Lecture 2: Effective Hamiltonian

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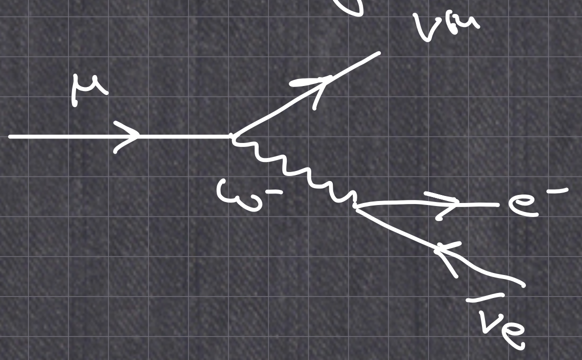
- hep-ph/9806471 Les Houches
- hep-ph/9512380 Review

weak decay: μ-decay



total decay rate

$$\tau = \frac{1}{\Gamma_\mu}$$

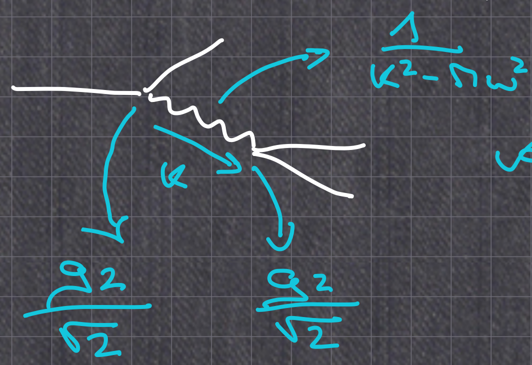
lifetime

Result:

$$\Gamma_\mu = \frac{G_F^2 \cdot m_\mu^5}{192\,\pi^3}\, f\left(\frac{m_e}{m_\mu}\right)$$

assume: $m_\nu = 0$

- Fermi-Constant G_F



$k^2 \ll M_W^2$

$$A \sim \frac{g_2^2}{2} \cdot \frac{1}{k^2 - M_W^2} \simeq -\frac{g_2^2}{2\, M_W^2}$$

$$G_F := \frac{g_2^2}{4\sqrt{2}\, M_W^2}$$

- Phase space function $f(x)$; $f(0) = 1$

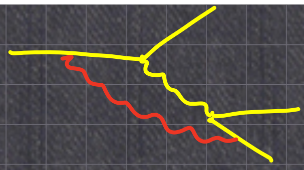
$$f(x) = 1 - 8x^2 + 8x^6 - x^8 - 24\, x^4 \ln x$$

$$x = \frac{m_e}{m_\mu} = \frac{511\ \text{keV}}{105\ \text{MeV}} \ll 1 \Rightarrow f(x) \simeq 1$$

Compare with experiment

Exp: $\tau_\mu^{\text{Exp}} = 2.1969811(22) \cdot 10^{-6}\ s$

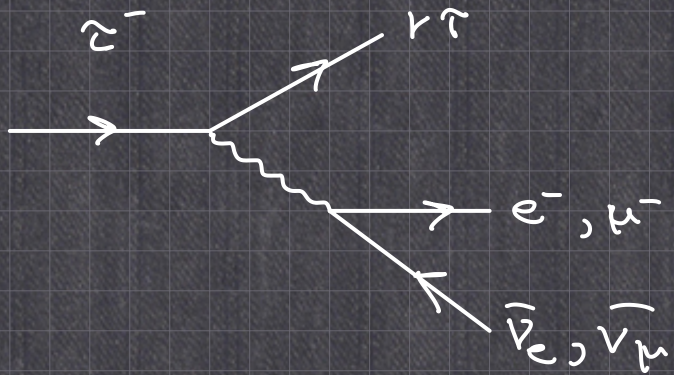
$\tau_\mu^{\text{Theo}} = 2.18776 \cdot 10^{-6}\ s$

