot \vec{\sigma},$$
  $$\frac{d\vec{a}(t)}{dt} = -\vec{X} \times \vec{a}(t).$$

  ▸ Precession around $\vec{X} = (\Delta m, 0, 0)$

- Example operator $\sigma_z|M\rangle = +|M\rangle, \quad \sigma_z|\bar{M}\rangle = -|\bar{M}\rangle$

  ▸ Flavor tagging at different times $\implies$ both σ_z and σ_y

y,z components from flavor asymmetry

- Decay final state that is a "flavor eigenstate" $(B^0 \to \ell^+ \nu X^-, \cdots)$

  ▸ $M \to f$ with $CP |f\rangle = |\bar{f}\rangle \neq |f\rangle$:   project to $|M\rangle, |\bar{M}\rangle$ with $P_{M/\bar{M}} = \dfrac{1 \pm b_z}{2}$
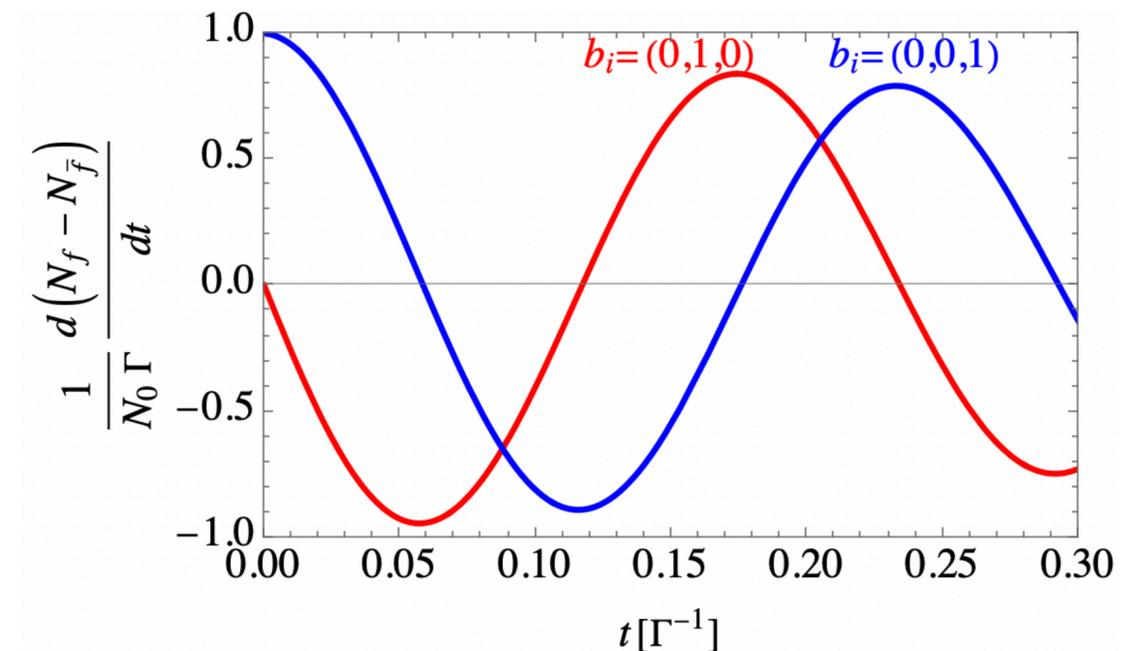
- Decay rate asymmetry to $f, \bar{f}$:

$$\left(N_{M(t) \to f} - N_{M(t) \to \bar{f}} \right) \sim \langle \sigma_z(t) \rangle = b_z(t)$$

- Observable:

$$\frac{1}{N_0} \frac{d(N_f(t) - N_{\bar{f}}(t))}{dt} = e^{-\Gamma t}(b_z \cos(\Delta m t) - b_y \sin(\Delta m t))\Gamma_{M \to f}$$

