Quantum Colliders

Y. Afik, JRMdN, EPJ Plus 136, 907 (2021)
Y. Afik, JRMdN, Quantum 6, 820 (2022)
Y. Afik, JRMdN, PRL 130, 221801 (2023)
Y. Afik, Y. Kats, JRMdN, A. Soffer, D. Uzan, arXiv 2406.04402 (2024)

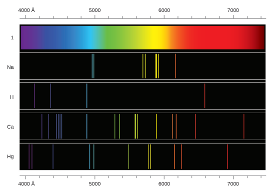
Juan Ramón Muñoz de Nova

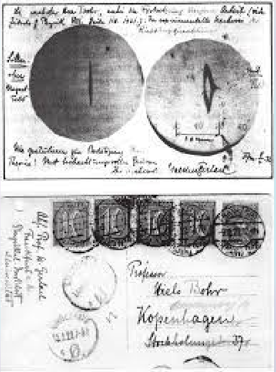
CERN, 06/02/2025

Part I: Quantum Theory

Quantum Theory: Quantization

- Quantum Mechanics was originally named after observation of quantized values:
  - Electromagnetic radiation (Black-body/Photoelectric effect)
  - Electron orbits (Atomic spectra)
  - Angular momentum (Stern-Gerlach)





Quantum Theory: Copenhagen Interpretation

- Copenhagen interpretation of Quantum Mechanics:
  1. Particles $\longleftrightarrow$ Waves $\rightarrow$ Superposition
  2. Outcomes of measurements: Observable eigenvalues $\rightarrow$ Quantization
  3. Probabilities of outcomes encoded in $|\Psi|^2$ $\rightarrow$ Interference

Quantum vs. Classical

- Quantum Mechanics: Superposition $\rightarrow$ *Fundamental* probabilistic description of measurements.
- Classical Mechanics: Random outputs using classical probability distributions resulting from *ignorance* (noise, experimental variations...)
- Is God *just* playing dice with the Universe? $\rightarrow$

Quantum vs. Classical

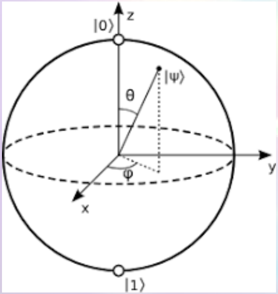
- Quantum Mechanics: Superposition $\rightarrow$ *Fundamental* probabilistic description of measurements.
- Classical Mechanics: Random outputs using classical probability distributions resulting from *ignorance* (noise, experimental variations...)
- Is God *just* playing dice with the Universe? $\rightarrow$ God is well beyond a mere croupier!
- Quantum Correlations=Correlations not accounted by classical probabilistic theories.

Quantum State

- **Pure state** $\rightarrow$ Wave function $|\Psi\rangle$
  1. $|\Psi\rangle = \sum_n \alpha_n \cdot |\phi_n\rangle \, , \; \langle\Psi|\Psi\rangle = \sum_n |\alpha_n|^2 = 1$
  2. Coherent mixture of quantum states $\rightarrow$ α_n are amplitudes
  3. Expectation values: $\langle A\rangle = \langle\Psi|A|\Psi\rangle = \sum_{n,m} \alpha_m^* \alpha_n \langle\phi_m|A|\phi_n\rangle$
- **Mixed state** $\rightarrow$ Generalization to density matrix ρ
  1. $\rho = \sum_n p_n \cdot |\phi_n\rangle\langle\phi_n| \, , \; \mathrm{tr}\rho = \sum_n p_n = 1$
  2. Incoherent mixture of quantum states $\rightarrow$ p_n are probabilities
  3. Expectation values: $\langle A\rangle = \mathrm{tr}(\rho A) = \sum_n p_n \langle\phi_n|A|\phi_n\rangle$

Qubits: Pure state

- Qubit: Two-level quantum system $|0\rangle, |1\rangle \rightarrow$ Most simple!
- Paradigmatic example: spin-1/2 particle. $|0\rangle \equiv |+\rangle, |1\rangle \equiv |-\rangle$
- General wave function: $|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle \equiv |\hat{\mathbf{n}}\rangle$
- $\rightarrow$ Eigenstate of spin projection: $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}|\hat{\mathbf{n}}\rangle = |\hat{\mathbf{n}}\rangle$
- Unit vector $\hat{\mathbf{n}} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$ labels quantum states $\rightarrow$ Surface of Bloch sphere.



Qubits: Density matrix

- General density matrix (2×2) for 1 qubit $\rightarrow$ 3 parameters B_i:

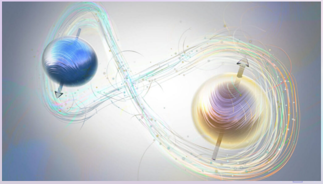
$$\rho = \frac{1 + \sum_i B_i \sigma^i}{2}, \ \mathbf{B} = \mathrm{tr}[\boldsymbol{\sigma}\rho], \ |\mathbf{B}| \leq 1$$

- Two qubits $\rightarrow$ Most simple example of quantum correlations.
- General density matrix (4×4) for 2 qubits $\rightarrow$ 15 parameters $B_i^{\pm}$, C_{ij}

$$\rho = \frac{\mathbb{1} + \sum_i \left(B_i^+ \sigma^i \otimes \mathbb{1} + B_i^- \mathbb{1} \otimes \sigma^i\right) + \sum_{i,j} C_{ij} \sigma^i \otimes \sigma^j}{4}$$

- Polarization vectors $\mathbf{B}^{\pm}$ and correlation matrix $\mathbf{C}$:

$$B_i^+ = \langle \sigma^i \otimes \mathbb{1} \rangle, \ B_i^- = \langle \mathbb{1} \otimes \sigma^i \rangle, \ C_{ij} = \langle \sigma^i \otimes \sigma^j \rangle$$



Quantum Discord

- Classically, two equivalent expressions for mutual information of bipartite system A and B (Alice and Bob):

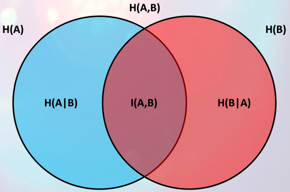
$$I(A,B) = H(A) + H(B) - H(A,B) = H(A) - H(A|B)$$
$$H(A,B) = -\sum_{x,y} p(x,y)\log_2 p(x,y)$$
$$H(A|B) = \sum_{y} p(y) H(A|B=y)$$

- Quantum mechanics can introduce a "discord" between both expressions:

$$\mathcal{D}(A,B) \equiv H(B) - H(A,B) + H(A|B) \neq 0$$

- Most basic form of quantum correlations!
- Quantum Discord is asymmetric $\mathcal{D}(A,B) \neq \mathcal{D}(B,A)$

PRL 88, 017901 (2001)

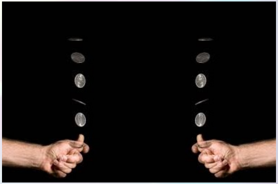


Quantum Discord: Classical states

- OK...but where is the Physics here? $\rightarrow$ Only classical states have zero discord!

