Decomposing Tensor Products of Two-dimensional Rational Conformal Field Theories with Transformers
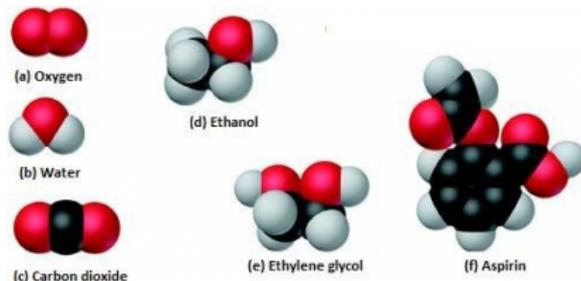
Scaling to Higher Central Charges and Unseen Classes

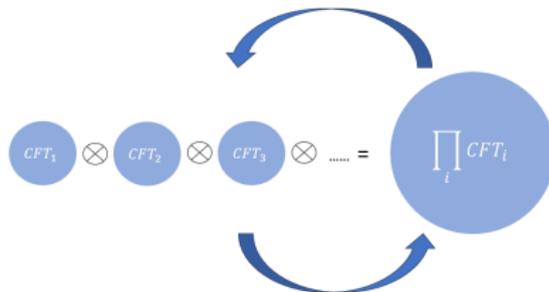Haotian Cao, Garrett Merz, Kyle Cranmer, and Gary Shiu

Parallel Session on *Lepton* & *Photon 2025*
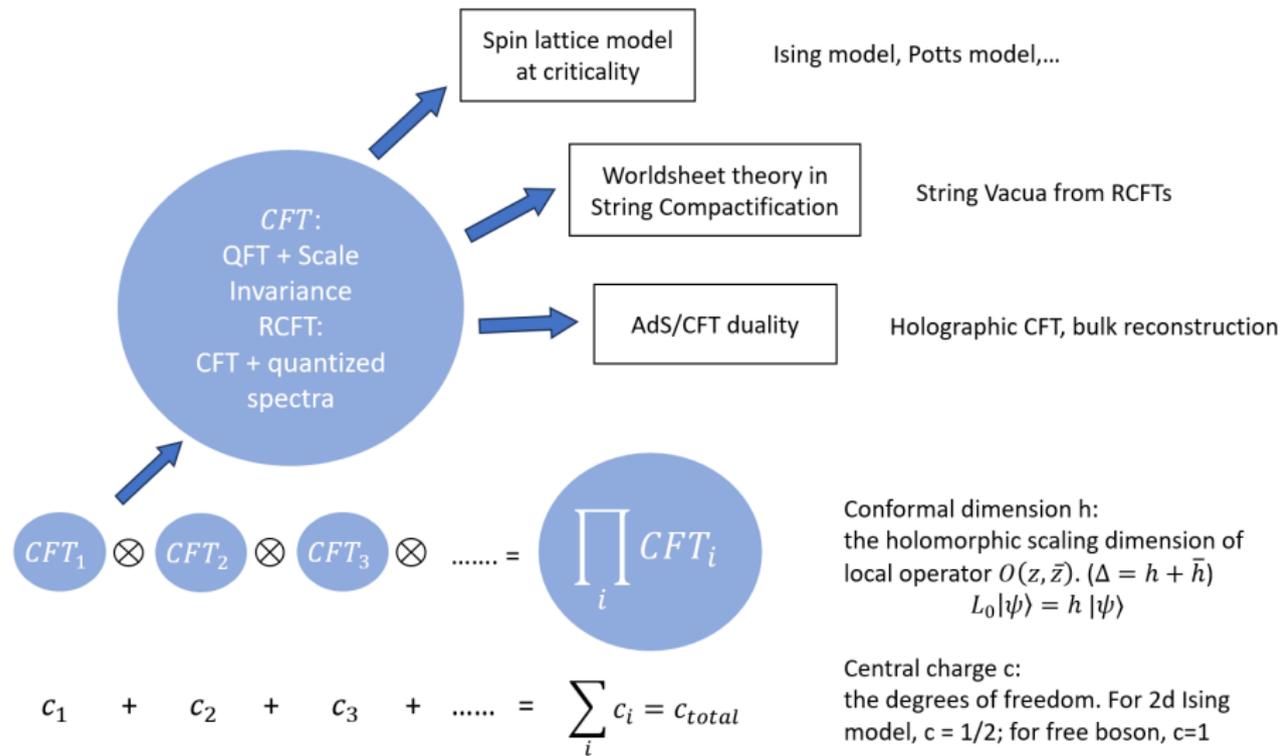University of Wisconsin Madison
25 August 2025

Inverse inference in atomic physics / biochemistry

Inverse inference in string theory and Conformal Field Theory (CFT)

