Introduction to Transverse Beam Dynamics

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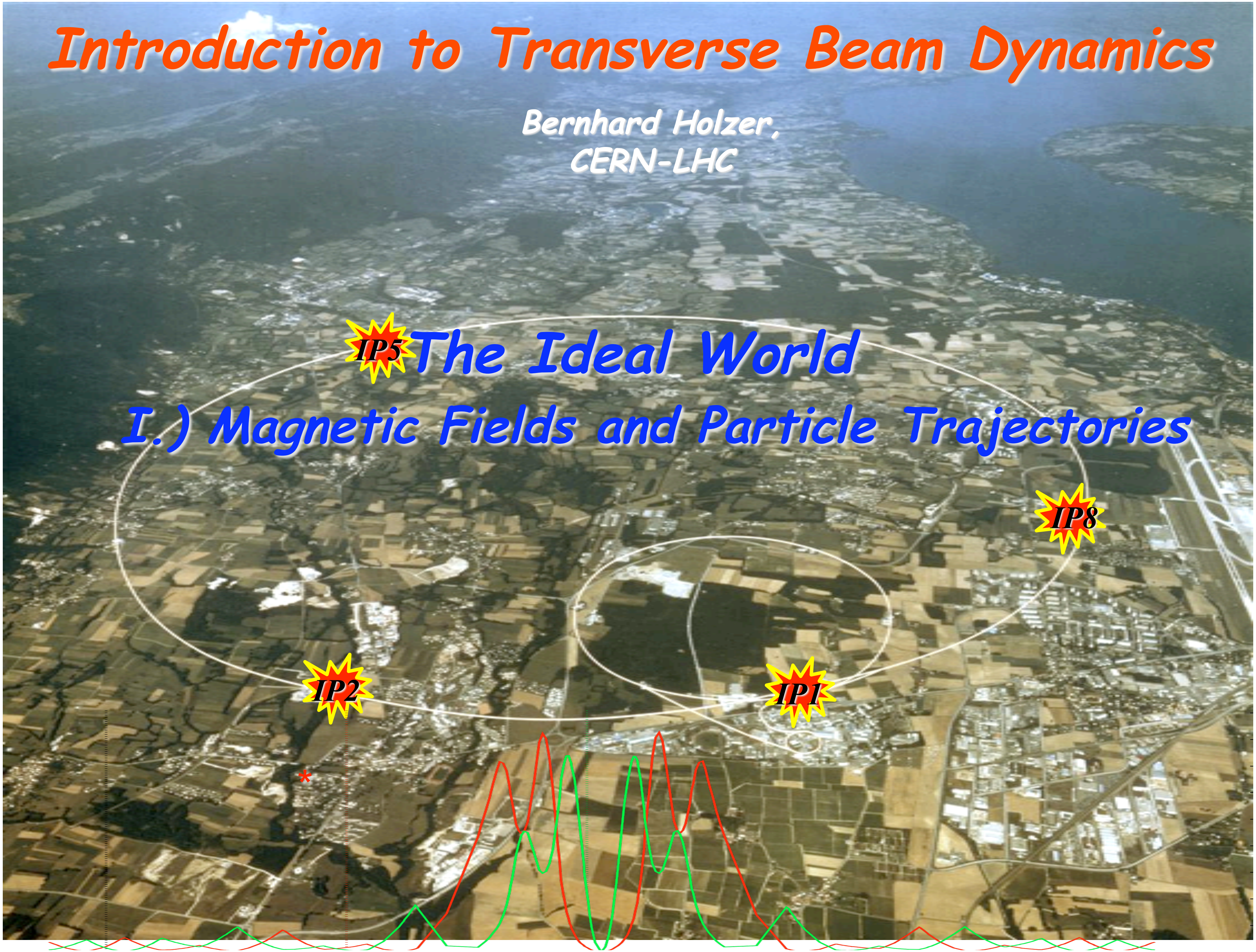
The Ideal World

I.) Magnetic Fields and Particle Trajectories



1.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine"