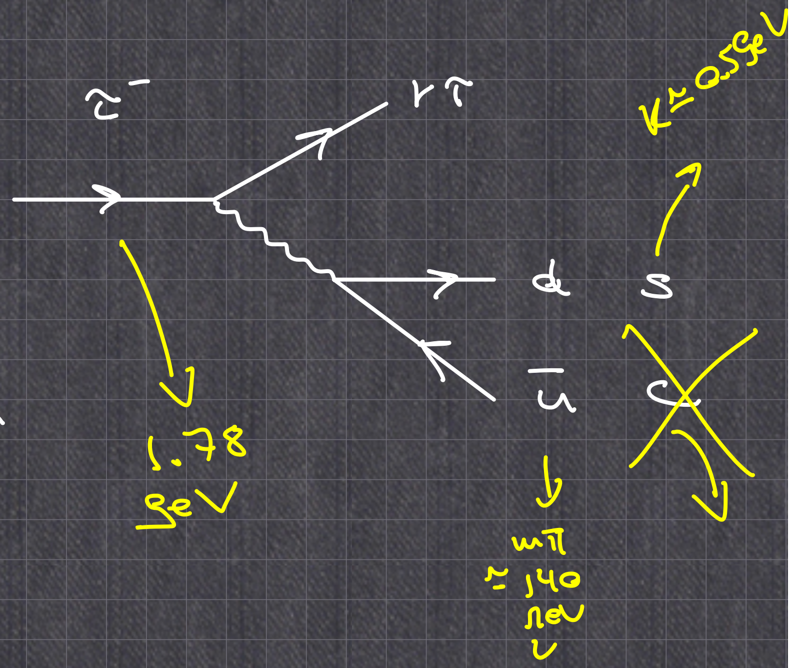
+ higher orders  + 

$+ \mathcal{O}(\alpha_{QED})$

$$\tau_\mu^{\text{Theo}} = 2.19699 \cdot 10^{-6}\,s$$

Tau-decay:





$$\Gamma_\tau = \frac{G_F^2 m_\tau^5}{192\pi^3}\left[f\left(\frac{m_e}{m_\tau}\right) + f\left(\frac{m_\mu}{m_\tau}\right) + |V_{ud}|^2 N_c\, g\left(\frac{m_u}{m_\tau}, \frac{m_d}{m_\tau}\right) + |V_{us}|^2 N_c\, g\left(\frac{m_u}{m_\tau}, \frac{m_s}{m_\tau}\right)\right]$$

Approx: $m_e, m_\mu, m_u, m_d, m_s \ll m_\tau \Rightarrow f, g \simeq 1$

$$[\dots] = 2 + N_c\left(|V_{ud}|^2 + |V_{us}|^2\right) \simeq \underline{\underline{5}}$$

$$\frac{\tau_\tau}{\tau_\mu} = \frac{\Gamma^\mu}{\Gamma^\tau} = \left(\frac{m_\mu}{m_\tau}\right)^5 \frac{1}{5}$$

Exp: $\quad \tau_\tau = 2.906(1)\cdot 10^{-13}\,s$

Theory: $\quad \tau_\tau = \left(\frac{m_\mu}{m_\tau}\right)^5 \frac{1}{5}\cdot \tau_\mu^{Exp/Th.}$

$$= 3.267\cdot 10^{-13}\,s$$

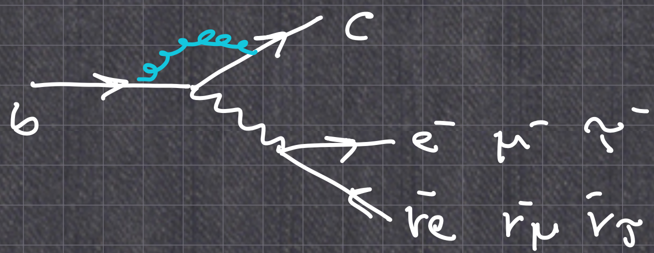
include QCD-effects:

$\alpha_s(m_\tau) \simeq 0.3$ $\qquad \alpha_{QED} \simeq \frac{1}{137}$

$\alpha_s(m_b) \simeq 0.2$

$\alpha_s(m_Z) \simeq 0.1$

b-quark decays



$$\Gamma_b = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\{\ldots\ e, \mu, \tau, \text{quarks})$$