Our task: given low-lying energy spectra of tensor product theories, we decompose them into factor/seed CFTs.
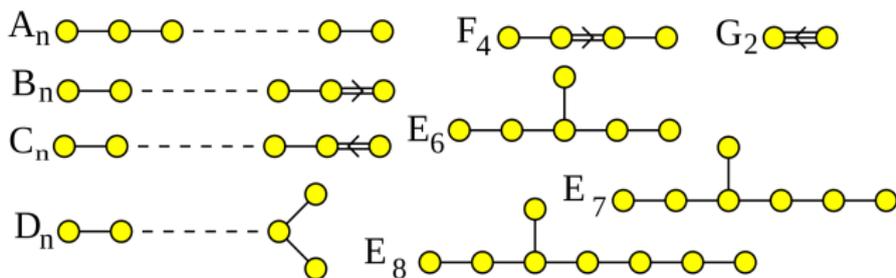
Spin lattice model at criticality

Ising model, Potts model,…

Worldsheet theory in String Compactification

String Vacua from RCFTs

AdS/CFT duality

Holographic CFT, bulk reconstruction

CFT:
QFT + Scale Invariance
RCFT:
CFT + quantized spectra

$CFT_1 \otimes CFT_2 \otimes CFT_3 \otimes \ldots\ldots = \prod_i CFT_i$

$c_1 + c_2 + c_3 + \ldots\ldots = \sum_i c_i = c_{total}$

Conformal dimension h:
the holomorphic scaling dimension of local operator $O(z, \bar{z})$. ($\Delta = h + \bar{h}$)
$$L_0 |\psi\rangle = h\, |\psi\rangle$$

Central charge c:
the degrees of freedom. For 2d Ising model, c = 1/2; for free boson, c=1

1. Experiment I: we train transformers on tensor products of 2d RCFTs generated by the affine **Kac-Moody algebras** where the central charge for tensor product theories are $c < 50$ and test on a held-out dataset. The model predicts the decomposition with 98% percent accuracy.

2. Experiment II: we train transformers on the same dataset as above, but test on tensor product theories that have central charge $50 \leq c \leq 100$ with the same **Kac-Moody algebras**. The initial model performance is around 80% but with priming, the model performance scales to 97%.

3. Experiment III: we train transformers on the same dataset as above, but test on tensor product theories that contain both **Kac-Moody algebras** and **coset models** while keeping the central charge $c < 50$. The initial model performance is poor but with priming, the model performance scales to 85%.

We work on two large classes of RCFTs: Kac-Moody (KM) algebra and cosets.

1. KM: $su(2)$, $su(3)$, $su(4)$, $su(5)$, $su(6)$, $so(7)$, $so(9)$, $so(11)$, $so(8)$, $so(10)$, $so(12)$, $sp(4)$, $sp(6)$, $sp(8)$, E_6, E_7, E_8, F_4, and G_2

2. Coset: unitary minimal models $\frac{su(2)_k \times su(2)_1}{su(2)_{k+1}}$, parafermionic theory $\frac{su(2)_k}{u(1)_k}$, $\mathcal{N} = 1$ superconformal theory $\frac{su(2)_k \times su(2)_2}{su(2)_{k+2}}$, and $\mathcal{N} = 2$ superconformal theory $\frac{su(2)_k \times u(1)_2}{u(1)_{k+2}}$.

Data Generation

The data features are conformal dimensions $\{h_<\}$ smaller than 1 and the labels are pairs of central charge and algebra letters $\{c, su2\}$. As an example, the conformal dimension and central charge takes the general form for RCFTs characeterized by KM algebra G:

$$h_r = \frac{C_r/\psi^2}{k + \tilde{h}_G} = \frac{(\lambda, \lambda + 2\rho)}{2(k + \tilde{h}_G)} \tag{1}$$

$$c_G = \frac{\tilde{k}|G|}{\tilde{k} + C_A/2} = \frac{k|G|}{k + \tilde{h}_G} \tag{2}$$

Here is an example of the KM CFT library:

```
Level 1: Dimensions = 0/1, 1/4, Central charge = su2,1/1
Level 2: Dimensions = 0/1, 1/3, Central charge = su3,2/1
Level 3: Dimensions = 0/1, 3/8, 1/2, Central charge = su4,3/1
Level 4: Dimensions = 0/1, 2/5, 3/5, Central charge = su5,4/1
Level 5: Dimensions = 0/1, 5/12, 37/84, 2/3, 5/7, 3/4, Central charge = su6,5/1
Level 6: Dimensions = 0/1, 7/16, 1/2, Central charge = so7,7/2
Level 7: Dimensions = 0/1, 1/2, Central charge = so8,4/1
Level 8: Dimensions = 0/1, 1/2, 9/16, Central charge = so9,9/2
Level 9: Dimensions = 0/1, 1/2, 5/8, Central charge = so10,5/1
Level 10: Dimensions = 0/1, 1/2, 7/8, Central charge = so11,11/2
```

