



Northern Illinois
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Modification Of Generating Functions For Dynamic Aperture Enlargement

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Background Theory

Generating Functions

Simulation Results

Conclusion

Background Theory

Dynamic Aperture (DA): region in phase space in which particles remain stable under perturbations.

1. Decrease loss of particles from the beam.
2. Induce higher intensity beams.

Large dynamic aperture approaches highly recommended in recent recommendations for the field [1, 2, 3].

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Our method: Finding a stable symplectic map close to the original lattice map.

All physical machines have a corresponding one-turn map, for which there exists a corresponding generating function of the form [4]:

$$F_0 = F_{\text{Quad}} + F_{\text{H}}. \quad (1)$$

Introduce a “stabilization function” χ :

$$\chi = 1 - \frac{\text{Invariant Surface}}{\text{Boundary Value}} \quad (2)$$

A new stable generating function is then found by:

$$F_S = F_{\text{Quad}} + \chi^2 F_{\text{H}} \quad (3)$$

All accelerators have invariant ellipses determined by linear map elements [5]:

$$\alpha_x = \frac{\langle x|x \rangle}{2 \sin(\mu)}, \quad \beta_x = \frac{\langle x|p_x \rangle}{\sin(\mu)}, \quad \gamma_x = \frac{\langle p_x|x \rangle}{\sin(\mu)}, \quad (4)$$

The resulting Courant-Snyder invariants are then found as

$$\begin{aligned} 1 &= \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2, \\ 1 &= \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2. \end{aligned}$$

Since all accelerators share this property, we will use the linear combination of these two invariant ellipses as our invariant surface.

Generating Functions

Let \mathcal{M} be a symplectic map. For every point z \exists neighborhood such that the map can be represented by the function F [6]:

$$\left(\vec{\nabla}F\right)^{\top} = \left(\alpha_1 \circ \begin{pmatrix} \mathcal{M} \\ \mathcal{I} \end{pmatrix}\right) \circ \left(\alpha_2 \circ \begin{pmatrix} \mathcal{M} \\ \mathcal{I} \end{pmatrix}\right)^{-1}, \quad (5)$$

where α is *any* conformal symplectic map and \mathcal{I} is the identity map. Conversely,

$$\mathcal{M} = \left(\alpha^1 \circ \begin{pmatrix} \left(\vec{\nabla}F\right)^{\top} \\ \mathcal{I} \end{pmatrix}\right) \circ \left(\alpha^2 \circ \begin{pmatrix} \left(\vec{\nabla}F\right)^{\top} \\ \mathcal{I} \end{pmatrix}\right)^{-1}. \quad (6)$$

Since \exists infinitely many conformal symplectic maps, infinitely many generating functions may be found for a given map.



Letting $S = S^T$, every generating type belongs to an equivalence class $[S]$ associated with [6]

$$\text{Jac}(\alpha) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -JM^{-1} & J \\ \frac{1}{2}(I + JS)M^{-1} & \frac{1}{2}(I - JS) \end{pmatrix}, \quad (7)$$

where M is the linear part of the symplectic map \mathcal{M} .



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There exist two main types of generating functions to consider:

1. Goldstein-type generating functions
2. Poincaré type generating functions



Require restoration of exact symplecticity for order n truncated map. Can be shown this is achieved with $S = 0$ [6]:

$$\text{Jac}(\alpha) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -JM^{-1} & J \\ \frac{1}{2}M^{-1} & \frac{1}{2}I \end{pmatrix}. \quad (8)$$

Defined as “Extended Poincaré Generating Function”, or *EXPO*.

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Note: Standard Poincaré generating function has $M^{-1} = I$:

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Need to verify is generating function is well-defined inside region of interest.

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1. Desire largest domain of definition to determine best generating function type.
2. If particles cross the domain of definition contours, we cannot trust the results of the tracking.