→ need transverse deflecting force

Lorentz force $\vec{F} = q * (\vec{E} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3*10^8 \, m/s$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q*3*10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q*300 \frac{MV}{m}$$

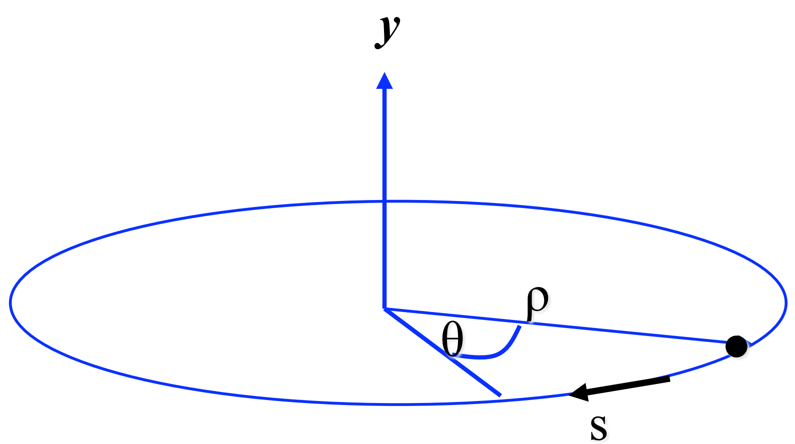
equivalent el. field ... E

technical limit for el. field:

$$E \leq 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $$F_L = e\,v\,B$$

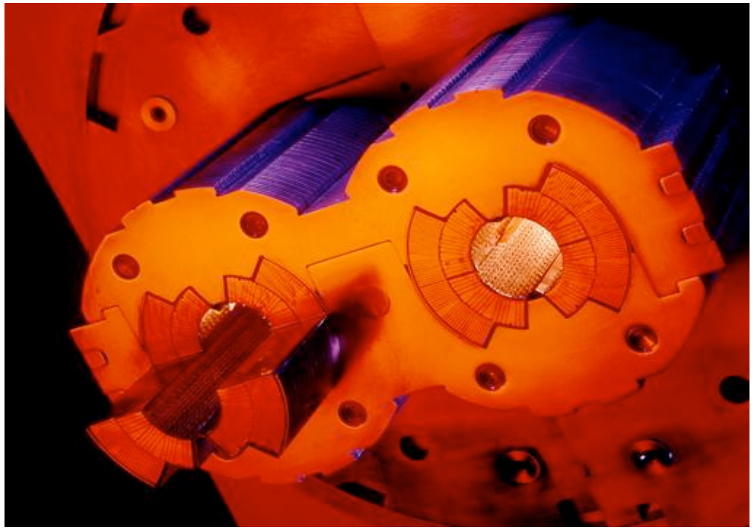
centrifugal force $$F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$$

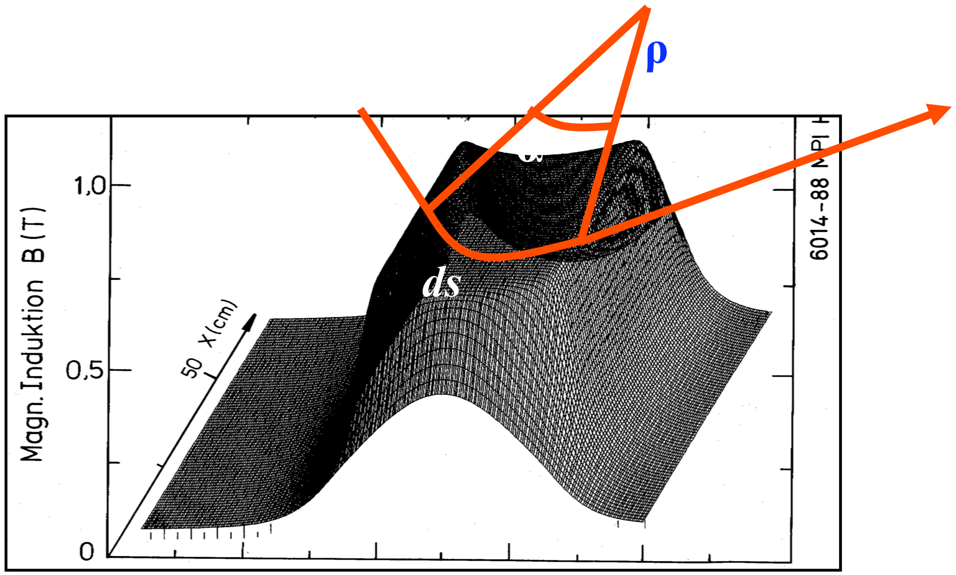
$$\frac{\gamma\, m_0\, v^2}{\rho} = e\,v\,B$$

