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How Matter Matters: The Story of Time Invariance Violation in Neutrino Oscillations

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While neutrino oscillations provide a well motivated probe for CP violation, non-trivial matter effects and our inability to build experiments in an anti-Earth limits our studies to improper tests of its effects. These limitations in turn motivate (from CPT theorem) time invariance studies, as under certain matter potential profiles, proper time invariance and improper time invariance are the same. With this in mind, the following talk will focus on revisiting the pedagogical study of time invariance in matter-based neutrino oscillations, providing potential consequences in the case where we have a new beam source (i.e. muon storage rings) which would allow for an experiment to make time invariance channel comparisons. We discuss the above for different types of matter potential profiles, in an effort to distinguish between intrinsic and matted-induced time invariance violation, if at all, in neutrino oscillation probabilities.

I. INTRODUCTION

Current and next generation of accelerator-based, long-baseline neutrino-oscillation experiments are well positioned to address the status of CP invariance in the neutrino sector by comparing the oscillations of: $P_{\mu e}$ versus $P_{\bar{\mu}\bar{e}}$ [1–4]. This is done, mostly, by making use of pion decays in flight at neutrino beam production. However, the fact that the neutrinos in such experiments usually traverse a non-trivial amount of ordinary matter makes the study of CP invariance more involved [5–7]. Given our inability to build experiments in an anti-Earth, we are stuck with comparing $P_{\mu e}^{\text{matter}}$ and $P_{\bar{\mu}\bar{e}}^{\text{matter}}$, both in matter as opposed to comparing $P_{\mu e}^{\text{matter}}$ with $P_{\bar{\mu}\bar{e}}^{\text{antimatter}}$. Hence, when neutrinos oscillate in matter, we cannot perform a direct, proper test of CP invariance, but must rely on an improper test of CP invariance and use our knowledge of the dynamics of neutrino flavor evolution in matter to study CP invariance in the lepton sector.

The study of T invariance in neutrino oscillations is less prevalent in the neutrino oscillation literature for a very practical reason. T invariance relates $P_{\mu e}$ with $P_{e\mu}$, and most intense, well characterized sources of electron-type neutrinos or antineutrinos (the Sun, nuclear reactors, etc.) yield neutrinos whose energies are low enough that the detection of muon-type neutrinos via charged-current weak interactions is kinematically forbidden. In the intermediate future, however, intense, high-energy muon storage rings – 'neutrino factories' – might be available and would yield intense, high-energy beams of ν_μ and $\bar{\nu}_e$ (or $\bar{\nu}_\mu$ and ν_e, depending on the charge of the stored muons) and allow for realistic studies of T invariance in neutrino oscillations (for overviews and recent studies, see [8–13]).

Similar to the case of CP-invariance studies, studies of T invariance in neutrino oscillations are also impacted by matter effects [7, 13–54]. For T invariance, what we want to compare is $P_{\mu e}$ and $P_{e\mu}$. However, we have to be careful by what we mean. For example, consider when a neutrino is produced as a ν_μ at position a and later detected as a ν_e at position b, let's call this the forward displacement (i.e. $P_{\mu\to e}^{\text{Forward}}$)to the process where a ν_e is produced at position b and detected later as a ν_μ at position a, let's call this the backward displacement (i.e. $P_{\mu\to e}^{\text{Back}}$). In practice, just like we can't shoot neutrinos through an anti-Earth, it is not reasonable to assume we can build both neutrino sources and detectors at two different positions on the Earth's surface such that we could shoot neutrinos forward ($a \to b$) and back ($b \to a$), so we are stuck with an improper test of T invariance, where we compare $P_{\mu\to e}^{\text{Forward}}$ and $P_{e\to\mu}^{\text{Forward}}$. Improper T-invariance tests, however, are qualitatively different from improper CP-invariance tests. If the forward trajectory is indistinguishable from the backward trajectory, the proper and improper tests of T invariance are identical. As far as neutrino oscillations are concerned, this trajectory is indistinguishable from one another if the matter potential is symmetric within the within the total displacement: $a \leftrightarrow b$ segment. In other words, matter-induced T-invariance violation (TV) is proportional to the asymmetry of the matter potential [14, 18, 19, 43, 44].

Here, we explore T invariance in neutrino oscillations, concentrating on the nontrivial role of matter effects. We aim at a pedagogical discussion and will repeat some well-known results for the sake of completeness and will restrict our discussion to the quantum-mechanical formalism of neutrino oscillations, see Section II. We discuss the three-flavor case in Section III, along with some practical implications of matter effects in possible future experimental setups. We summarize our findings and present concluding remarks in Section IV.