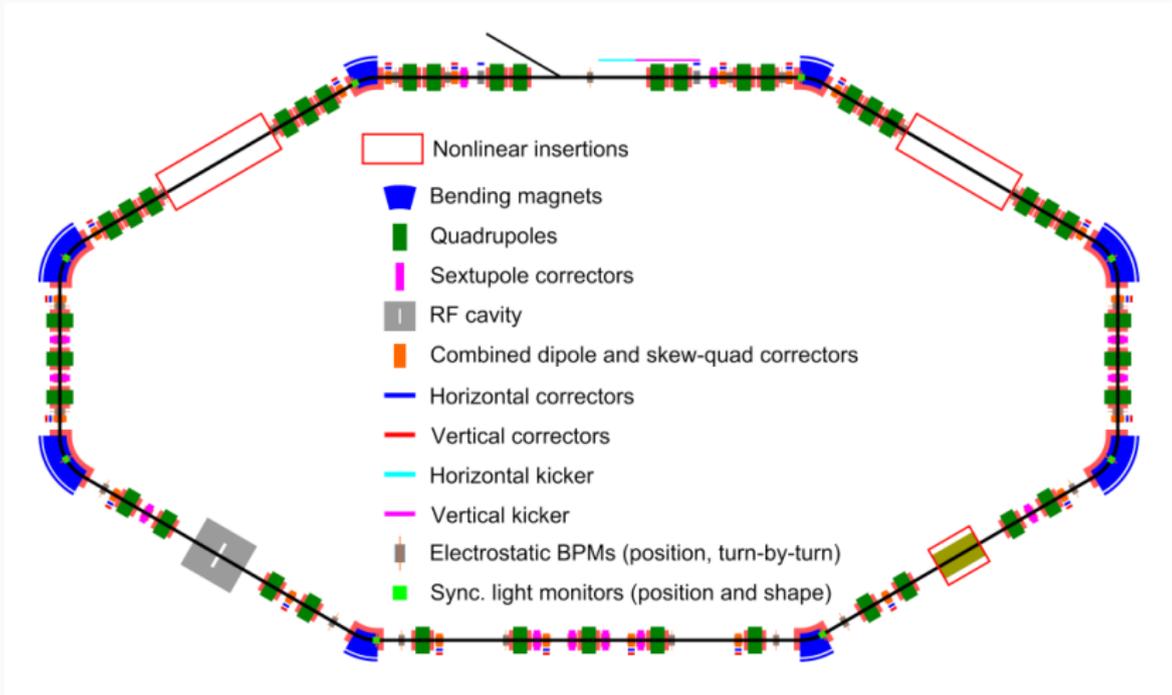


1. Start with known truncated lattice map \mathcal{M} at order n .
2. Calculate generating function from map:

$$\left(\vec{\nabla}F\right)^{\top} = \left(\alpha_1 \circ \begin{pmatrix} \mathcal{M} \\ \mathcal{I} \end{pmatrix}\right) \circ \left(\alpha_2 \circ \begin{pmatrix} \mathcal{M} \\ \mathcal{I} \end{pmatrix}\right)^{-1}.$$

3. Use stabilizing function ($\chi = 1 - (S/B)$) to calculate new generating function: $F_{\text{New}} = F_{\text{Quad}} + \chi^2 F_H$.
4. Find domain of definition.
5. Calculate stable map from the new generating function.
6. Track electrons for 10,000 turns to estimate dynamic aperture for new map.

Simulation Results



Circumference	39.9831 m
Beam Energy	150 MeV
Maximum (β_x, β_y)	(8.5 m, 4 m)
Emittance (ϵ_x, ϵ_y)	(0.04 μm , 0.04 μm)
Tune (ν_x, ν_y)	(0.3,0.3)
Beam size (σ_x, σ_y)	(0.0005831 m, 0.0004 m)
Nonlinear insert length	1.7 m
c-parameter	0.03
t-parameter	0.223, 0.4

Table 1: IOTA Electron Simulation Parameters in COSY Infinity [8].



