Topological Aspects of Particle Production: Applications to Cosmology

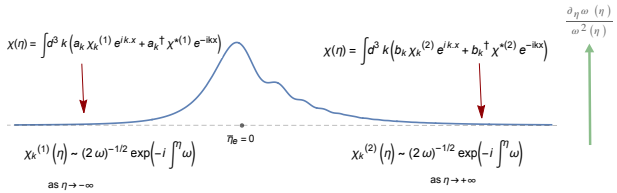
Nidhi Sudhir (work in collaboration with Daniel Chung)
Department of Physics, University of Wisconsin-Madison

1) Motivation

- **Particle production (PP)** sourced by background time dependence is a simple consequence of **broken time translation invariance.**
- In cosmology, PP due to expansion of space-time is ubiquitous and serves a production mechanism in superheavy dark matter models.
- Particle mode functions get a time dependence from the i.e.

$$\chi_k'' + \omega^2(\eta, k)\,\chi_k = 0$$

and the PP may be understood as a result of mismatch between vacuum states in the asymptotic past and the future.



$$N_k = \langle 0|\hat{N}_k|0\rangle = \langle 0|b_k^\dagger b_k|0\rangle = |\beta_k|^2$$

- Analytic estimates for the Bogoliubov coefficient β_k can only be obtained for simple $\omega^2(\eta)$ mainly through saddle point approximations.
- Here, we obtain new analytic insights from Stokes phenomenon using global analytic properties of the dispersion relation.

2) Mathematical Set-up

Bogoliubov transformation method - *propagator matrices* $U(\eta_1, \eta_2)$

- Canonical transformation to 'coefficients of WKB modes'

$$\chi_k(\eta) = \alpha_k(\eta) f_-(\eta) + \beta_k(\eta) f_+(\eta)$$
$$\partial_\eta \chi_k(\eta) = i\,\omega(\eta)\left[\beta_k(\eta) f_+(\eta) - \alpha_k(\eta) f_-(\eta)\right]$$

where

$$f_\pm(\eta) = \left(2\omega(\eta)\right)^{-1/2} \exp\left(\pm i \int_{\eta_0}^{\eta} d\eta'\,\omega(\eta')\right).$$

- The mode equation can be re-written as

$$\partial_\eta \begin{bmatrix}\alpha_k(\eta)\\ \beta_k(\eta)\end{bmatrix} = \underbrace{\frac{\partial_\eta \omega(\eta)}{2\omega(\eta)} \begin{bmatrix} 0 & \exp\left(+2i\int_{\eta_0}^{\eta}\omega\right)\\ \exp\left(+2i\int_{\eta_0}^{\eta}\omega\right) & 0\end{bmatrix}}_{M(\eta)} \begin{bmatrix}\alpha_k(\eta)\\ \beta_k(\eta)\end{bmatrix};$$

which gives

$$\begin{pmatrix}\alpha_k(\eta_1)\\ \beta_k(\eta_1)\end{pmatrix} = U(\eta_1,\eta_2)\cdot\begin{pmatrix}\alpha_k(\eta_2)\\ \beta_k(\eta_2)\end{pmatrix}, \quad U(\eta_1,\eta_2) \equiv \mathbb{P}\left[\exp\left(\int_{\eta_2}^{\eta_1} d\eta\, M(\eta)\right)\right].$$

Take-Away

"Long wavelength particle modes produced from coupling to a time dependent background measure topology associated with WKB asymptotic expansions"

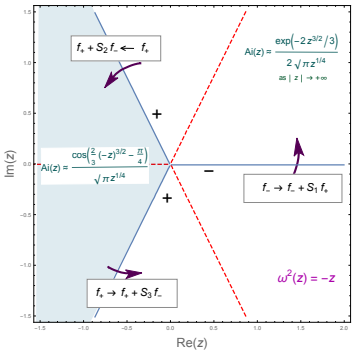
3) Stokes Phenomenon

"Given an exact solution to a complex Schrodinger-like differential equation, it's WKB series jumps discretely over certain boundaries in the complex plane called Stokes lines."

- **Stokes lines (SL):** contours in $\mathbb{C}$ such that

$$\mathrm{Im}\left(i\int_0^z dz'\,\omega(z')\right) = 0.$$

- WKB series transforms across an SL such that the **dominant WKB function gains** a contribution from the **subdominant** one **times a (Stokes) constant**.



- This implies the following approximate propagator matrices across Stokes lines along which $f_\pm(z)$ are respectively dominant

$$U_+ \approx \begin{bmatrix}1 & 0\\ S & 1\end{bmatrix} \approx U_-^T$$

- Determining the Stokes constants for generic $\omega^2(\eta)$ is difficult. For simple monomial dependencies, symmetries of mode equation fix it.

Determining Stokes constants for $\omega^2(z) = A z^n$

- **Symmetries:** For $\omega^2(z) = A z^n$, mode equation is symmetric under $z \to \exp\left(\frac{2i\pi}{n+2}\right) z$ ($\mathbb{Z}_{n+2}$ symmetry). This equates all Stokes constants, i.e. $S_i = S, \quad \forall i$.
- **Single Valuedness:** $\chi_k(z)$is single valued in $\mathbb{C}$, i.e. propagation along a circular contour in the complex plane should return identity.

$$\implies S_n = 2i\cos\left(\frac{\pi}{n+2}\right)$$

4) New Insights: $\beta_{k\to 0}$ for simple $\omega^2(\eta)$

(arXiv:2408.11452)

- Monomial $\omega^2(\eta)$ is obtained in the $k \to 0$ limit of many physical dispersion relations

$$\omega^2(\eta) \approx k^2 + A\eta^n \xrightarrow{k\to 0} \omega^2(\eta) \approx A\eta^n$$

- Combining connection matrices from $\mathbb{R}_-$ to $\mathbb{R}_+$

$$|\beta| = |\beta(+\infty)| = \cot\left[\frac{\pi}{n+2}\right] \qquad (1)$$

- **Counts the $n+2$ Stokes (asymptotic expansion) sectors!**

5) Ongoing work: Extending to $k \neq 0$

(w. Junqi Zhang)

- Determining Stokes constants for $\omega^2(z) = k^2 + z^n$.
- **Rotational Symmetry :** The mode equation is symmetric under ($\mathbb{Z}_{n+2}$ symmetry)

$$z \to \gamma z, k \to \gamma^{-1} k, \quad \text{where } \gamma = e^{2\pi i/(n+2)}.$$

This relates all the Stokes constants $S_i(k)$ as

$$\implies S_i(k) = S_1\left(\gamma^{j-1}k\right)$$

- **Time Reversal Symmetry:** $\omega^2(\eta) = \omega^2(-\eta), \ \eta \in \mathbb{R}$

$$\implies \beta^* = -\beta$$

where β is a known function in terms of the Stokes constants.

- **Single Valuedness:** Propagation along a circular contour in $\mathbb{C}$ equals identity.
- The above severely constrain the Stokes constants $S_i(k)$.

Can these constraints be solved for general n ?

6) Cosmological Models

- May be applied to models with time dependent part of the dispersion relation passing through a zero.
- In many of these models the topological contribution noted in Eq. (1) to total particle production is negligible due to phase space suppression in the $k \to 0$ limit.
- An example where Eq. (1) presents a significant contribution to the total number density is a **constant spatial curvature model** with

$$\omega^2 = k^2 + \underbrace{a^2(\eta) f(\phi)}_{E\eta^n} - \underbrace{a''(\eta)/a(\eta)}_{K^2}$$

obtainable in scenarios of two spectator fields - one coupled to the background of the other. (cf. arXiv:2408.11452)