$$\rho_{\text{class}} = \sum_{n,m} p_{n,m} \left|n\right\rangle \otimes \left|m\right\rangle \left\langle n\right| \otimes \left\langle m\right|$$

- $\left|n\right\rangle, \left|m\right\rangle$ form an *orthonormal* basis for A, B
- $p_{n,m}$: classical probability of being $\left|n\right\rangle \otimes \left|m\right\rangle$
- Qubits $\rightarrow$ Tails and heads between two coins!



$$\begin{aligned} \rho_{\text{class}} &= p_{++} \left|++\right\rangle \left\langle ++\right| + p_{+-} \left|+-\right\rangle \left\langle +-\right| \\ &+ p_{-+} \left|-+\right\rangle \left\langle -+\right| + p_{--} \left|--\right\rangle \left\langle --\right| \end{aligned}$$

Entanglement

- What if we generalize the previous idea? $\rightarrow$ Separability:

$$\rho_{\text{sep}} = \sum_{n,m} p_{n,m} \left|n\right\rangle \otimes \left|m\right\rangle \left\langle n\right| \otimes \left\langle m\right| = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)}$$

- $\left|n\right\rangle, \left|m\right\rangle$ not necessarily *orthonormal* now $\rightarrow$ $p_{n,m}$ are quasi-probabilities (not disjoint events)
- Any classically correlated state (classical probability) is separable.
- **Entanglement**: Non-separability of a bipartite quantum state.

Separable

Non-Separable

Entanglement: Two qubits

- Two qubits: Separability=Positive P-representation $P(\mathbf{n}_A, \mathbf{n}_B) \geq 0$:

$$\rho = \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \, P(\mathbf{n}_A, \mathbf{n}_B) \, |\mathbf{n}_A \mathbf{n}_B\rangle \langle \mathbf{n}_A \mathbf{n}_B| \, , \; \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \, P(\mathbf{n}_A, \mathbf{n}_B) = 1$$

- $P(\mathbf{n}_A, \mathbf{n}_B)$ is a quasi-probability: Overlap $|\langle \mathbf{n}_A | \mathbf{n}_B \rangle|^2 \neq 0$
- Separability=Purely classical spins pointing at directions $\mathbf{n}_A, \mathbf{n}_B$

$$C_{ij} = \langle \sigma^i \otimes \sigma^j \rangle = \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \; P(\mathbf{n}_A, \mathbf{n}_B) n_A^i n_B^j$$

- Entanglement=NO positive P-representation $\rightarrow$ Genuine non-classical!

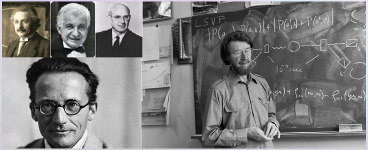
Steering and Bell nonlocality

- EPR Paradox: Quantum Mechanics challenges local realism!
- Schrödinger: QM should hold but also locality $\rightarrow$ Local quantum states $\rightarrow$ Bob can "steer" Alice state $\rightarrow$ Steering
- Bell: Local realism $\rightarrow$ Joint Alice and Bob measurements M_A, M_B accounted by local hidden-variable model

$$p(a,b|M_A M_B) = \int d\lambda \; p(a|M_A\lambda)p(b|M_B\lambda)p(\lambda)$$

- Bell Theorem: Local realistic theories satisfy Bell (CHSH) inequality

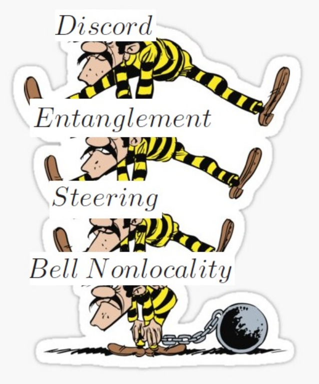
$$\begin{aligned} |C(M_A,M_B) - C(M_A,M'_B) &+ C(M'_A,M_B) + C(M'_A,M'_B)| \leq 2 \\ C(M_A,M_B) &= \sum_{a,b=\pm 1} (a \cdot b)p(a,b|M_A M_B) \end{aligned}$$



Hierarchy of Quantum Correlations

- Steering and Discord can be asymmetric between Alice and Bob.
- Bell Nonlocality and Entanglement are always symmetric.
- Quantum Hierarchy:

$$\textit{Bell Nonlocality} \subset \textit{Steering} \subset \textit{Entanglement} \subset \textit{Discord}$$



Part II: Quantum Field Theory

Quantum Optics

- Light has motivated the major advancements in modern Physics:
  - EM radiation: Maxwell equations
  - Special relativity: Michelson-Morley experiment
  - Quantum mechanics: Black-body spectrum & Photoelectric effect
  - Quantum field theory: First quantized field

. . . and God said: "let there be light" . . .

Quantum field theory

- Set of classical fields $\{\phi_A\}$ described by Lagrangian density $\mathcal{L} : (\phi_A, \partial_\mu \phi_A, x)$

$$L = \int \mathrm{d}\mathbf{x}\ \mathcal{L}(\phi_A, \partial_\mu \phi_A, x),\ S = \int \mathrm{d}t\ L = \int \mathrm{d}^4 x\ \mathcal{L}$$

- Equations of motion: $\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} = \frac{\delta \mathcal{L}}{\delta \phi_A}$
- Canonical momentum: $\Pi_A(x) \equiv \frac{\delta \mathcal{L}}{\delta \partial_t \phi_A}$
- Poisson brackets from Hamiltonian theory:

$$\{Q, P\} = \sum_A \int \mathrm{d}\mathbf{x}\ \frac{\delta Q}{\delta \phi_A(\mathbf{x})} \frac{\delta P}{\delta \Pi_A(\mathbf{x})} - \frac{\delta Q}{\delta \Pi_A(\mathbf{x})} \frac{\delta P}{\delta \phi_A(\mathbf{x})}$$

- Quantization à la Dirac: promotion of magnitudes Q, P to operators $\hat{Q}, \hat{P}$ satisfying commutation relations:

$$[\hat{Q}, \hat{P}] = i\hbar \{Q, P\}$$

Quantum KG field

- Simplest example: massless Hermitian scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$.
- Field $\phi \to$ Doublet $\Phi = \begin{bmatrix} \phi \\ i\Pi \end{bmatrix}$
- KG equation $\Box\phi = \partial_\mu\partial^\mu\phi = 0 \to$ Linear equation $i\hbar\partial_t\Phi = M\Phi$:
  - Expansion of the field in terms of eigenmodes: $M\Phi_n = \epsilon_n\Phi_n$
  - Conjugate solutions: $\bar{\Phi}_n = \sigma_z\Phi_n^*$
  - Conserved inner product:

$$(\Phi_n|\Phi_m) \equiv i\int \mathrm{d}^3\mathbf{x}\ (\phi_n^*\Pi_m - \phi_m\Pi_n^*) = \langle\Phi_n|\sigma_x|\Phi_m\rangle$$

- Careful: both M and inner product are not positive definite!