II. FORMALISM

We assume all neutrino sources and detectors of interest are such that we can treat neutrinos as ultra-relativistic coherent linear superpositions of the different neutrino mass eigenstates ν_i with masses m_i for $i = 1, 2, 3, \ldots, N$ and a common, well-defined energy E. We will be interested in circumstances where the neutrinos are produced and detected as flavor eigenstates. As usual, the weak-interaction or flavor states are related to the mass eigenstates via

$$|\nu_\alpha\rangle = U_{\alpha i}^*|\nu_i\rangle, \tag{II.1}$$

where $\alpha = e, \mu, \tau, \ldots$, and $U_{\alpha i}$ are the elements of a unitary matrix. We assume the standard 3 active neutrinos limit of

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our formalism. Antineutrino flavor eigenstates follow similarly. By taking CPT invariance as a fundamental symmetry, this ensures that the neutrino and antineutrino mass eigenstates have identical masses and that the same $U_{\alpha i}$ appears in the two equations above (modulo the complex conjugation). It also defines a direct relation between CP invariance and T invariance which we will exploit in this study. Flavor evolution along the source–detector trajectory (coordinate x, baseline L) is governed by the following Schrödinger-like equation:

$$i\frac{\mathrm{d}}{\mathrm{d}x}|\nu\rangle = H|\nu\rangle, \tag{II.2}$$

where H (the Hamiltonian) is a 3×3 Hermitian matrix. Assuming no new interactions other than those of the standard model, we can write, for neutrinos,

$$H = \sum_i \frac{m_i^2}{2E}|\nu_i\rangle\langle\nu_i| + A(x)|\nu_e\rangle\langle\nu_e|, \tag{II.3}$$

where $A = \sqrt{2}G_F n_e(x)$, G_F is the Fermi constant, and $n_e(x)$ is the (in general, position dependent) electron-number density of the medium, which we assume is neutral. Given the computational challenges to calculating the neutrino evolution under a non-zero matter potential A, we have chosen to approximate $n_e(x)$ by studying two different classes of matter density profiles: a symmetric or constant matter profile characterized by an average density, and a non-symmetric or non-constant profile in the form of a two step piecewise constant potential. An example of this is shown in figure 1.

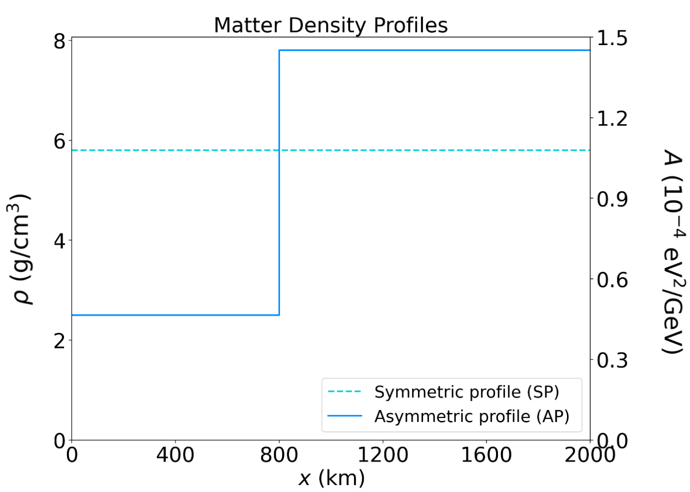

FIG. 1: Above shows an example symmetric (dashed cyan SP) and a asymmetric (solid blue AP) matter profiles used in our analysis.

III. THREE FLAVORS AND SOME PRACTICAL RESULTS

To begin, we present a few concrete examples in order to illustrate how matter effects can impact studies of TV. For this work, we make use of the standard PDG parameterization for the 3×3 mixing matrix and concentrate on the normal neutrino mass ordering (NO), $m_1^2 < m_2^2 < m_3^2$. When computing oscillation probabilities, we use the best-fit values of the oscillation parameters reported by the NuFit collaboration (2024 results) [55]: $\sin^2\theta_{12} = 0.308$, $\sin^2\theta_{23} = 0.470$, $\sin^2\theta_{13} = 0.02215$, $\Delta m_{21}^2 = 7.49\times10^{-5}$ eV^2, and $\Delta m_{31}^2 = 2.513\times10^{-3}$ eV^2.