Meson flavor state when it is produced, $t = 0$

$b_i = (0,1,0)$ $b_i = (0,0,1)$

- Both b_y and b_z are obtained as they oscillated into each other.
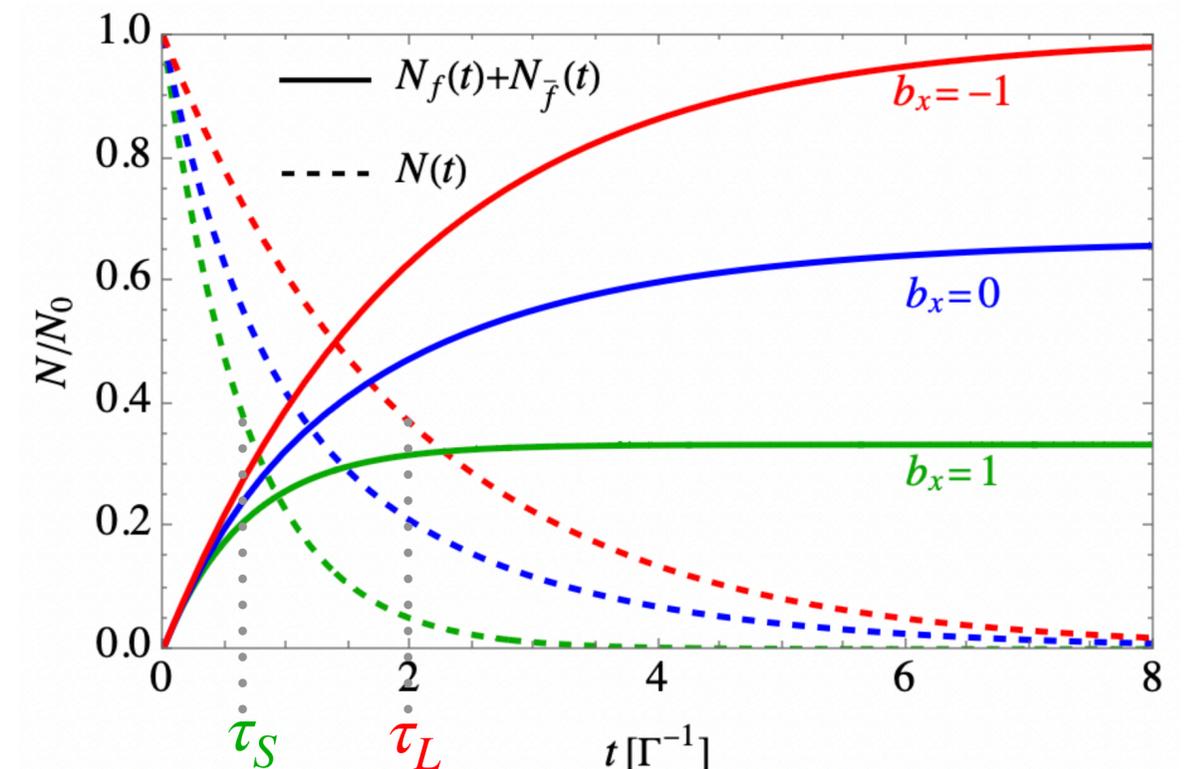
$\Delta m \approx 27\,\Gamma$,
slightly damped oscillation

x components from decay rates

- Most meson can also decay to CP eigenstate $|f_\pm\rangle$ with $CP|f_\pm\rangle = \pm|f_\pm\rangle$, (e.g. $B_s^0 \to J/\psi\eta$) leading to a decay life time difference of $|M_1\rangle$ and $|M_2\rangle$

$$\Gamma_{M(t)\to f_+} = \frac{1 + b_x(t)}{2}\Gamma_{M_1 \to f_+}$$

- The decay to both "CP final state" ($M(t) \to f_+$) or "flavor final state" ($M(t) \to f/\bar{f}$) depends on b_x

  ▸ $b_x = 1$, M_1 can decay to f_+, less $f/\bar{f}$ final state

  ▸ $b_x = -1$, M_2 can't decay to f_+

- The total decay rate to flavor eigenstate (e.g. semileptonic decay):

$$\frac{1}{N_0}\frac{d(N_f(t) + N_{\bar{f}}(t))}{dt} = e^{-\Gamma t}(\cosh(\Delta\Gamma t/2) - \boxed{b_x}\sinh(\Delta\Gamma t/2))\Gamma_{M\to f}$$

difference between each lines $\propto \Delta\Gamma$ i.e., decay to CP eigenstates.

Observables in semileptonic decay channel

- One meson: $\rho_M = \dfrac{\mathbb{1}_2 + b_i \sigma_i}{2}$

  ▸ $N_f - N_{\bar{f}} \Longrightarrow b_y, b_z$

  $$\frac{1}{N_0} \frac{d(N_f(t) - N_{\bar{f}}(t))}{dt} = e^{-\Gamma t}(b_z \cos(\Delta m t) - b_y \sin(\Delta m t))\Gamma_{M \to f}$$

  ▸ $N_f + N_{\bar{f}} \Longrightarrow b_x$

  $$\frac{1}{N_0} \frac{d(N_f(t) + N_{\bar{f}}(t))}{dt} = e^{-\Gamma t}(\cosh(\Delta\Gamma t/2) - b_x \sinh(\Delta\Gamma t/2))\Gamma_{M \to f}$$