$$M\bar{\Phi}_n = -\epsilon_n^*\bar{\Phi}_n,\ (\Phi_n|\Phi_m) = -(\bar{\Phi}_m|\bar{\Phi}_n)$$

- Plane waves provide complete orthonormal basis with positive/negative frequency (norm)

$$M\Phi_\mathbf{k} = \hbar\omega_\mathbf{k}\Phi_\mathbf{k},\ \Phi_\mathbf{k}(\mathbf{x}) = \frac{e^{i\mathbf{k}\mathbf{x}}}{\sqrt{2\omega_\mathbf{k}(2\pi)^3}}\begin{bmatrix} 1 \\ \omega_\mathbf{k} \end{bmatrix},\ M\bar{\Phi}_\mathbf{k} = -\hbar\omega_\mathbf{k}\bar{\Phi}_\mathbf{k}$$

Harmonic oscillators

- Fourier amplitudes promoted to operators: $\hat{a}(\mathbf{k}) = (\Phi_{\mathbf{k}}|\hat{\Phi}) \rightarrow$

$$\hat{\Phi}(\mathbf{x}) = \sqrt{\hbar}\int \mathrm{d}^3\mathbf{k}\ [\hat{a}(\mathbf{k})\Phi_{\mathbf{k}}(\mathbf{x}) + \hat{a}^\dagger(\mathbf{k})\Phi^*_{\mathbf{k}}(\mathbf{x})]$$

- Canonical commutation rules $\rightarrow [\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (\Phi_{\mathbf{k}}|\Phi^*_{\mathbf{k}'}) = \delta(\mathbf{k}-\mathbf{k}')$
- QFT described in terms of harmonic oscillators $\rightarrow$ We can import all our knowledge from textbooks:
  - Phase space variables: $\hat{X} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}$, $\hat{P} = i\frac{\hat{a}^\dagger - \hat{a}}{\sqrt{2}}$
  - Number states: $\hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle$
  - Coherent states: $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, $\alpha\in\mathbb{C}$
  - Squeezed states: Coherent states of Bogoliubov transformation

$$U = e^{\lambda[(\hat{a}^\dagger)^2 - \hat{a}^2]},\ \hat{b} = U^\dagger\hat{a}U = (\cosh\lambda)\hat{a} + (\sinh\lambda)\hat{a}^\dagger$$

Number vs. P representations

- Number states form a complete orthonormal basis $\rightarrow$ Any quantum state ρ on the Fock space can be written:

$$\rho = \sum_{n,m} \rho_{nm} \left|n\right\rangle \left\langle m\right|$$

- Coherent states form an overcomplete basis $1 = \frac{1}{\pi}\int \mathrm{d}^2\alpha \ \left|\alpha\right\rangle \left\langle\alpha\right| \rightarrow$ Glauber-Sudarshan P-representation:

$$\rho = \int \mathrm{d}^2\alpha \ P(\alpha) \left|\alpha\right\rangle \left\langle\alpha\right|, \ \int \mathrm{d}^2\alpha \ P(\alpha) = 1,$$

- Normal-ordered expectation values

$$\langle (\hat{a}^\dagger)^m (\hat{a})^n \rangle = \int \mathrm{d}^2\alpha \ (\alpha^*)^m \alpha^n P(\alpha)$$

- $P(\alpha)$ is a quasi-probability distribution: Overlap $|\left\langle\alpha|\beta\right\rangle|^2 \neq 0$
- No need for $P(\alpha)$ to be positive or even well-defined!

Quantum Correlations

- General quantum state of EM field:

$$\rho = \int \mathrm{d}^2\{\alpha_{\mathbf{k}}\}\ P(\{\alpha_{\mathbf{k}}\}) \prod_{\mathbf{k}} \otimes \left|\alpha_{\mathbf{k}}\right\rangle \left\langle\alpha_{\mathbf{k}}\right|$$

- Measurements=Normal-ordered expectation values $\rightarrow$ Positive P-representation yields classical states of light $\rightarrow$ Quantum correlations require negative-valued P-distributions!
- Simplest example: bipartite correlations
- Bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$: Product space of Fock spaces of two modes $\hat{a}_i, \hat{a}_j$

$$\rho = \int \mathrm{d}^2\alpha_i \mathrm{d}^2\alpha_j\ P(\alpha_i, \alpha_j) \left|\alpha_i\alpha_j\right\rangle \left\langle\alpha_i\alpha_j\right|$$

Cauchy-Schwarz violation

- Quantum correlations tested via first, second-order correlations:

$$g_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle\,,\; c_{ij} = \langle \hat{a}_i \hat{a}_j \rangle\,,\; \Gamma_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_i \rangle$$

- Classical light obeys:
  - $|c_{ij}|^2 \leq g_{ii} g_{jj}$
  - $\Gamma_{ij} \leq \sqrt{\Gamma_{ii}\Gamma_{jj}}$
- Proofs explicitly based on positivity of P $\rightarrow$ Effective scalar product $\rightarrow$ Cauchy-Schwarz inequalities!
- Operators A, B do satisfy mathematical Cauchy-Schwarz inequality associated to trace scalar product

$$\langle A|B\rangle \equiv \mathrm{tr}(\rho A^\dagger B) = \langle A^\dagger B\rangle$$

- Any quantum state obeys:
  - $|c_{ij}|^2 \leq (g_{ii}+1) g_{jj}$
  - $\Gamma_{ij} \leq \sqrt{(\Gamma_{ii}+g_{ii})(\Gamma_{jj}+g_{jj})}$
- There is room for violation of the *classical* CS inequalities!

Quantum Many-Body Field Theory

- Second-quantization many-body Hamiltonian (fermions and bosons):

$$\hat{H} = \int \mathrm{d}\mathbf{x}\ \hat{\Psi}^\dagger(\mathbf{x})\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right]\hat{\Psi}(\mathbf{x}) + \frac{1}{2}\int \mathrm{d}\mathbf{x}\int \mathrm{d}\mathbf{x}'\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}')V(\mathbf{x}-\mathbf{x}')\hat{\Psi}(\mathbf{x}')\hat{\Psi}(\mathbf{x})$$

- Field operator: $\hat{\Psi}(\mathbf{x}) = \sum_n \hat{a}_n\phi_n(\mathbf{x})$, $\hat{a}_n = \langle\phi_n|\hat{\Psi}\rangle$
- Fermionic or bosonic annihilation operators: $[\hat{a}_n, \hat{a}_m^\dagger]_\pm = \delta_{nm}$
- Heisenberg equation of motion of field operator:

$$i\hbar\partial_t\hat{\Psi}(\mathbf{x},t) = [\hat{\Psi}(\mathbf{x},t),\hat{H}] = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) + \int \mathrm{d}\mathbf{x}'\hat{\Psi}^\dagger(\mathbf{x}',t)V(\mathbf{x}-\mathbf{x}')\hat{\Psi}(\mathbf{x}',t)\right]\hat{\Psi}(\mathbf{x},t)$$