Fig. 2 (top) depicts $P_{\mu e}$ and $P_{e\mu}$ as a function of the neutrino energy for $L = 2000$ km and $\delta_{\rm CP} = 0$. The green and gold lines correspond to vacuum oscillations while the purple and blue correspond to ordinary matter with constant $\rho = 5.7$ g/cm^3), equivalent matter potential term: $A = 1.1\times10^{-3}$ eV^2/GeV. For each set of probabilities, the oscillation probabilities are identical and the curves lie on top of one another. The main conclusion is that T invariance is preserved ($P_{e\mu} = P_{\mu e}$) .

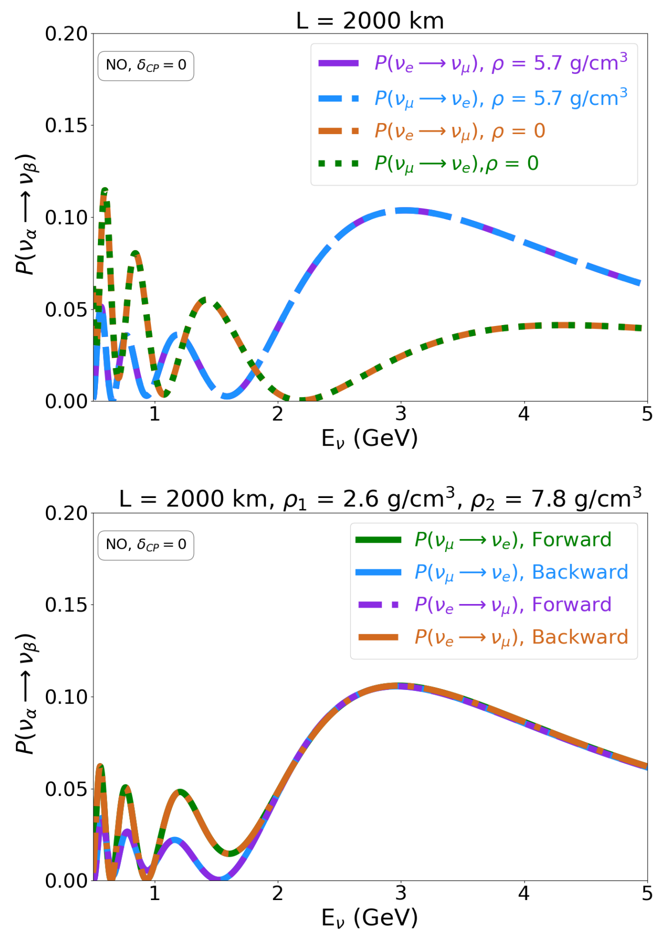

FIG. 2: Neutrino oscillation probabilities for $L = 2000$ km and $\delta = 0$. Top: The vacuum lines are identical and the matter lines are separately also identical. We assume a density for constant matter of $\rho = 5.7$ g/cm^3. Bottom: The colored lines correspond to oscillations in piecewise constant matter with $A_1 = 5\times10^{-4}$ eV^2/GeV, $x_1 = 800$ km and $A_2 = 1.5\times10^{-3}$ eV^2/GeV, $x_2 = 800 + 1200$ km. This corresponds to densities of $\rho = 2.6$ g/cm^3 and $\rho = 7.8$ g/cm^3 respectively. The Forward superscript indicates neutrinos produced either at a and detected at b (Forward), while Backward indicates the reverse.

In contrast, we consider a non-constant/asymmetric matter potential in Fig. 2 (bottom). We still focus on $\delta_{\rm CP} = 0$, where T invariance is conserved in vacuum, depicting now just the neutrino oscillation probabilities $P_{\mu e}$ and $P_{e\mu}$. The non-constant matter potential: $A_1 = 5\times10^{-4}$ eV^2/GeV, $x_1 = 800$ km and $A_2 = 1.5\times10^{-3}$ eV^2/GeV, $x_2 = L = 800 + 1200 = 2000$ km which corresponds to densities of

$\rho = 2.6$ g/cm^3 and $\rho = 7.8$ g/cm^3 respectively. The Forward superscript indicates neutrinos produced either at a and detected at b (Forward), while Backward indicates the reverse. As a consequence of intrinsic T-invariance conservation, $P_{e\mu}^{Forward} = P_{\mu e}^{Backward}$. However, since the matter potential is asymmetric, there are matter-induced, improper T-violating effects: $P_{e\mu}^{Froward} \neq P_{\mu e}^{Froward}$. These effects are largest at lower energies, when three-flavor effects are most visible – at energies below the first oscillation maximum. For higher energies, "solar" effects are very small and the system can be described as if there were only two neutrino flavors, where $P_{e\mu}^{Forward}$ and $P_{\mu e}^{Froward}$ are guaranteed to be identical.