1. Purely Linear Elements (Dipoles, Quadrupoles, and Drifts)
2. Linear Elements + Sextupoles
3. Linear Elements + Octupole Channel
4. Linear Elements + Sextupoles & Octupoles
5. Linear Elements + Nonlinear Insert
6. Linear Elements + Nonlinear Insert & Sextupoles
7. Linear Elements + Nonlinear Insert & Octupoles
8. Linear Elements + Nonlinear Insert & Sextupoles & Octupoles



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- Original map truncated to order 5.
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- Stable map calculated to order 9.



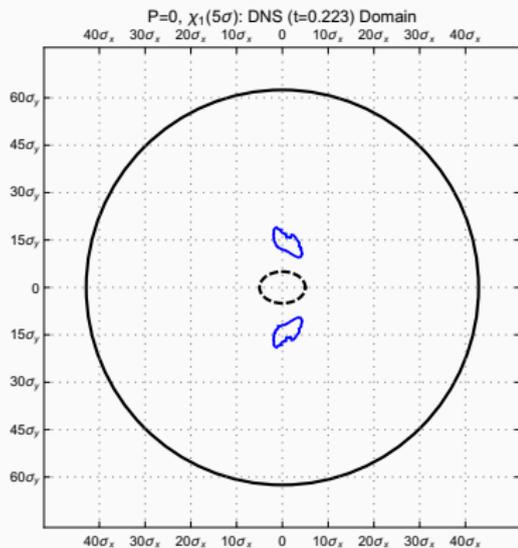
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- Invariant surface evaluated at different values of σ : 2.5σ , 5σ , 7.5σ , 10σ , and 25σ .

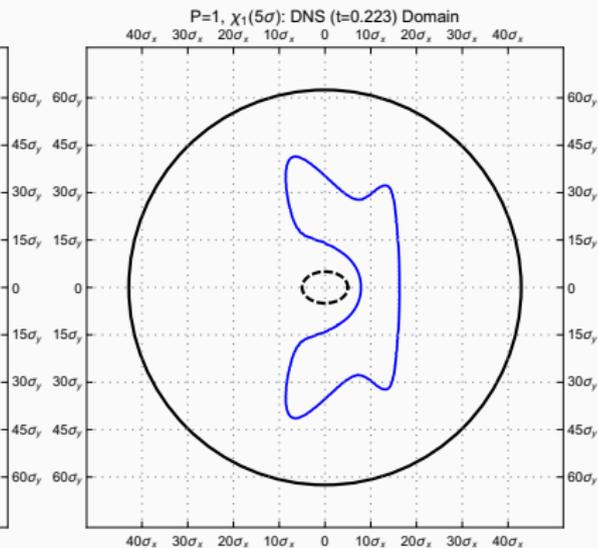


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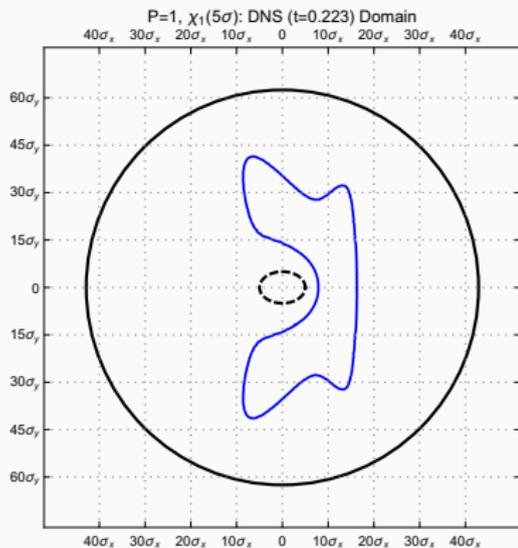
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- From Eq. (3), stable GF calculated to order 10.
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- Invariant surface evaluated at different values of σ : 2.5σ , 5σ , 7.5σ , 10σ , and 25σ .
- DA tracking consists of 8,100 evenly spaced particles between 0 and 10σ (no initial transverse momentum) in the first quadrant.