- Same equation of motion derived from a classical Lagrangian:

$$L = \int \mathrm{d}\mathbf{x}\ i\Psi^*\partial_t\Psi - H$$

Non-relativistic vs. relativistic QFT

- Differences with respect relativistic QFT:
  - Not Lorentz invariant
  - Fixed cutoffs (IR $\rightarrow$ System size; UV $\rightarrow$ Microscopic physics)
  - Interactions are (typically) classical
  - Effective and approximate theories
  - Second-quantization is not fundamental (trick to rewrite symmetric/antisymmetric wavefunctions)
- Features:
  - Highly tunable (interactions, external potential, mass)
  - Non-trivial spatiotemporal dependence leads to rich physics (Analogue Gravity, Floquet systems)
  - Variety of internal degrees of freedom (high spin, isospin, mixtures)
  - Many-body systems lead to spontaneous symmetry breaking (supersolids, time crystals)
  - Topological physics and synthetic dimensions

Condensates: GP mean-field

- Bose-Einstein condensate close to $T = 0$
- Field operator: $\hat{\Psi}(\mathbf{x}, t) = \Psi(\mathbf{x}, t) + \hat{\varphi}(\mathbf{x}, t) \rightarrow$ Higgs-mechanism
- Contact interaction $V(\mathbf{x} - \mathbf{x}') = g\delta(\mathbf{x} - \mathbf{x}') \rightarrow$ Non-relativistic Higgs boson!

$$\mathcal{L} = i\Psi^* \partial_t \Psi - \frac{|\nabla\Psi|^2}{2m} - V(\mathbf{x})|\Psi|^2 - g\frac{|\Psi|^4}{2}$$

- Heisenberg e.o.m. $\rightarrow$ Gross-Pitaevskii equation:

$$i\hbar\partial_t\Psi(\mathbf{x}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) + g|\Psi(\mathbf{x}, t)|^2\right]\Psi(\mathbf{x}, t)$$

- Condensate wavefunction=Coherent state spontaneously breaking $U(1)$ symmetry

Condensates: Bogoliubov approximation

- Stationary condensate: $\hat{\Psi}(\mathbf{x},t)=[\Psi_0(\mathbf{x})+\hat{\varphi}(\mathbf{x},t)]e^{-i\mu t/\hbar}$+Quantum fluctuations at linear order $\rightarrow$ Bogoliubov-de Gennes equations:

$$\begin{aligned} M\hat{\Phi} &= i\hbar\partial_t\hat{\Phi},\ M=\begin{bmatrix} N & A \\ -A^* & -N^* \end{bmatrix},\ \hat{\Phi}\equiv\begin{bmatrix} \hat{\varphi}(\mathbf{x},t) \\ \hat{\varphi}^\dagger(\mathbf{x},t) \end{bmatrix} \\ N &= -\frac{\hbar^2\nabla^2}{2m}+V(\mathbf{x})+2g|\Psi_0(\mathbf{x})|^2-\mu,\ A=g\Psi_0^2(\mathbf{x}) \end{aligned}$$

- Eigenmodes: $Mz_n=\epsilon_n z_n,\ z_n=\begin{bmatrix} u_n \\ v_n \end{bmatrix},\ M\bar{z}_n=-\epsilon_n^*\bar{z}_n,\ \bar{z}_n=\sigma_x z_n^*$
- Conserved inner product:

$$(z_n|z_m)\equiv\langle z_n|\sigma_z|z_m\rangle=\int \mathrm{d}\mathbf{x}\ z_n^\dagger\sigma_z z_m=\int \mathrm{d}\mathbf{x}\ u_n^*u_m-v_n^*v_m$$

- Complete basis $(z_n|z_m)=\delta_{nm}\Longrightarrow$ Quantization mimics KG field:

$$\hat{\Phi}=\sum_n z_n\hat{b}_n+\bar{z}_n\hat{b}_n^\dagger$$

Bogoliubov vs. Higgs

- Amplitude-phase (Madelung) decomposition:

$$\hat{\Psi}(\mathbf{x},t)=\sqrt{\hat{n}(\mathbf{x},t)}e^{i\hat{\theta}(\mathbf{x},t)}=\sqrt{n(\mathbf{x},t)+\delta\hat{n}(\mathbf{x},t)}e^{i(\theta(\mathbf{x},t)+\delta\hat{\theta}(\mathbf{x},t))}$$

- Density/phase fluctuations=Higgs/Goldstone modes:

$$\begin{aligned}\delta\hat{n}(\mathbf{x},t) &= \Psi_0^*(\mathbf{x})\hat{\varphi}(\mathbf{x},t)+\Psi_0(\mathbf{x})\hat{\varphi}^\dagger(\mathbf{x},t)\\ \delta\hat{\theta}(\mathbf{x},t) &= i\frac{\Psi_0(\mathbf{x})\hat{\varphi}^\dagger(\mathbf{x},t)-\Psi_0^*(\mathbf{x})\hat{\varphi}(\mathbf{x},t)}{2|\Psi_0(\mathbf{x})|^2}\end{aligned}$$

- Homogeneous condensate: $\Psi_0(\mathbf{x})=\sqrt{n_0}\rightarrow$ Massless superluminal dispersion relation for both Higgs and Goldstone modes:

$$\omega^2=c^2k^2\left(1+\frac{k^2}{4\Lambda^2}\right),\ c^2=\frac{gn_0}{m},\ \Lambda=\frac{1}{\xi_0}=\frac{mc}{\hbar}$$

- Actual Higgs: $\omega^2=c^2k^2$ (Goldstone), $\omega^2=\frac{m_H^2c^4}{\hbar^2}+c^2k^2$ (Higgs)

Analogue gravity

- GP/Schrödinger equation similar to Euler equation for potential flow!

$$\partial_t n + \nabla(n\mathbf{v}) = 0,\ \mathbf{v}(\mathbf{x},t) = \frac{\hbar\nabla\theta(\mathbf{x},t)}{m} \rightarrow \text{Potential flow!}$$

$$\hbar\partial_t\theta = \frac{\hbar^2\nabla^2\sqrt{n}}{2m\sqrt{n}} - \frac{1}{2}m\mathbf{v}^2 - gn - V$$

- Neglecting quantum ($\hbar$) potential in smooth background $\rightarrow$ Ideal potential flow $\rightarrow$ Analogue gravity (massless Hermitian field in curved spacetime)

$$\Box\delta\hat{\theta} \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\delta\hat{\theta}) = 0$$

$$g_{\mu\nu}(x) = \frac{n(x)}{c(x)}\begin{bmatrix} -[c^2(x)-v^2(x)] & -\mathbf{v}^T(x) \\ -\mathbf{v}(x) & \delta_{ij}\end{bmatrix},$$

- Gravitational phenomena (e.g., Hawking radiation) in the lab!

W. G. Unruh, PRL 46, 1351 (1981)

L. J. Garay, J. R. Anglin, J. I. Cirac, P. Zoller, PRL 85, 4643 (2000)

How did I end up here????