While the analysis from the above plots focused on $\delta_{\rm CP} = 0$ to enforce T-invariance conservation in vacuum and in constant (or symmetric) matter, we also explored the impact of matter effects, constant or not, when $\delta_{\rm CP} \neq 0$. For details on this discussion, see the full publication [56]. However the main conclusion from that exercise is that when $\delta_{\rm CP} \neq 0$, this simply reveals that the magnitude of the T-invariance violation is impacted by matter effects. In summary, for a symmetric matter potential, matter effects will not lead to matter-induced T-violating effects. If there is intrinsic TV (e.g., $\delta_{\rm CP} \neq 0, \pi$) matter effects will modify "how much" T invariance is violated.

A. T-invariance Violation in Realistic Environments

Throughout this section, we have explored the level of TV manifest in different situations, using the symmetric and asymmetric potentials of Fig. 1 as a demonstrative case. The matter densities associated to these potentials are relatively large compared to those relevant for long-baseline terrestrial experiments. The same is true of the total baseline length of 2000 km. In this subsection, we briefly explore the level of TV in a more realistic scenario, focusing on the baseline length of the DUNE experiment (1300 km) as well as realistic matter density profiles. In particular, we considered the matter density profile corresponding to the Crustal [57] profile, analyzed in Refs. [58, 59]. While this profile is not symmetric, the asymmetries in it are small but appropriate for terrestrial long-baseline neutrino experiments.

The oscillation-probability differences $P_{\mu e} - P_{e\mu}$ for the Crustal matter density profile is depicted in Fig. 3(left). we consider $\delta_{\rm CP} = 0, \pm\pi/2$. For $\delta_{\rm CP} = 0$, $P_{\mu e} - P_{e\mu}$ is effectively zero for all energies. It is not, however, *exactly* zero as we will see in the coming discussion, but, for all practical purposes, the matter density profiles here are symmetric enough not to induce any meaningful TV if there is no intrinsic TV. While Fig. 3 (top) reveals that $P_{\mu e} - P_{e\mu}$ may be observably large for $\delta_{\rm CP} = \pm\pi/2$, it bears distinguishing the impact of the *size* of the matter density profile (with average densities of $\sim$3 g/cm^3) from that of the asymmetries in the profile over the 1300 km of travel. To inspect this, for a given profile $\rho(x)$, we define an average matter density $\bar{\rho}$ over a distance L. Then we calculate the oscillation probabilities inside of the matter density $\rho(x)$ as well as in our averaged-matter density, assuming a constant profile with uniform density $\bar{\rho}$. The asymmetry-induced TV, $\Delta P^{\rm assym.}$ is then defined by

$$\Delta P^{\rm assym.} \equiv (P_{\mu e} - P_{e\mu})|_{\rho(x)} - (P_{\mu e} - P_{e\mu})|_{\bar{\rho}}. \qquad \text{(III.1)}$$

By subtracting off the constant-matter-density difference, we isolate the amount of TV induced by the profile's asymmetry.