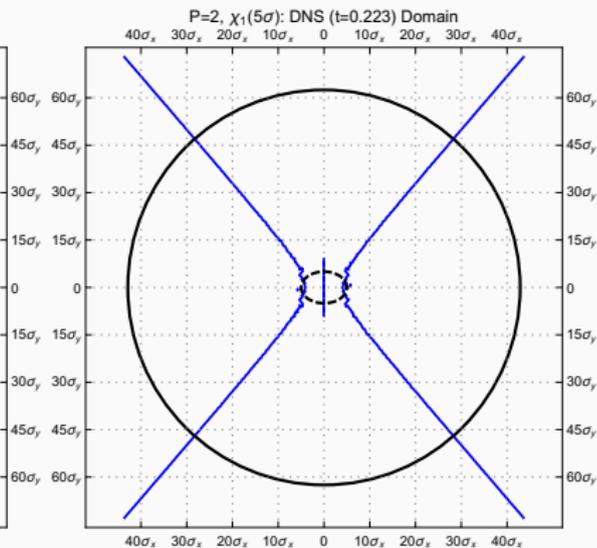
(a) Poincaré



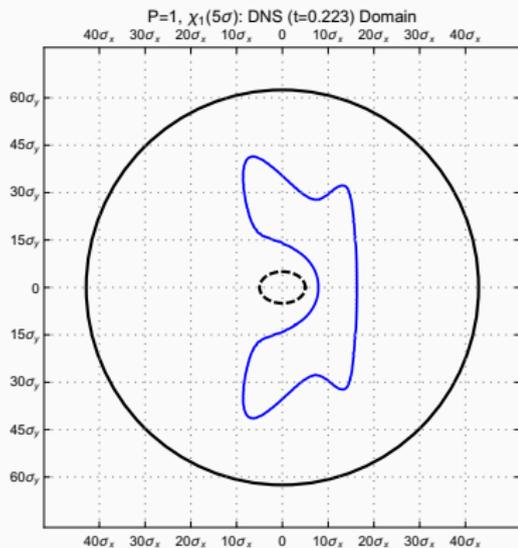
(b) Golstein Type 1



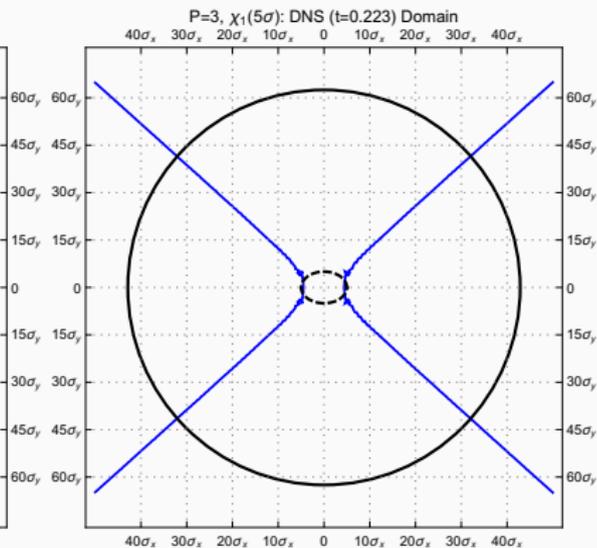
(a) Goldstein Type 1



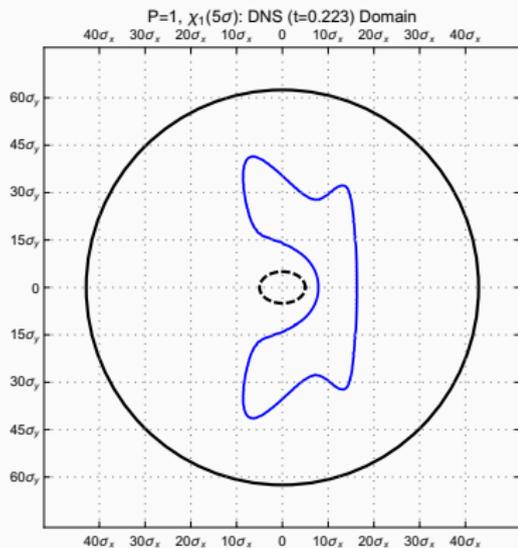
(b) Golstein Type 2



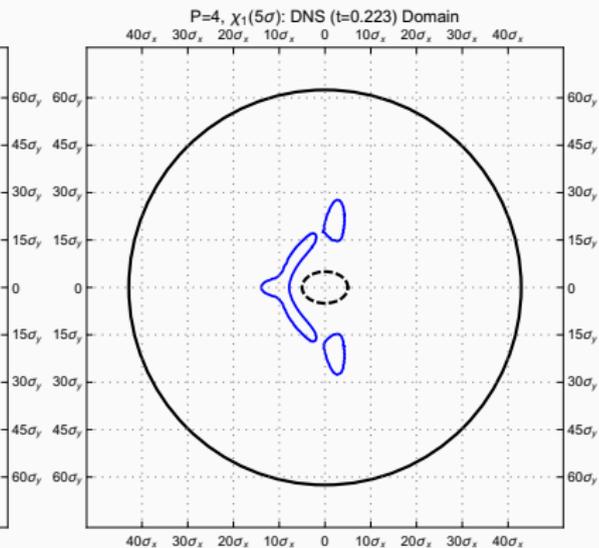
(a) Goldstein Type 1



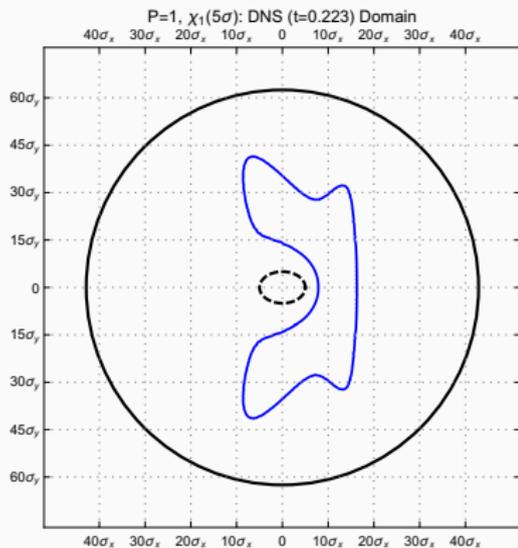
(b) Golstein Type 3



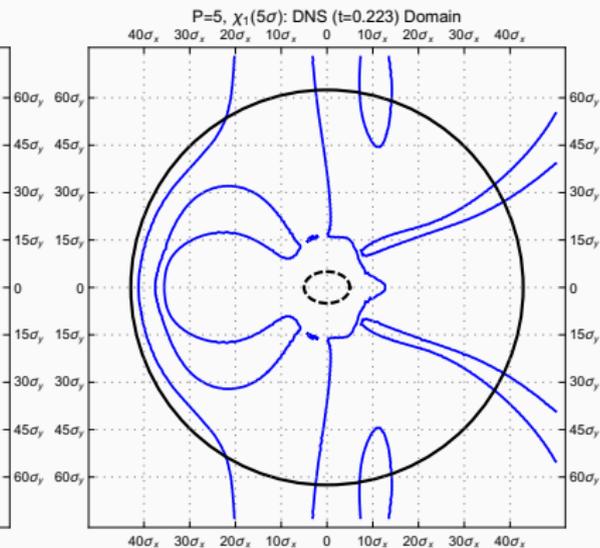
(a) Goldstein Type 1



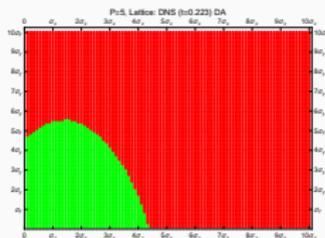
(b) Golstein Type 4



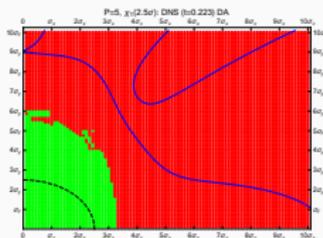
(a) Goldstein Type 1



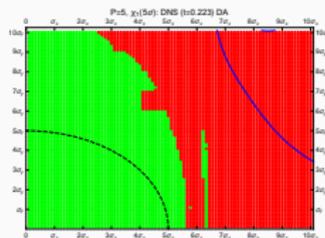
(b) EXPO



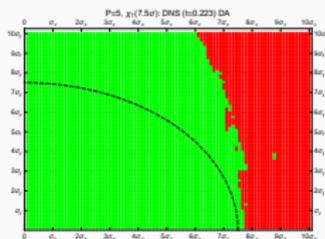
(a) Original Lattice



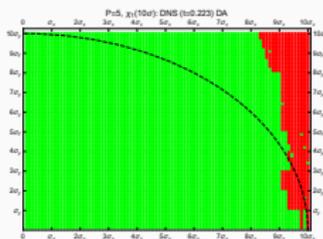
(b) 2.5σ Surface



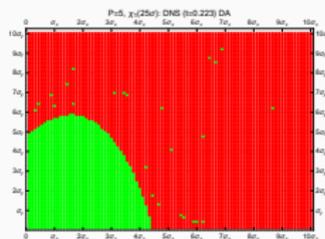
(c) 5σ Surface



(d) 7.5σ Surface



(e) 10σ Surface



(f) 25σ Surface

Figure 6: Dynamic Aperture estimation for Nonlinear Insert ($t=0.223$) with sextupoles turned on in IOTA lattice.

Conclusion