- Theoretical PhD: CS violation and entanglement in Hawking radiation

PHYSICAL REVIEW A **89**, 043808 (2014)

Violation of Cauchy-Schwarz inequalities by spontaneous Hawking radiation in resonant boson structures

J. R. M. de Nova, F. Sols, and I. Zapata
Departamento de Física de Materiales, Universidad Complutense de Madrid, E-28040 Madrid, Spain
(Received 7 November 2012; revised manuscript received 20 February 2014; published 7 April 2014)

New Journal of Physics
The open access journal at the forefront of physics
DPG IOP Institute of Physics
Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

PAPER

Entanglement and violation of classical inequalities in the Hawking radiation of flowing atom condensates

J R M de Nova, F Sols and I Zapata
Departamento de Física de Materiales, Universidad Complutense de Madrid, E-28040 Madrid, Spain

- Experimental postdoc at the Technion: first observation of Hawking effect!

LETTER
https://doi.org/10.1038/s41586-019-1241-0

Observation of thermal Hawking radiation and its temperature in an analogue black hole

Juan Ramón Muñoz de Nova[1], Katrine Golubkov[1], Victor I. Kolobov[1] & Jeff Steinhauer[1]*

nature physics
ARTICLES
https://doi.org/10.1038/s41567-020-01076-0
Check for updates

Observation of stationary spontaneous Hawking radiation and the time evolution of an analogue black hole

Victor I. Kolobov, Katrine Golubkov, Juan Ramón Muñoz de Nova and Jeff Steinhauer

- Exciting prospects in analogue gravity: black-hole laser, Quantum Chronodynamics!

PHYSICAL REVIEW RESEARCH **5**, 043282 (2023)

Black-hole laser to Bogoliubov-Cherenkov-Landau crossover: From nonlinear to linear quantum amplification

Juan Ramón Muñoz de Nova and Fernando Sols
Departamento de Física de Materiales, Universidad Complutense de Madrid, E-28040 Madrid, Spain

Simultaneous symmetry breaking in spontaneous Floquet states: Floquet-Nambu-Goldstone modes, Floquet thermodynamics, and the time operator

Juan Ramón Muñoz de Nova and Fernando Sols
Departamento de Física de Materiales, Universidad Complutense de Madrid, E-28040 Madrid, Spain
(Dated: April 22, 2024)

Part III: Quantum Colliders

Fateful coffee!

- So many analogies...why not studying quantum problems in High-Energy Physics, a relativistic QFT?
- One day, after years of coffee breaks at the Technion with my friend Yoav Afik...

1. Juan: Mmmm...Could you measure a CS violation in a collider? It is like entanglement but not the same and...
2. Yoav: This Cauchy-Schwarz thing is weird...Entanglement is interesting. Tell me more. With fermions.
3. Juan: Oh, well, then there is the spin, it is a qubit you see, there are products of Pauli matrices...
4. Yoav: Better. Give me a sec.

Dramatization

Top Drosophila

- Yoav eventually came to my office and showed me one single equation on top quarks...
- Top quarks are a drosophila of relativistic two-qubit system $\rightarrow$ Huge potential!

$$R^I_{\alpha_1\alpha_2,\ \beta_1\beta_2} = \overline{\sum} \langle t(k_1,\alpha_2), \bar{t}(k_2,\beta_2)|\mathcal{T}|a(p_1), b(p_2)\rangle^* \times \langle t(k_1,\alpha_1), \bar{t}(k_2,\beta_1)|\mathcal{T}|a(p_1), b(p_2)\rangle \quad (2.3)$$

where $I \equiv ab = gg, q\bar{q}$ and the bar denotes averaging over the spins and colors of I and summing over the colors of $t, \bar{t}$. Moreover, α, β are spin labels referring to t and $\bar{t}$, respectively. The matrices R^I can be decomposed in the spin spaces of t and $\bar{t}$ as follows:

$$R^I = f_I \left[A^I \mathbb{1} \otimes \mathbb{1} + \tilde{B}^{I+}_i \sigma^i \otimes \mathbb{1} + \tilde{B}^{I-}_i \mathbb{1} \otimes \sigma^i + \tilde{C}^I_{ij} \sigma^i \otimes \sigma^j \right], \quad (2.4)$$

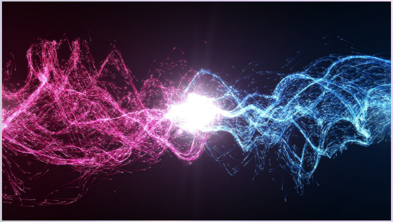
$$f_{gg} = \frac{(4\pi\alpha_s)^2}{N_c(N_c^2-1)}, \qquad f_{q\bar{q}} = \frac{(N_c^2-1)(4\pi\alpha_s)^2}{N_c^2},$$

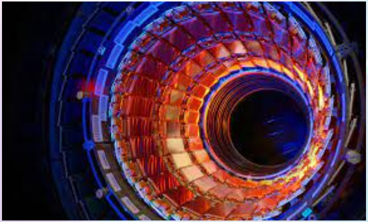
where N_c denotes the number of colors. The first (second) factor in the tensor products of the 2×2 unit matrix $\mathbb{1}$ and of the Pauli matrices σ^i refers to the t ($\bar{t}$) spin space.

W. Bernreuther, D. Heisler, Z.-G. Si, JHEP 12, 026 (2015)

Quantum High-Energy Colliders?

- Naively, Quantum Correlations should be easily studied in colliders...right? $\rightarrow$ Not so fast!
  1. Momentum measurement $\rightarrow$ Decoherence
  2. Lack of control of internal d.o.f. in initial state $\rightarrow$ Decoherence
  3. Most relevant observables in colliders: cross-sections, lifetimes. . . $\rightarrow$ Classical probabilistic objects





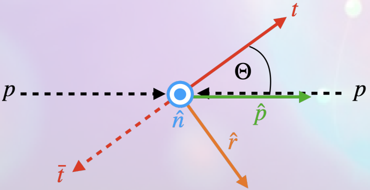
Top-pair production: Kinematics

- Kinematics determined by invariant mass $M_{t\bar{t}}$ and top direction $\hat{k}$ in c.m. frame

$$\begin{aligned} k^{\mu} &= (k^0, \mathbf{k}), \bar{k}^{\mu} = (k^0, -\mathbf{k}) \\ M^2_{t\bar{t}} &\equiv s \equiv (k + \bar{k})^2 \end{aligned}$$

- Invariant mass M is simply related to top c. m. velocity β

$$M_{t\bar{t}} = \frac{2m_t}{\sqrt{1-\beta^2}} \to \beta = 0 \to M_{t\bar{t}} = 2m_t$$



Top-pair production: Quantum State

- Production process from initial state I with internal degrees of freedom λ: $|I\lambda\rangle \to t + \bar{t}$
- $t\bar{t}$ spins described by production spin density matrix:

$$R^{I\lambda}_{\alpha\beta,\alpha'\beta'}(M_{t\bar{t}}, \hat{k}) \equiv \langle M_{t\bar{t}}\hat{k}\alpha\beta| \, T \, |I\lambda\rangle \, \langle I\lambda| \, T^{\dagger} \, |M_{t\bar{t}}\hat{k}\alpha'\beta'\rangle$$

- Experiment: Momentum measurements+Average over events $\to$ Genuine density-matrix description!