Tensor product: multiplication of diagonal modular invariant partition function.

$$Z_{total} = \prod_i^L Z_i = \prod_{i=1}^L \sum_{j\bar{j}} N_{j\bar{j},i} \chi_{j,i}(q) \bar{\chi}_{j\bar{j},i}(\bar{q}) \tag{3}$$

where $\chi_j(q)$ is the generating function called the character

$$\chi_{\mathcal{R}}(q) = \text{Tr}_{\mathcal{R}} q^{L_0 - \frac{c}{24}} = q^{h - \frac{c}{24}}(a_0 + a_1 q + ...) \tag{4}$$

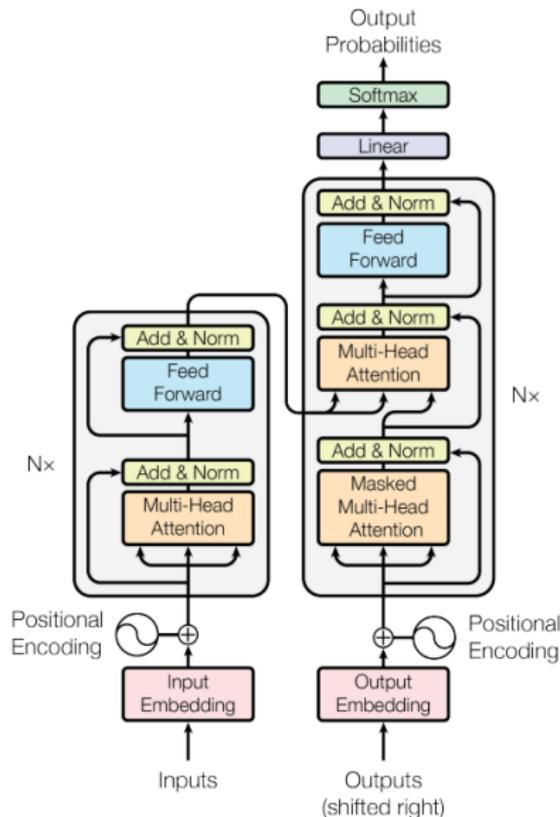Here is an example of the training data:

```
"1|0/1 1/4 1/2   su2,1/1 su2,1/1"
"2|0/1 1/4 1/3 7/12 su3,2/1 su2,1/1"
"3|0/1 1/4 3/8 1/2 5/8 3/4   su4,3/1 su2,1/1"
"4|0/1 1/4 2/5 3/5 13/20 17/20  su5,4/1 su2,1/1"
"5|0/1 1/4 5/12 37/84 2/3 29/42 5/7 3/4 11/12 27/28 su6,5/1 su2,1/1"
"6|0/1 1/4 7/16 1/2 11/16 3/4    su2,1/1 so7,7/2"
"7|0/1 1/4 1/2 3/4   su2,1/1 so8,4/1"
"8|0/1 1/4 1/2 9/16 3/4 13/16    su2,1/1 so9,9/2"
"9|0/1 1/4 1/2 5/8 3/4 7/8  su2,1/1 so10,5/1"
"10|0/1 1/4 1/2 3/4 7/8 su2,1/1 so11,11/2"
```

We train encoder-decoder Transformers with 8 attention heads, 2 encoder-decoder layers, 256 embedding dimensions, and 128 batch dimensions on 3 million datasets. We used three main metrics to capture the learning dynamics:

❶ Full accuracy: the model predicts both central charge and algebra letters correctly.

❷ Letter accuracy: the model predicts the algebra letters alone correctly.

❸ c accuracy: the model predicts the central charge alone correctly.

Vaswani et al. [2023]