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3. Simplified form of surface utilized:

$$S_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2,$$

$$S_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2,$$

$$S_{\text{Tot}} = S_x + S_y.$$

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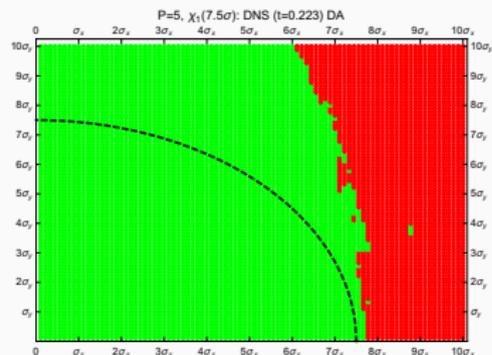
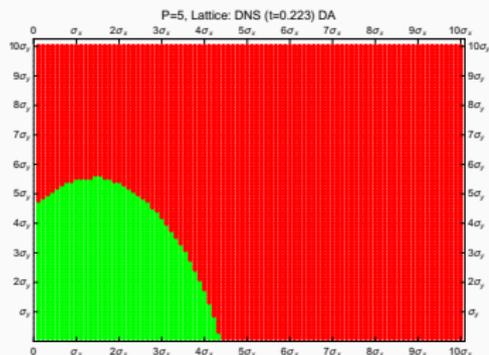
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$$S_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2,$$

$$S_{\text{Tot}} = S_x + S_y.$$

4. Converting stabilized map back into physical machine fine tuning of magnetic elements.

While particles do not always stay on/inside invariant surface as intended, the method does show some promise to increase the estimated DA of an accelerator with careful choosing of the original lattice and invariant surface.





1. Conduct simulations for the same configurations of IOTA with proton beam.
2. Consideration of more complex and varied forms for the invariant surface.
3. Feasibility of using higher precision floating points.
 - Increase the initial order used to calculate original lattice map.
4. Develop optimization scheme to determine method to fine tune physical machine parameters to match the stabilized one-turn map.

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-  S. Gourlay et al.
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-  S. Nagaitsev et al.
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-  P J Channell.
Stable symplectic maps near arbitrary lattice maps.
[AIP Conference Proceedings \(American Institute of Physics\); \(United States\), 326:1, 02 1995.](#)

-  M. Berz, K. Makino, and W. Wan.
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CRC Press, 2015.
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-  Jorge Valenzuela José and Eugene Jerome Saletan.
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Iota (integrable optics test accelerator): facility and experimental beam physics program.