$$R^{I}(M_{t\bar{t}}, \hat{k}) = \frac{1}{N_{\lambda}} \sum_{\lambda} R^{I\lambda}(M_{t\bar{t}}, \hat{k})$$

$$\rho^{I}(M_{t\bar{t}}, \hat{k}) = \frac{R^{I}(M_{t\bar{t}}, \hat{k})}{\mathrm{tr}\left[R^{I}(M_{t\bar{t}}, \hat{k})\right]}$$

LO QCD $t\bar{t}$ Quantum state

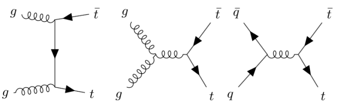
- $t\bar{t}$ production from most elementary QCD processes:

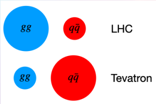
$$\begin{aligned} q + \bar{q} &\rightarrow t + \bar{t},\ q = u, d\ldots \\ g + g &\rightarrow t + \bar{t} \end{aligned}$$

- Initial state $I = q\bar{q}, gg \rightarrow \rho^I(M_{t\bar{t}}, \hat{k})$
- LHC $\rightarrow$ Total quantum state: *Incoherent* mixture of $I = q\bar{q}, gg$ processes with (PDF-dependent) probability w_I

$$\rho(M_{t\bar{t}}, \hat{k}) = \sum_{I=q\bar{q},gg} w_I(M_{t\bar{t}})\rho^I(M_{t\bar{t}}, \hat{k})$$

- QCD Input: $w_I(M_{t\bar{t}}), \rho^I(M_{t\bar{t}}, \hat{k}) \rightarrow$ QI Output: Textbook problem of *convex sum* of quantum states!





Y. Afik, JRMdN, EPJ Plus 136, 907 (2021), Quantum 6, 820 (2022)

$t\bar{t}$ Quantum Correlations

- Quantum state $\rho(M_{t\bar{t}}, \hat{k})$: Function of scattering angle Θ and $M_{t\bar{t}}$.
- Two main regions of quantumness:
  - Ultrarelativistic high-p_T for both $q\bar{q}$ and gg (spin triplet)
  - Threshold for gg (spin singlet).

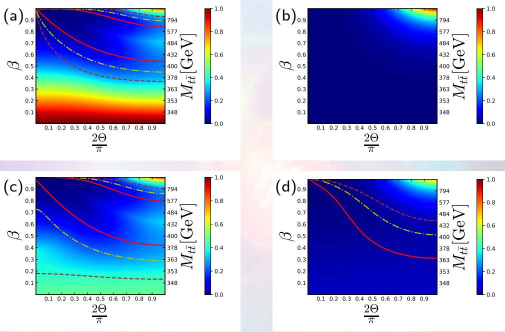
- Colorbar: Discord.
- Solid, dashed-dotted, dashed: Boundaries of Entanglement, Steering, Bell Nonlocality $\rightarrow$ Hierarchy!

a) $gg \rightarrow t\bar{t}$

b) $q\bar{q} \rightarrow t\bar{t}$

c) Run 2 LHC $\sqrt{s} = 13$ TeV

d) Tevatron $\sqrt{s} = 1.96$ TeV



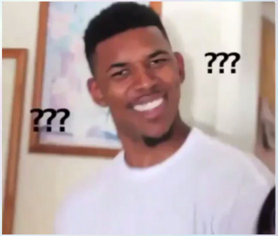
Y. Afik, JRMdN, PRL 130, 221801 (2023)

Cauchy-Schwarz violation

- Simple criterion of entanglement: Cauchy-Schwarz violation

$$|\mathrm{tr}\ \mathbf{C}| = |\langle \boldsymbol{\sigma}_+ \cdot \boldsymbol{\sigma}_- \rangle| = \left| \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \ P(\mathbf{n}_A, \mathbf{n}_B) \mathbf{n}_A \cdot \mathbf{n}_B \right|$$
$$\leq \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \ P(\mathbf{n}_A, \mathbf{n}_B) \, |\mathbf{n}_A \cdot \mathbf{n}_B| \leq \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \ P(\mathbf{n}_A, \mathbf{n}_B) = 1$$

- $D = \dfrac{\mathrm{tr}\ \mathbf{C}}{3} < -1/3 \rightarrow$ CS Violation $\rightarrow$ Entanglement
- Wait a minute...A cosine average larger than one???
- D=Quantum observable with a genuine quantum range of values $-1 < D < -1/3$



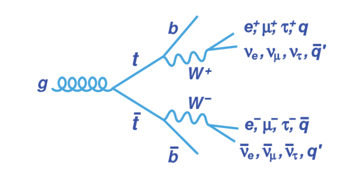
Y. Afik, JRMdN, EPJ Plus 136, 907 (2021), Quantum 6, 820 (2022)

Quantum Tomography: Two qubits, two tops

- **Quantum Tomography**: Reconstruction of quantum state from measurement of a set of observables.
- Two-qubits $\to$ Quantum tomography $=$ Measurement of spin polarizations and spin correlations.
- Top quarks: Spin polarizations $\mathbf{B}^{\pm}$ and spin correlation matrix $\mathbf{C}$ extracted from cross-section $\sigma_{\ell\bar{\ell}}$ of dileptonic decay

$$\frac{1}{\sigma_{\ell\bar{\ell}}}\frac{d\sigma_{\ell\bar{\ell}}}{d\Omega_{+}d\Omega_{-}}=\frac{1}{(4\pi)^2}\left[1+\mathbf{B}^{+}\cdot\hat{\ell}_{+}-\mathbf{B}^{-}\cdot\hat{\ell}_{-}-\hat{\ell}_{+}\cdot\mathbf{C}\cdot\hat{\ell}_{-}\right]$$

- $\hat{\ell}_{\pm}$: lepton directions in each top (antitop) rest frames.

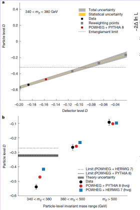


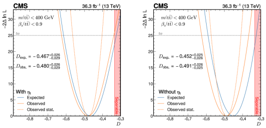
Experimental entanglement observation

- D directly measurable from decay cross-section of angular separation of lepton products:

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1}{2}(1 - D\cos\varphi), \;\; \cos\varphi \equiv \hat{\ell}_+ \cdot \hat{\ell}_-$$

- Entanglement detection from one single magnitude $\to$ No need for Quantum Tomography!
- Entanglement observed by ATLAS and CMS with more than 5σ!
- Toponium signatures?





$D = -0.537 \pm 0.002$ (stat.) ± 0.019 (syst.) (-0.470 ± 0.002 (stat.) ± 0.017 (syst.))