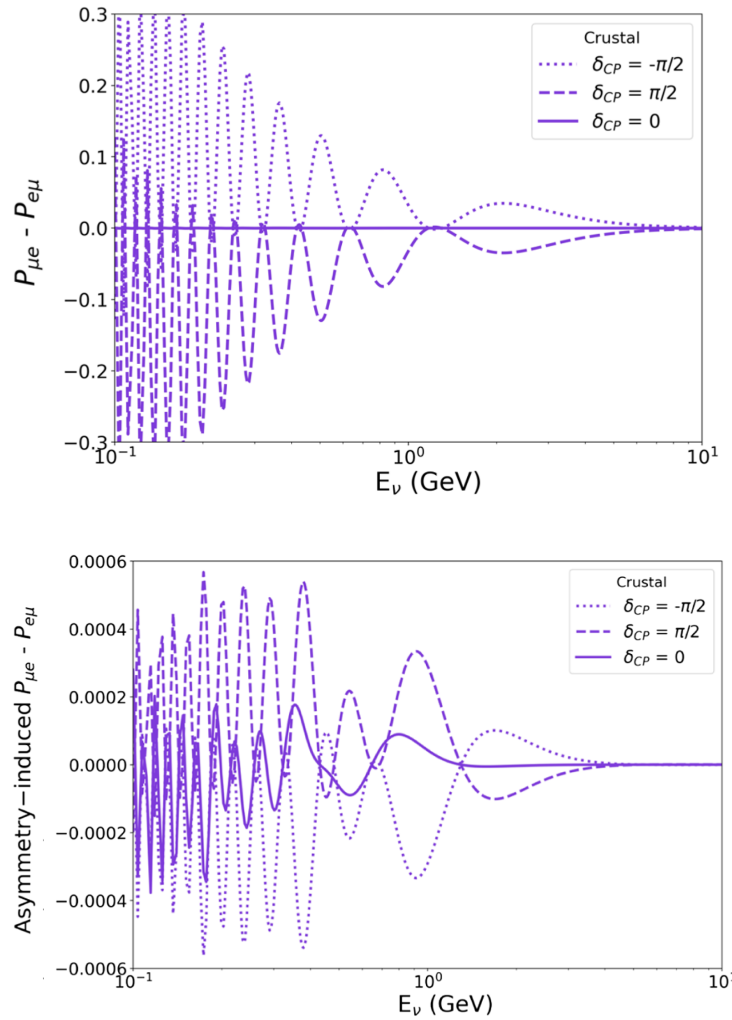

FIG. 3: Top: Improper time-invariance violation present for DUNE baselines and energies ($L = 1300$ km) using the Crustal [57]) matter density profile. We display the difference of oscillation probabilities $P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu)$ for three different choices of $\delta_{\rm CP}$: 0 (solid), $+\pi/2$ (dashed), and $-\pi/2$ (dotted). Bottom: the amount of asymmetry-induced T-invariance violation (see text for definition) at DUNE baseline/energies is shown for three choices of $\delta_{\rm CP}$; other oscillation parameters are fixed to their best-fit values.

This quantity is depicted for the same matter density profile in Fig. 3 (bottom). We present the asymmetry-induced violation for three choices of $\delta_{\rm CP}$: 0 and $\pm\pi/2$. We see that for any choice of $\delta_{\rm CP}$ the asymmetry introduces very small amounts of matter-induced TV – $\Delta P^{\rm assym.} \lesssim 6 \times 10^{-4}$ for all energies, where the oscillation probabilities themselves are at the level of $\sim$5% for these energies and baseline lengths – an insignificant, immeasurable amount.

Given the above discussion, we see that while matter may introduce relatively large induced TV (as long as the lepton sector has inherent CP violation), the amount of induced violation due to the matter density profile's asymmetry is very small – at least when considering realistic amounts of asymmetry for terrestrial long-baseline accelerator experiments.

IV. CONCLUDING REMARKS

Investigations of CP invariance and T invariance using neutrino oscillation experiments are impacted by matter effects. Here, we investigated the impact of matter effects on T-odd observables (e.g., $P_{\mu e} - P_{e\mu}$). We made use of the quantum-mechanical formalism of neutrino-flavor evolution and attempted to be comprehensive and pedagogical. Our main conclusions are as follows. Matter effects impact T-odd observables, but matter-induced T-invariance violation (TV) is qualitatively different from, and more subtle than, matter-induced CP-invariance violation. If the matter distribution is symmetric relative to the neutrino production and detection points, matter effects will not introduce any "extra" TV. On the other hand, if there is intrinsic TV, matter effects can modify the size of the T-odd observable. For Earth-bound long-baseline oscillation experiments, we argued, mostly via concrete examples, that these effects are quantitatively small. If the matter distribution is not symmetric relative to the neutrino production and detection points, there is genuine matter-induced TV. For Earth-bound long-baseline oscillation experiments, we argued, mostly via concrete examples, that these effects are quantitatively very small.

While our results indicate that T invariance violation tests are of limited practical use due to current beam technology, such studies are fascinating in their own right to provide a fuller picture of neutrinos with the fundamental symmetries of the universe. With future intense, well-characterized, high-energy electron-neutrino beams, circumstances may chance dramatically. This would be the case if neutrino beams from high-energy muon storage rings – so-called 'neutrino factories' – became available and allowed for precision studies of T-invariance violation in neutrino oscillations.

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