- Meson pair: $\rho_{MM} = \dfrac{\mathbb{1}_4 + b_i^{\mathcal{A}} \sigma_i \otimes \mathbb{1}_2 + b_i^{\mathcal{B}} \mathbb{1}_2 \otimes \sigma_i + C_{ij} \sigma_i \otimes \sigma_j}{4}$

  ▸ Four observables from the correlation between the above two

  $\mathscr{H}_A \otimes \mathscr{H}_B \quad N_{f\bar{f}}$: meson A decay to f **and** meson B decay to $\bar{f}$

  $I_2 \otimes I_2 \longrightarrow \quad N_{\text{tot}} = N_{ff} + N_{\bar{f}f} + N_{f\bar{f}} + N_{\bar{f}\bar{f}}$

  $\sigma_z \otimes \sigma_z \longrightarrow \quad A_{ff} = N_{ff} - N_{\bar{f}f} - N_{f\bar{f}} + N_{\bar{f}\bar{f}} = N_{\text{like}} - N_{\text{unlike}}$

  $\sigma_z \otimes I_2 \longrightarrow \quad A_f^{\mathcal{A}} = N_{ff} + N_{f\bar{f}} - N_{\bar{f}f} - N_{\bar{f}\bar{f}}$

  $I_2 \otimes \sigma_z \longrightarrow \quad A_f^{\mathcal{B}} = N_{ff} - N_{f\bar{f}} + N_{\bar{f}f} - N_{\bar{f}\bar{f}}$

Observables in semileptonic decay channel

- One meson: $\rho_M = \dfrac{\mathbb{1}_2 + b_i \sigma_i}{2}$

  - $N_f - N_{\bar{f}} \Longrightarrow b_y, b_z$

    $$\frac{1}{N_0}\frac{d(N_f(t) - N_{\bar{f}}(t))}{dt} = e^{-\Gamma t}(b_z \cos(\Delta m t) - b_y \sin(\Delta m t))\Gamma_{M \to f}$$

  - $N_f + N_{\bar{f}} \Longrightarrow b_x$

    $$\frac{1}{N_0}\frac{d(N_f(t) + N_{\bar{f}}(t))}{dt} = e^{-\Gamma t}(\cosh(\Delta\Gamma t/2) - b_x \sinh(\Delta\Gamma t/2))\Gamma_{M \to f}$$

- Meson pair: $\rho_{MM} = \dfrac{\mathbb{1}_4 + b_i^{\mathcal{A}}\sigma_i \otimes \mathbb{1}_2 + b_i^{\mathcal{B}}\mathbb{1}_2 \otimes \sigma_i + C_{ij}\sigma_i \otimes \sigma_j}{4}$

  - Four observables from the correlation between the above two

$$\frac{d^2 N_{\text{tot}}}{dt_1 dt_2} = N_0 \Gamma_{B_0 \to f}^2 e^{-\Gamma(t_1+t_2)}\left(\mathrm{ch}_{t_1}\mathrm{ch}_{t_2} - \mathrm{ch}_{t_1}\mathrm{sh}_{t_2} b_x^{\mathcal{A}} - \mathrm{sh}_{t_1}\mathrm{ch}_{t_2} b_x^{\mathcal{B}} + \mathrm{sh}_{t_1}\mathrm{sh}_{t_2} C_{xx}\right) + \mathcal{O}(\epsilon),$$

$$\frac{d^2 A_{ff}}{dt_1 dt_2} = N_0 \Gamma_{B_0 \to f}^2 e^{-\Gamma(t_1+t_2)}\left(\mathrm{s}_{t_1}\mathrm{s}_{t_2} C_{yy} - \mathrm{c}_{t_1}\mathrm{s}_{t_2} C_{zy} - \mathrm{s}_{t_1}\mathrm{c}_{t_2} C_{yz} + \mathrm{c}_{t_1}\mathrm{c}_{t_2} C_{zz}\right) + \mathcal{O}(\epsilon),$$

$$\frac{d^2 A_f^{\mathcal{A}}}{dt_1 dt_2} = N_0 \Gamma_{B_0 \to f}^2 e^{-\Gamma(t_1+t_2)}\left(\mathrm{ch}_{t_2}(\mathrm{c}_{t_1} b_z^{\mathcal{A}} - \mathrm{s}_{t_1} b_y^{\mathcal{A}}) - \mathrm{sh}_{t_2}(\mathrm{c}_{t_1} C_{zx} - \mathrm{s}_{t_1} C_{yx})\right) + \mathcal{O}(\epsilon),$$

$$\frac{d^2 A_f^{\mathcal{B}}}{dt_1 dt_2} = N_0 \Gamma_{B_0 \to f}^2 e^{-\Gamma(t_1+t_2)}\left(\mathrm{ch}_{t_1}(\mathrm{c}_{t_2} b_z^{\mathcal{B}} - \mathrm{s}_{t_2} b_y^{\mathcal{B}}) - \mathrm{sh}_{t_1}(\mathrm{c}_{t_2} C_{xz} - \mathrm{s}_{t_2} C_{xy})\right) + \mathcal{O}(\epsilon),$$