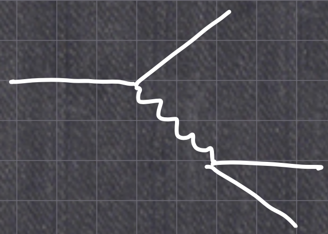
$\tau^B_{Exp} \simeq 1.5\ ps$

$\tau^b_{Theo} \simeq 0.9\ldots 3.7\ ps$

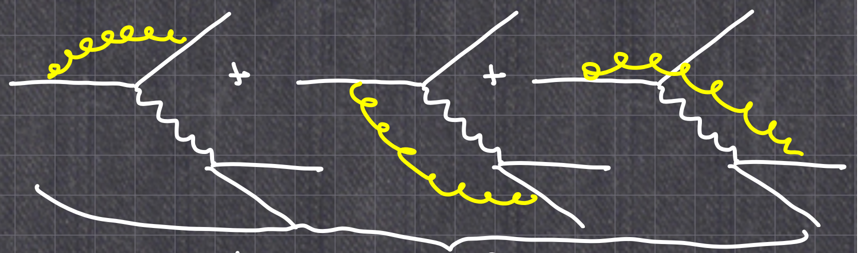
take QCD effects serious:

$\rightarrow$ 1 Def. of quark mass
- pole mass
- $\overline{MS}$ mass
- potential subtr. 1S, kinetic ...

② pure QCD corrections



$+$



Tree-level   1-loop-level

$\mathcal{O}(1)$ naive exp: $\mathcal{O}(\alpha_s(m_b)) \mathrel{\hat{=}} 0.2$

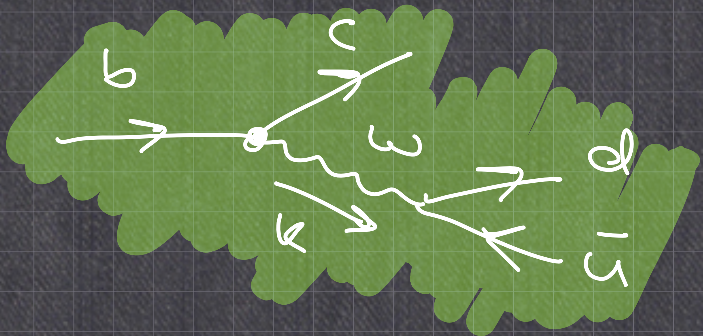
Real calc. $\mathcal{O}(1)$ $+ \ldots \alpha_s(m_b) + \ldots \underbrace{\alpha_s(m_b) \underbrace{\ln\left(\frac{m_b^2}{m_W^2}\right)}_{6-7}}_{1.2}$

	leading logs	NLL	NNLL	
Tree-level	1			
1-loop	$\alpha_s \ln$	α_s		
2-loop	$\alpha_s^2 \ln^2$	$\alpha_s^2 \ln$	α_s^2	
3-loop	$\alpha_s^3 \ln^3$	$\alpha_s^3 \ln^2$	$\alpha_s^3 \ln$	α_s^3

Effective theories + Renormalisation group equation

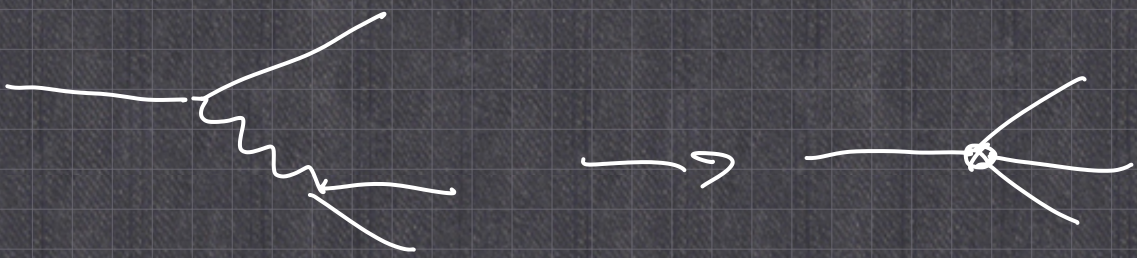
$+$  $+ \ldots$

This procedure is formalised in the derivation of the effective Hamiltonian

① Start: 