ATLAS, Nature 633, 542 (2024)
CMS, ROPP 87, 117801 (2024)

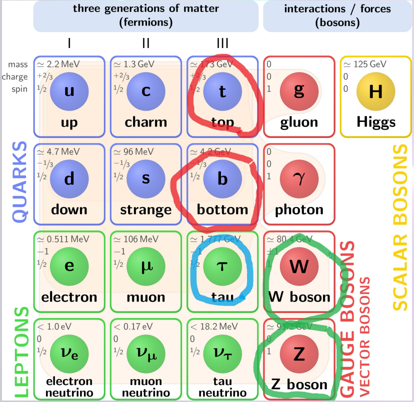
New Physics Witnesses

- Approximate *CP*-invariance of Standard Model $\rightarrow \mathbf{C} = \mathbf{C}^{\mathrm{T}}$, $\mathbf{B}^{+} = \mathbf{B}^{-}$ $\rightarrow$ Symmetric Discord and Steering!
- Therefore: Discord and/or Steering asymmetry $\rightarrow$ New Physics!
- New physics witnesses: Symmetry protected observables by SM, only non-zero for New Physics:
  - $\Delta\mathcal{D}_{t\bar{t}} \equiv \mathcal{D}_t - \mathcal{D}_{\bar{t}}$
  - Asymmetries in steering measurements.
- No SM contribution to New Physics witnesses!

Y. Afik, JRMdN, PRL 130, 221801 (2023)

Alternative and complementary approaches

- Quantum spin-correlations in alternative qubits and qutrits:

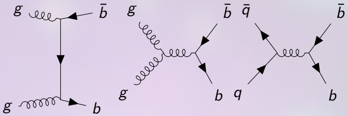


1. t (EPJ Plus 136, 907 (2021))
2. b (arXiv 2406.04402 (2024))
3. τ (EPJC 83, 162 (2023))
4. $W^{\pm}$(PLB 825, 136866 (2022))
5. Z^0(PRD 107, 016012 (2023)).

- Complementary approaches QI-HEP:
  1. Flavor entanglement in mesons: PRL 99, 131802 (2007)
  2. Neutrino oscillations: PRL 117, 050402 (2016)
  3. QI techniques to study QCD interactions: PRL 124, 062001 (2020)

Quantum Bottoms

- *Mutatis mutandis*: Quantum information with $b\bar{b}$!
- $b\bar{b}$ quantum tomography: $\Lambda_b(udb), \bar{\Lambda}_b(\bar{u}\bar{d}\bar{b})$ decays retain $b\bar{b}$ spin information Y. Kats, D. Uzan, JHEP 03 (2024) 063.
- Experimentally challenging $\longleftrightarrow$ Theoretically interesting:
  - Spin correlations in $b\bar{b}$ not measured yet $\rightarrow$ Uncharted territory!
  - Ultrarelativistic $b\bar{b}$ at LHC
  - ATLAS, CMS and also LHCb can play the game!
  - Paves the way to study quantum correlations in hadronizing systems $\rightarrow$ Quark-Gluon Plasma STAR Collaboration, Nature 548, 62 (2017).



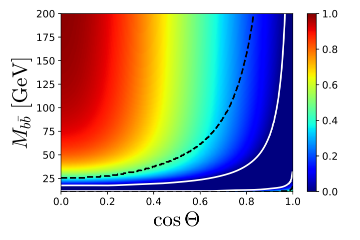
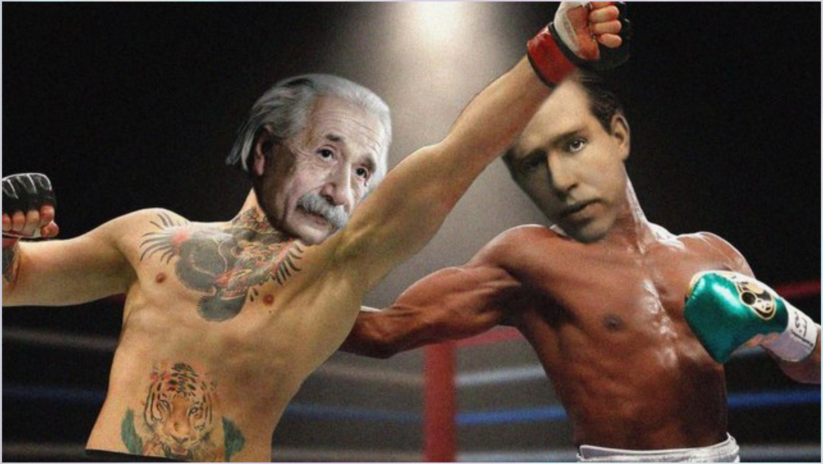


Figure: $b\bar{b}$ concurrence. Y. Afik, Y. Kats, JRMdN, A. Soffer, D. Uzan arXiv 2406.04402 (2024)

Conclusions and outlook

- Quantum Information theory $\longleftrightarrow$ High-Energy Physics. Interdisciplinary, huge potential and great interest!
- QI perspective:
  1. Highest-energy observation of entanglement ever!
  2. Genuinely relativistic, exotic symmetries and interactions, fundamental nature $\rightarrow$ Frontier of known Physics!
  3. Highly-demanding measurements naturally implemented at LHC.
- HEP perspective:
  1. Quantum Tomography: Novel experimental tool.
  2. QI measurements can be used to understand HEP (e.g., toponium, hadronization).
  3. QI techniques can inspire new approaches for searching New Physics: PRD 106, 055007 (2022), JHEP 148, (2023), EPJC 83, 162 (2023)
- Already first measurements of $t\bar{t}$ entanglement by ATLAS and CMS. Highest-energy entanglement ever!
- Many more measurements on its way!

Thank You



Backup

Quantum Theory: Measurement

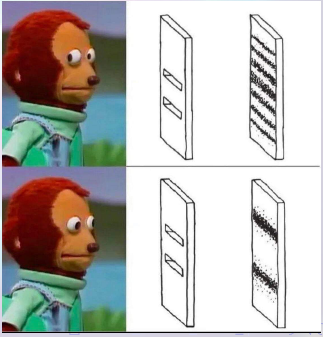
- Copenhagen interpretation of Quantum Mechanics:
  - Wavefunction collapses by measurement process: Quantum state projected to measured state.
- Simple model of measurement: System+Apparatus get entangled

$$|\Psi\rangle \otimes |A\rangle \to \sum_n c_n |\phi_n\rangle \otimes |A_n\rangle \to \rho = \sum_{n,n} c_n c_m^* |\phi_n\rangle \otimes |A_n\rangle \langle\phi_m| \otimes \langle A_m|$$

- Decoherence: Tracing out environment $\to$

$$\rho_S = \mathrm{tr}_E(\rho) = \sum_n |c_n|^2 |\phi_n\rangle \otimes |A_n\rangle \langle\phi_n| \otimes \langle A_n|$$

- Probabilities according to Born rule!
- Collapse is irreversible and... who is irreversible??? Exactly!
- CAUTION: Still not solved



Steering: Two qubits

- Measurements of Bob can "steer" quantum state of Alice.
- Steering: Original conception of Schrödinger of EPR paradox → Only well-defined in 2007! (Wiseman, Jones, Doherty, PRL 98, 140402 (2007))
- Alice post-measurement state described by local-hidden states:

$$\tilde{\rho}_{\hat{\mathbf{n}}} = \Pi^B_{\hat{\mathbf{n}}} \rho \Pi^B_{\hat{\mathbf{n}}} = \int d\lambda \; p(1|\hat{\mathbf{n}}\lambda) p(\lambda) \rho_B(\lambda)$$

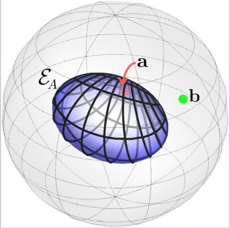
- If not, quantum state is **steerable**.