[Journal of Instrumentation](#), 12(03):T03002, mar 2017.

-  Michigan State University.
Cosy.

Thank You!

Backup Slides



Type F_1 is the solution of the implicit relations:

$$\left(\vec{\nabla}F_1\right)^T \begin{pmatrix} \vec{q} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \vec{p} \\ -\vec{P} \end{pmatrix} \quad (10)$$

α can be chosen as a linear map, and this expression may be written as:

$$\left(\vec{\nabla}F\right)^T \left(C \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} + D \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \right) = A \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} + B \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \quad (11)$$

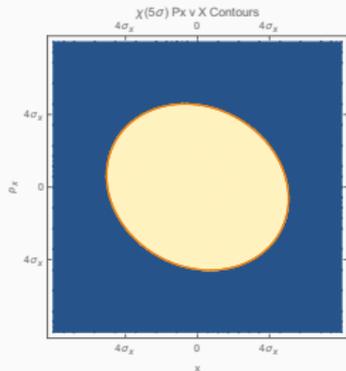
In order to match these two expressions, clearly:

$$\text{Jac}(\alpha_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0_n & 0_n & 0_n & I_n \\ 0_n & -I_n & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \\ I_n & 0_n & 0_n & 0_n \end{pmatrix} \quad (12)$$

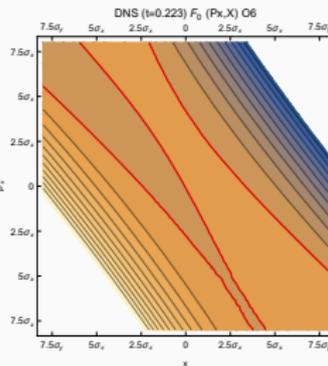
$$\begin{aligned} \text{Jac}(\alpha_1) &= \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \end{pmatrix}, \text{Jac}(\alpha_2) = \begin{pmatrix} 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \\ \text{Jac}(\alpha_3) &= \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{pmatrix}, \text{Jac}(\alpha_4) = \begin{pmatrix} 0 & 0 & -I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{pmatrix}. \end{aligned}$$

$$\text{Jac}(\alpha_{\text{PO}}) = \begin{pmatrix} -J & J \\ \frac{1}{2}I & \frac{1}{2}I \end{pmatrix} = \begin{pmatrix} 0 & -I & 0 & I \\ I & 0 & -I & 0 \\ \frac{1}{2}I & 0 & \frac{1}{2}I & 0 \\ 0 & \frac{1}{2}I & 0 & \frac{1}{2}I \end{pmatrix},$$

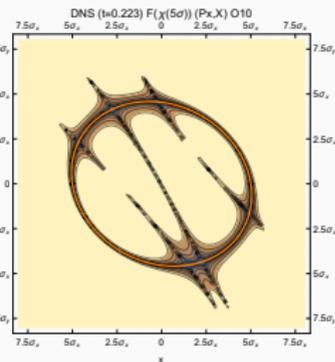
$$\text{Jac}(\alpha_{\text{EXPO}}) = \begin{pmatrix} -JM^{-1} & J \\ \frac{1}{2}M^{-1} & \frac{1}{2}I \end{pmatrix}.$$



(a) $\chi = 1 - \frac{S}{B}$



(b) Original GF

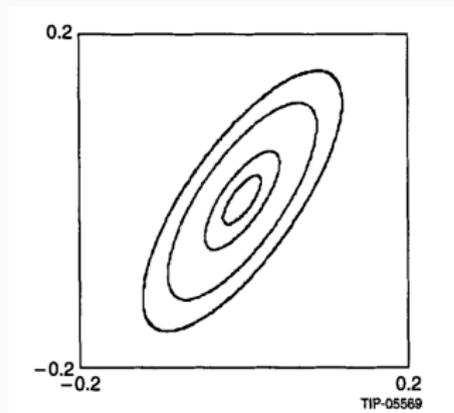


(c) Stabilized GF

Figure 7: Zero-Contours of the stabilization function (a), original lattice generating function (b), and the stabilized generating function (c).