② "Integrate out" heavy W

$$\frac{1}{k^2 - M_W^2} \xrightarrow{k^2 \ll M_W^2} \frac{1}{-M_W^2}$$



Feynman-rules for $b \to c\bar{u}d$

$$= c\,\frac{i}{2}\frac{g_2}{\sqrt{2}} V_{cb}^* \gamma_\mu(1-\gamma_5) b \cdot \frac{1}{k^2 - M_W^2} \cdot \bar{d}\,\frac{i}{2}\frac{g_2}{\sqrt{2}} V_{ud}\gamma^\mu(1-\gamma_5) u$$

$k^2 \ll M_W^2$

$$\underbrace{\left(\frac{g_2}{2\sqrt{2}}\right) \cdot \frac{1}{M_W^2}}_{\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}}} \cdot \underbrace{V_{cb}^* V_{ub}}_{V_{CKM}} \cdot \underbrace{1}_{C_2} \cdot \underbrace{\bar{c}\gamma_\mu(1-\gamma_5) b \cdot \bar{d}\gamma^\mu(1-\gamma_5) u}_{Q_2}$$

$C_2 \to$ Wilson coefficient

$Q_2 \to$ 4-quark operator

Add QCD:

① new operator: → color indices

$$Q_2 = (\bar{c}^{\alpha} b^{\alpha})_{V-A} (\bar{d}^{\beta} u^{\beta})_{V-A} \longrightarrow \text{colour-singlett operator}$$

$$Q_1 = (\bar{c}^{\alpha} b^{\beta})_{V-A} (\bar{d}^{\beta} u^{\alpha})_{V-A} \longrightarrow \text{colour-rearranged operator}$$

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{ckm} \left(C_1 Q_1 + C_2 Q_2 \right)$$

②

no QCD	+ QCD:
$C_2 = 1$	$C_2 = 1 + \mathcal{O}(\alpha_s) \simeq 1.1$
$C_1 = 0$	$C_1 = 0 + \mathcal{O}(\alpha_s) \simeq -0.2$