$\epsilon = 10^{-3} \sim 10^{-5}$

Sensitivity estimation

Tab I. Statistical uncertainty with 10^7 events

- Example 1: Bell state $(|M\bar{M}\rangle - |\bar{M}M\rangle)/\sqrt{2}$

$$\rho_{\text{Bell}}: \quad b_i^{\mathcal{A}/\mathcal{B}} = 0, \; C_{ij} = -\delta_{ij}$$

- Example 2:

$$\rho_{\kappa} = (1-\kappa)\rho_{\text{Bell}} + \kappa\frac{\mathbb{1}_4}{4}, \qquad \kappa = 0.2$$

- B_s pair: same sensitivity on y and z components; x components are harder to measure.

- K_0 pair: same order sensitivity on all components. C_{xx} can be reconstruct almost equally well with others.

	$B_s^0\bar{B}_s^0$ fitted ρ_{Bell} (ρ_κ)	$K^0\bar{K}^0$ fitted ρ_{Bell} (ρ_κ)	Obs.
$b_x^{\mathcal{A}}$	0 ± 0.020 (0 ± 0.017)	0 ± 0.0015 (0 ± 0.0018)	N_{tot}
$b_x^{\mathcal{B}}$	0 ± 0.019 (0 ± 0.016)	0 ± 0.0017 (0 ± 0.0018)	
$b_y^{\mathcal{A}}$	0 ± 0.0009 (0 ± 0.0010)	0 ± 0.0019 (0 ± 0.0021)	$A_f^{\mathcal{A}}$
$b_z^{\mathcal{A}}$	0 ± 0.0010 (0 ± 0.0009)	0 ± 0.0014 (0 ± 0.0016)	
$b_y^{\mathcal{B}}$	0 ± 0.0010 (0 ± 0.0008)	0 ± 0.0020 (0 ± 0.0023)	$A_f^{\mathcal{B}}$
$b_z^{\mathcal{B}}$	0 ± 0.0009 (0 ± 0.0008)	0 ± 0.0015 (0 ± 0.0016)	
C_{xx}	-1 ± 0.36 (-0.8 ± 0.29)	-1 ± 0.0031 (-0.8 ± 0.0032)	N_{tot}
C_{yx}	0 ± 0.016 (0 ± 0.018)	0 ± 0.0023 (0 ± 0.0026)	$A_f^{\mathcal{A}}$
C_{zx}	0 ± 0.018 (0 ± 0.017)	0 ± 0.0017 (0 ± 0.0019)	
C_{xy}	0 ± 0.017 (0 ± 0.015)	0 ± 0.0025 (0 ± 0.0029)	$A_f^{\mathcal{B}}$
C_{xz}	0 ± 0.016 (0 ± 0.016)	0 ± 0.0018 (0 ± 0.0020)	
C_{yy}	-1 ± 0.0011 (-0.8 ± 0.0010)	-1 ± 0.0027 (-0.8 ± 0.0023)	
C_{yz}	0 ± 0.0007 (0 ± 0.0008)	0 ± 0.0022 (0 ± 0.0020)	A_{ff}
C_{zy}	0 ± 0.0008 (0 ± 0.0008)	0 ± 0.0021 (0 ± 0.0017)	
C_{zz}	-1 ± 0.0011 (-0.8 ± 0.0009)	-1 ± 0.0010 (-0.8 ± 0.0011)	

Example quantum information variables

- **Concurrence** measures how much two subsystems are non-separable.

- **Bell variable** is constructed as a normalized measure for the Bell inequality.

- **Steerability** indicate the system does not admit a local hidden state model.

- **The second stabilizer Renyi entropy** (SSRE) quantifies the computation advantage this system would have on a quantum computer over a classical computer.