Steering: Two qubits

- Alice post-measurement state: same as for quantum discord.

$$\rho_{\hat{\mathbf{n}}} = \frac{\tilde{\rho}_{\hat{\mathbf{n}}}}{\mathrm{Tr}\tilde{\rho}_{\hat{\mathbf{n}}}} = \frac{1 + \mathbf{B}^{+}_{\hat{\mathbf{n}}} \cdot \sigma}{2}, \quad \mathbf{B}^{+}_{\hat{\mathbf{n}}} = \frac{\mathbf{B}^{+} + \mathbf{C} \cdot \hat{\mathbf{n}}}{1 + \hat{\mathbf{n}} \cdot \mathbf{B}^{-}}$$

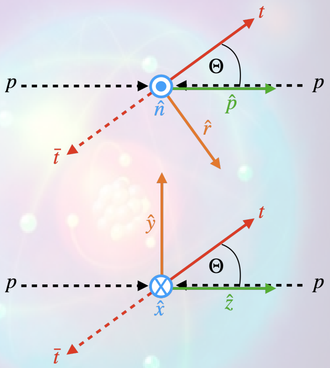
- Set of conditional polarizations $\mathbf{B}^{+}_{\hat{\mathbf{n}}}$ describes an ellipsoid.
- Steering ellipsoid: Fundamental QI object, containing all information about the system.
- Similar for Bob $\rightarrow$ Steering: also asymmetric between Alice and Bob.



Jevtic, Pusey, Jennings, Rudolph
PRL 113, 020402 (2014)

Basis Selection

- Different basis for computing spin polarization and spin correlations characterizing the quantum state.
- Helicity basis: $\{\hat{k}, \hat{r}, \hat{n}\}$:
  - $\hat{k}$ - top direction in $t\bar{t}$ c.m. frame.
  - $\hat{p}$ - beam direction ($\cos\Theta = \hat{k}\cdot\hat{p}$).
  - $\hat{r} = (\hat{p} - \cos\Theta\hat{k})/\sin\Theta$.
  - $\hat{n} = \hat{r}\times\hat{k}$.
  - Study of individual $t\bar{t}$ production with **fixed energy and direction**.
- Beam basis: $\{\hat{x}, \hat{y}, \hat{z}\}$:
  - $\hat{z}$ along beam axis.
  - $\hat{x}, \hat{y}$ transverse directions to beam.
  - Fixed in space: no change with $\hat{k}$.
  - Study of **total integrated quantum state**.



LO QCD Quantum State

- Most general 2-qubit density matrix (15 parameters):

$$\rho(M_{t\bar{t}}, \hat{k}) = \frac{1 + \sum_i \left(B_i^+ \sigma^i + B_i^- \bar{\sigma}^i\right) + \sum_{i,j} C_{ij} \sigma^i \bar{\sigma}^j}{4}$$

- Standard Model $\rightarrow \mathbf{B}^+ = \mathbf{B}^-$, $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$
- LO QCD $\rightarrow$
  1. $\rho(M_{t\bar{t}}, \hat{k})$ is a T-state (unpolarized) $\rightarrow \mathbf{B}^\pm = 0$
  2. Spin along $\hat{n}$ is uncorrelated to other directions
- Only 4 parameters in SM LO QCD: $C_{kk}, C_{rr}, C_{nn}, C_{kr}$

$$\mathbf{B}^\pm = 0, \ \mathbf{C} = \begin{bmatrix} C_{kk} & C_{kr} & 0 \\ C_{kr} & C_{rr} & 0 \\ 0 & 0 & C_{nn} \end{bmatrix}$$

Two-qubit Quantum Criteria

- In general, evaluation of all quantum correlations is a complicated problem (discord and steering).
- However, due to the simple form of $\rho(M_{t\bar{t}}, \hat{k})$ in SM LO QCD:

1. Quantum Discord: Analytical (T-states).
2. Entanglement: Concurrence $0 \leq \mathcal{C}[\rho] \leq 1$, $\mathcal{C}[\rho] > 0$ iff ρ entangled:

$$\mathcal{C}[\rho] = \max(\Delta, 0), \; \Delta \equiv \frac{-C_{nn} + |C_{kk} + C_{rr}| - 1}{2}$$

3. Steerability *iff* $\int \mathrm{d}\hat{\mathbf{n}} \sqrt{\hat{\mathbf{n}}^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{C} \hat{\mathbf{n}}} > 2\pi$ (T-state).
4. CHSH violation *iff* $\mu_1 + \mu_2 > 1$ ($\mu_{1,2}$ largest eigenvalues of $\mathbf{C}^{\mathrm{T}}\mathbf{C}$).

Steering ellipsoid

- Normalized dileptonic cross-section → Angular distribution:

$$p(\hat{\ell}_+, \hat{\ell}_-) = \frac{1}{\sigma_{\ell\bar{\ell}}} \frac{d\sigma_{\ell\bar{\ell}}}{d\Omega_+ d\Omega_-} = \frac{1 + \mathbf{B}^+ \cdot \hat{\ell}_+ - \mathbf{B}^- \cdot \hat{\ell}_- - \hat{\ell}_+ \cdot \mathbf{C} \cdot \hat{\ell}_-}{(4\pi)^2}$$

- Conditional quantum states:

$$\rho_{\hat{\mathbf{n}}}^{(\pm)} = \frac{1 + \mathbf{B}_{\hat{\mathbf{n}}}^{\pm} \cdot \boldsymbol{\sigma}_{\pm}}{2}, \quad \mathbf{B}_{\hat{\mathbf{n}}}^{\pm} = \frac{\mathbf{B}^{\pm} + \mathbf{C}^{\pm} \cdot \hat{\mathbf{n}}}{1 + \mathbf{B}^{\mp} \cdot \hat{\mathbf{n}}}, \quad \mathbf{C}^+ = \mathbf{C}, \ \mathbf{C}^- = \mathbf{C}^T$$

- Direct conditional quantum tomography:

$$p(\hat{\ell}_{\pm} | \hat{\ell}_{\mp} = \mp\hat{\mathbf{n}}) = \frac{p(\hat{\ell}_{\pm}, \hat{\ell}_{\mp} = \mp\hat{\mathbf{n}})}{p(\hat{\ell}_{\mp} = \mp\hat{\mathbf{n}})} = \frac{1 \pm \mathbf{B}_{\hat{\mathbf{n}}}^{\pm} \cdot \hat{\ell}_{\pm}}{4\pi}$$

- Discord → Minimization over conditional entropies.
- $\mathbf{B}_{\hat{\mathbf{n}}}^{\pm}$ → Steering ellipsoid. PRL 113, 020402 (2014)
- Highly-challenging measurements in conventional setups → Natural implementation in colliders!