Experiment I: $c < 50$

We first train Transformers on tensor product of RCFTs characterized by Kac-Moody algebra with total central charge smaller than 50 and test on a held-out dataset unseen during training.
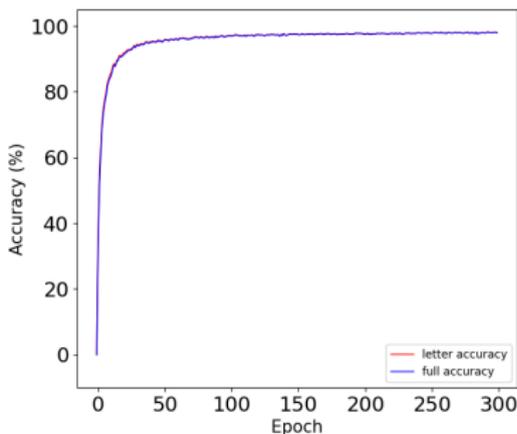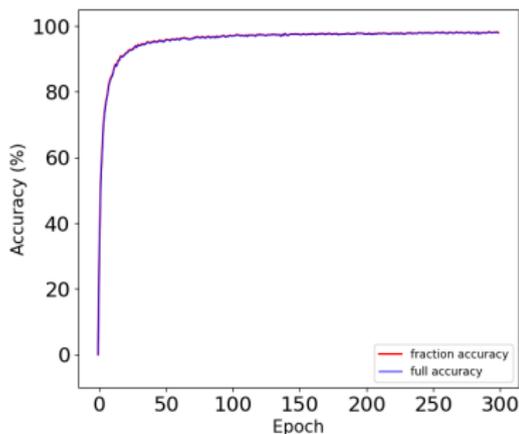

Figure: The highest accuracy score is 98.02%. The model learns c and letters at the same rate.

Next, we test model performance on out of distribution dataset. One generalization task is that for a fixed family of CFT, we extend our method to larger value of central charge.
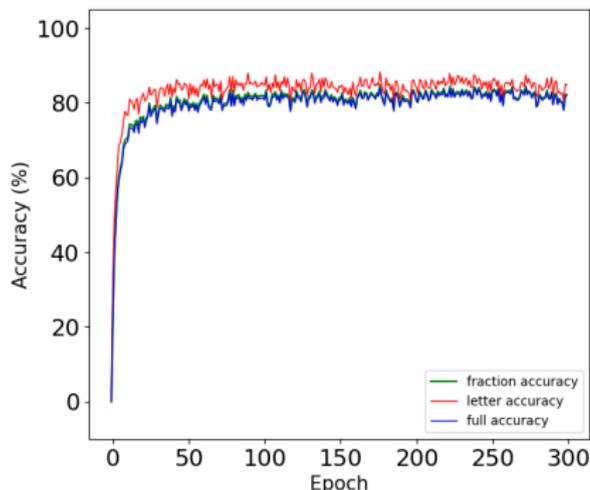

Figure: The model is learning algebra letters quicker because the class of CFT remains unchanged while the central charge range grows out of distribution.

We also study the learning behavior of the model for distinct ranges of central charge.

Train on $0 < c < 50$		Train on $0 < c < 100$	
Test Data	**Accuracy (%)**	**Test Data**	**Accuracy (%)**
$50 \leq c \leq 75$	**91.29**	$0 \leq c \leq 50$	**96.48**
$75 \leq c \leq 100$	47.20	$50 \leq c \leq 100$	**99.37**
$50 \leq c \leq 100$	84.23		

Table: Accuracy for two training ranges shown side-by-side.

The accuracy on the left shows that the training behavior is consistent with extrapolation decay in distribution-shift settings. The training data follows a distribution with boundary that is directly correlated with the value of central charges.

Priming: adding even a small number of examples from the out-of-range interval $50 \leq c \leq 100$ significantly improves performance on that interval.

With $30,000$ such examples (1% of the original dataset), accuracy rises from 82.9% to 97.01%, indicating that limited priming can largely overcome extrapolation decay.

Train data ($c < 50$, 3M)	Examples added	Accuracy (%)
No added examples	0	82.9
$50 \leq c \leq 100$	30	84.71
$50 \leq c \leq 100$	300	85.41
$50 \leq c \leq 100$	3000	93.33
$50 \leq c \leq 100$	30000	97.01

Table: Priming table for c generalization.

We add coset models to our KM CFT library and make tensor product models out of them.
Again, we added a small number of out-of-distribution data to the training data. The initial data performance is poor but the performance rises as we include more examples.

Train data (3M)	Examples added	Accuracy (%)
No added examples	0	0
coset+KM	30	1.16
coset+KM	300	17.47
coset+KM	3000	52.4
coset+KM	30000	83.02

Table: Priming table for unseen class generalization.

Conclusions and Future Directions

Conclusions:

1. The model successfully decompose the tensor product into their seeds: low-lying spectra contain rich information about the central charge and symmetry algebra.

2. There is an universality: low-lying h information spans across different ranges of central charges and unseen classes.

3. The model is able to extend outside the training domain with reasonable amount of priming.

Future Directions:

1. We can test on single RCFTs to see if the model can tell whether a theory is prime or not.

2. We can include orbifolds that add twisted sectors in the spectrum when generalizing to unseen classes.

Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones,
    Aidan N. Gomez, Lukasz Kaiser, and Illia Polosukhin. Attention is all you need,
    2023. URL https://arxiv.org/abs/1706.03762.