- ……

Tab II. Example quantum information variables of $\rho_\kappa(\kappa = 0.2)$, measured from with 10^7 events

	$B_s^0 \bar{B}_s^0$	$K^0 \bar{K}^0$
Concurrence	0.69 ± 0.14	0.702 ± 0.0024
Bell Variable	0.1854 ± 0.0014	0.188 ± 0.003
Quantum Discord	0.58 ± 0.10	0.617 ± 0.0023
Steerability Variable	0.48 ± 0.07	0.4822 ± 0.0015
Conditional Entropy	-0.15 ± 0.29	-0.157 ± 0.005
SSRE	0.38 ± 0.15	0.388 ± 0.003

Can be measured without the full denstiy matrix.

B_d^0: [Belle, J.Mod.Opt (2004); Belle PRL (2007)],

Conclusion and discussion

- We provide a systematic reconstruction of the flavor density matrix of meson pair

  ▸ In the Bloch-vector space, measureing y, z components requires a sizable $\Delta m/\Gamma$, while measuring x components requires a sizable $\Delta\Gamma/\Gamma$. (Ideally order 1)

	B_s^0	B_d^0	D^0	K^0
Γ (ps^{-1})	0.662	0.658	2.44	5.59×10^{-3}
$\Delta m/\Gamma$	26.8	0.769	4.6×10^{-3}	0.95
$\Delta\Gamma/\Gamma$	0.135	4.0×10^{-3}	0.012	1.99

  ▸ The complete flavor density matrix of B_s^0 pair and K^0 pair is easier to measure. The (y, z) components of the density matrix of B_d^0 can also be measured.

- With the complete flavor state reconstructed, any quantum information variable can be measured. Some variables (such as Bell variable) can be measured from a partial density matrix.

- Generally, there is no first principle prediction of the meson flavor state (e.g. at LHC). Measuring the flavor state can facilitate the production mechanism.

Backup: definitions

- Concurrence: $\mathcal{C} = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ $\qquad R_\rho = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}, \quad \tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)$

  $\mathcal{C} > 0$: entangle; $\quad \mathcal{C} = 0$: separable

- Bell variable: $\mathcal{B} = |C_{yy} + C_{zz}| - \sqrt{2}$

  $\mathcal{B} > 0$: Bell non-local; $\quad \mathcal{B} < 0$: Bell local

- Discord: qantum discord captures a broader class of quantum correlations than entanglement; it can be non-zero even for separable states

$$
\begin{aligned}
\mathcal{D} = 1 &+ \frac{1}{4}(1 - C_{zz} - C_{xx} - C_{yy})\log_2\left(\frac{1 - C_{zz} - C_{xx} - C_{yy}}{4}\right) \\
&+ \frac{1}{4}(1 - C_{zz} + C_{xx} + C_{yy})\log_2\left(\frac{1 - C_{zz} + C_{xx} + C_{yy}}{4}\right) \\
&+ \frac{1}{4}(1 + C_{zz} - C_{xx} + C_{yy})\log_2\left(\frac{1 + C_{zz} - C_{xx} + C_{yy}}{4}\right) \\
&+ \frac{1}{4}(1 + C_{zz} + C_{xx} - C_{yy})\log_2\left(\frac{1 + C_{zz} + C_{xx} - C_{yy}}{4}\right) \\
&- \frac{1}{2}(1 + \lambda)\log_2\left(\frac{1 + \lambda}{2}\right) - \frac{1}{2}(1 - \lambda)\log_2\left(\frac{1 - \lambda}{2}\right),
\end{aligned}
$$

Backup: definitions

- Steerability condition is $\mathcal{S} > 0$ excludes local hidden state model

$$\mathcal{S} = \frac{1}{2\pi^2}\int d\hat{\mathbf{n}}\sqrt{\hat{\mathbf{n}}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\hat{\mathbf{n}}} - \frac{1}{\pi}$$

- Conditional entropy quantifies the number of bits needed for subsystem A to reconstruct subsystem B

$$S(\rho_{\mathcal{A}}|\rho_{\mathcal{B}}) = S(\rho_{\mathcal{AB}}) - S(\rho_{\mathcal{B}}), \quad S(\rho) = -\mathrm{tr}(\rho\log_2\rho).$$

- The second stabilizer Renyi entropy (SSRE) quantifies quantum magic

$$\mathcal{M}_2 = -\log_2\left(\frac{1 + \sum_i (b_i^{\mathcal{A}})^4 + \sum_j (b_j^{\mathcal{B}})^4 + \sum_{i,j} C_{ij}^4}{1 + \sum_i (b_i^{\mathcal{A}})^2 + \sum_j (b_j^{\mathcal{B}})^2 + \sum_{i,j} C_{ij}^2}\right)$$