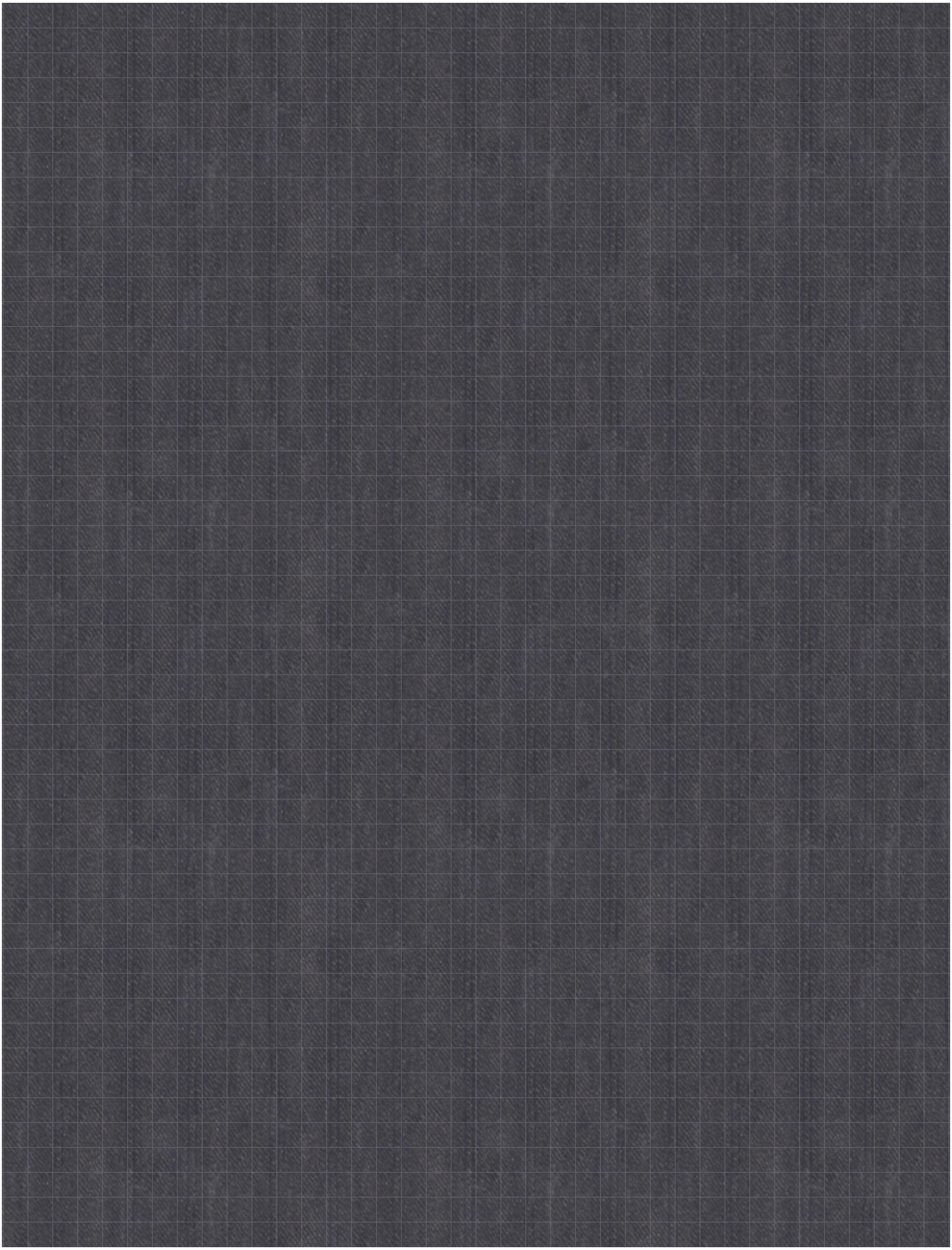
③ sum up all the logs with the renormalisation group

④ $\mathcal{H}_{eff}$ separates the scales

short distance physics is in $C_i(\mu)$

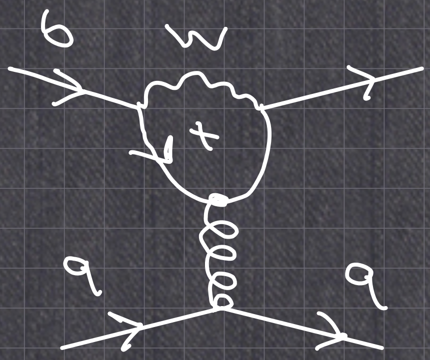
long — " — $\langle Q_i \rangle (\mu)$

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{ckm} \left(C_1 Q_1 + C_2 Q_2 \right)$$



Penguin operators

QCD penguin operators



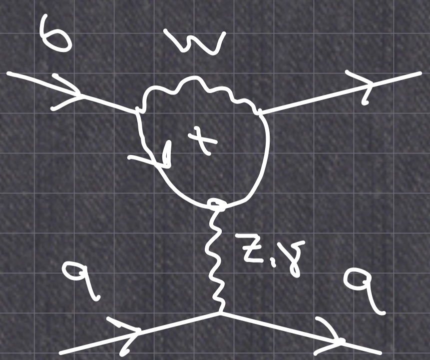
$$Q_3 = (\bar{s}\, b)_{V-A} \sum_{q=u,\ldots,b} (\bar{q}\, q)_{V-A}$$

$$Q_4 = (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\ldots,b} (\bar{q}^{\beta} q^{\alpha})_{V-A}$$

$$Q_5 = (\bar{s}\, b)_{V-A} \sum_{q=u,\ldots,b} (\bar{q}\, q)_{V+A}$$

$$Q_6 = (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\ldots,b} (\bar{q}^{\beta} q^{\alpha})_{V+A}$$