(a)



(b)

Figure 8: From [4], the (x, p_x) phase space plots are shown for a FODO cell with fringe fields: (a) before stabilization, (b) after stabilization.



All maps have associated generating functions of different types [6]. Define:

1. Let $\alpha = (\alpha_1, \alpha_2)^\top$ and $\alpha^{-1} = (\alpha^1, \alpha^2)^\top$ be a diffeomorphism.
2. Let the $4n \times 4n$ Jacobian matrix of α be split into $2n \times 2n$ blocks:

$$\text{Jac}(\alpha) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (13)$$

3. A map α is *conformal symplectic* if it satisfies

$$(\text{Jac}(\alpha))^\top J_{4n} \text{Jac}(\alpha) = \mu \tilde{J}_{4n} \quad (14)$$

Note:

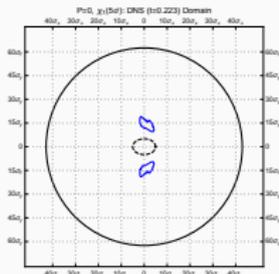
$$J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \quad J_{4n} = \begin{pmatrix} 0_{2n} & I_{2n} \\ -I_{2n} & 0_{2n} \end{pmatrix}, \quad \tilde{J}_{4n} = \begin{pmatrix} J_{2n} & 0_{2n} \\ 0_{2n} & -J_{2n} \end{pmatrix}$$



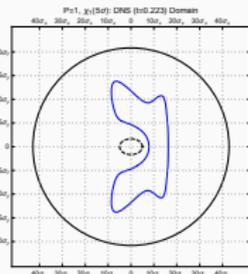
	Lattice	1σ	2.5σ	5σ	7.5σ	10σ	25σ
L	8100	853	3772	4451	8100	8100	8100
LS	5364	316	1500	3836	5601	7165	6972
LO	1747	418	1580	3754	4576	2408	
LSO	849	289	1201	2994	1755	1030	
DN (t=0.223)	8100	590	2861	7155	8100	8100	8100
DNS (t=0.223)	1489	209	1224	3876	5801	7330	1656
DNO (t=0.223)	1358	322	1346	3581	4800	2202	
DNSO (t=0.223)	1158	206	1130	2989	2982	799	
DN (t=0.4)	8100	426	2131	6033	8100	8100	8100
DNS (t=0.4)	1839	224	1102	3321	5424	6847	2152
DNO (t=0.4)	715	262	1094	2877	2326	822	
DNSO (t=0.4)	640	165	868	2363	1408	799	

Table 2: Total Number of Stable Particles

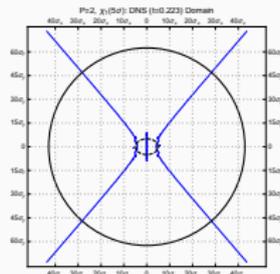
Domain of Definition: Comparison (5σ Invariant Surface)



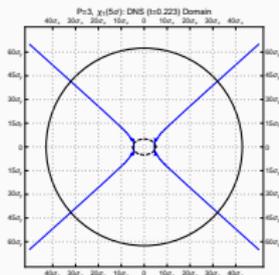
(a) Poincaré



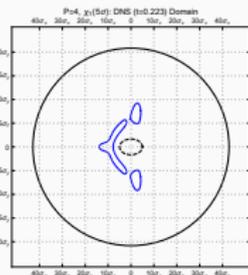
(b) Goldstein Type 1



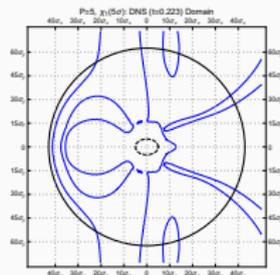
(c) Goldstein Type 2



(d) Goldstein Type 3



(e) Goldstein Type 4



(f) EXPO