Electro-weak penguin operators



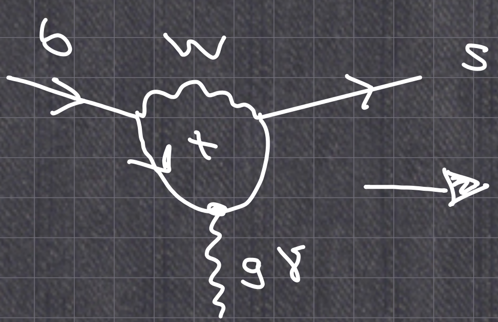
$$Q_7 = \frac{3}{2} (\bar{s}\, b)_{V-A} \sum_{q=u,\ldots,b} e_q (\bar{q}\, q)_{V+A}$$

$$Q_8 = \frac{3}{2} (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\ldots,b} e_q (\bar{q}^{\beta} q^{\alpha})_{V+A}$$

$$Q_9 = \frac{3}{2} (\bar{s}\, b)_{V-A} \sum_{q=u,\ldots,b} e_q (\bar{q}\, q)_{V-A}$$

$$Q_{10} = \frac{3}{2} (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\ldots,b} e_q (\bar{q}^{\beta} q^{\alpha})_{V-A}$$

Magnetic penguin operators

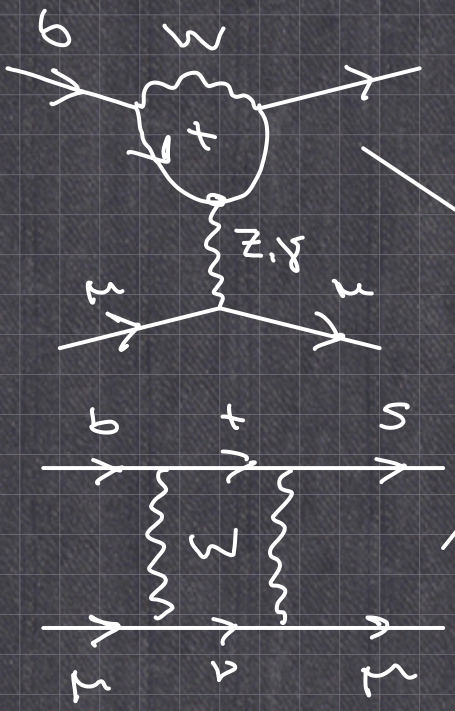


$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} (1+\gamma_5)\, b_\alpha F_{\mu\nu}$$

$$Q_{8g} = \frac{g}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} (1+\gamma_5) T^a_{\alpha\beta} b_\beta G^a_{\mu\nu}$$

Semileptonic penguin operators

12:12



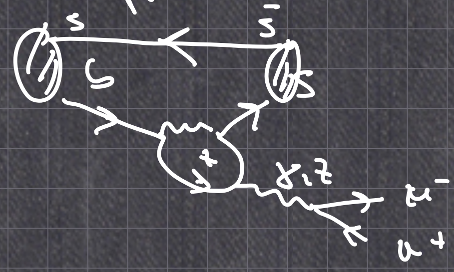
q^2

$$Q_{9V} = (\bar{s} b)_{V-A} (\bar{\mu} \mu)_V$$

$$Q_{10A} = (\bar{s} b)_{V-A} (\bar{\mu} \mu)_A$$

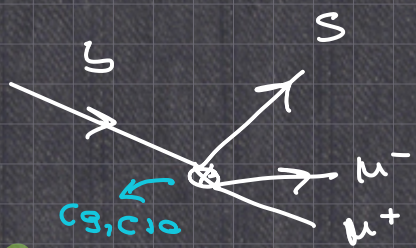
$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \Big[\sum_{\substack{q = u,c \\ q' = u,c \\ q'' = d,s}} V_c \left\{ C_1(\mu) Q_1^{qq'q''} + C_2(\mu) Q_2^{qq'q''} \right\} - V_p \sum_{j \geq 3} C_j(\mu) Q_j \Big]$$

How to apply the $\mathcal{H}_{eff}$ to $B_s \to \phi\, \mu\mu$



in the SM: $b \to s\, \mu\, \mu$

in the $\mathcal{H}_{eff}$:



Q_{9V}
Q_{10A} +