$$\frac{p}{e} = B\,\rho$$

$B\rho$ = *"beam rigidity"*

The Magnetic Guide Field





field map of a storage ring dipole magnet

Example LHC:

$$\left.\begin{array}{l} B = 8.3\,T \\ p = 7000\,\dfrac{GeV}{c} \end{array}\right\}$$

$$\rho = 2.53\,km \longrightarrow 2\pi\rho = 17.6\,km$$

$$\approx 66\%$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

rule of thumb:

$$\frac{1}{\rho} \approx 0.3\,\frac{B[T]}{p[GeV/c]}$$

„normalised bending strength“

Example LHC:

7000 GeV Proton storage ring

dipole magnets N = 1232

l = 15 m

q = +1 e

$$\int \boldsymbol{B}\, d\boldsymbol{l} \approx \boldsymbol{N}\ \boldsymbol{l}\ \boldsymbol{B} = 2\pi\ \boldsymbol{p}/\boldsymbol{e}$$

$$\boldsymbol{B} \approx \frac{2\pi\ \ 7000\ \ 10^{9}\, \boldsymbol{eV}}{1232\ \ 15\, \boldsymbol{m}\ \ 3\ 10^{8}\ \frac{\boldsymbol{m}}{\boldsymbol{s}}\ \ \boldsymbol{e}} = 8.3\ \boldsymbol{Tesla}$$

2.) Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

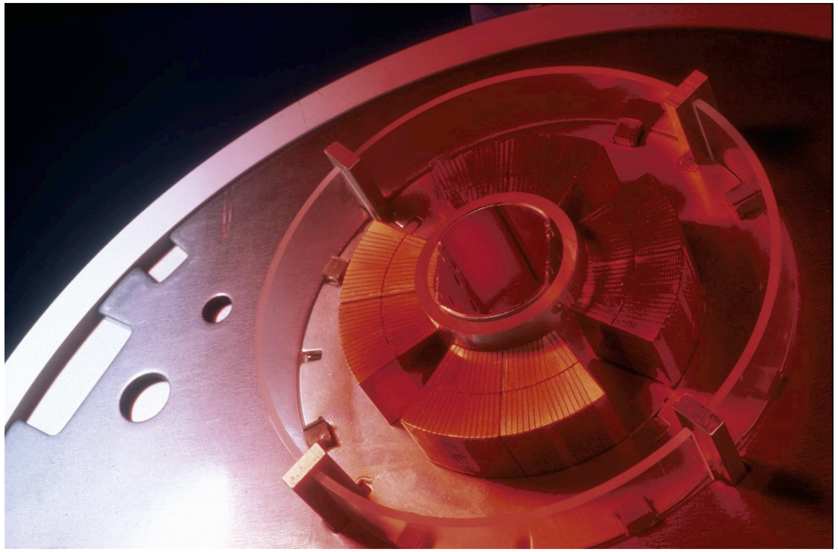
linear increasing Lorentz force

linear increasing magnetic field $B_y = g\,x \qquad B_x = g\,y$

normalised quadrupole field:

gradient of a quadrupole magnet: $g = \dfrac{2\mu_0 n I}{r^2}$

$\longrightarrow \quad k = \dfrac{g}{p/e}$



LHC main quadrupole magnet

$g \approx 25 \ldots 220 \ T/m$

simple rule: $k = 0.3 \dfrac{g(T/m)}{p(GeV/c)}$

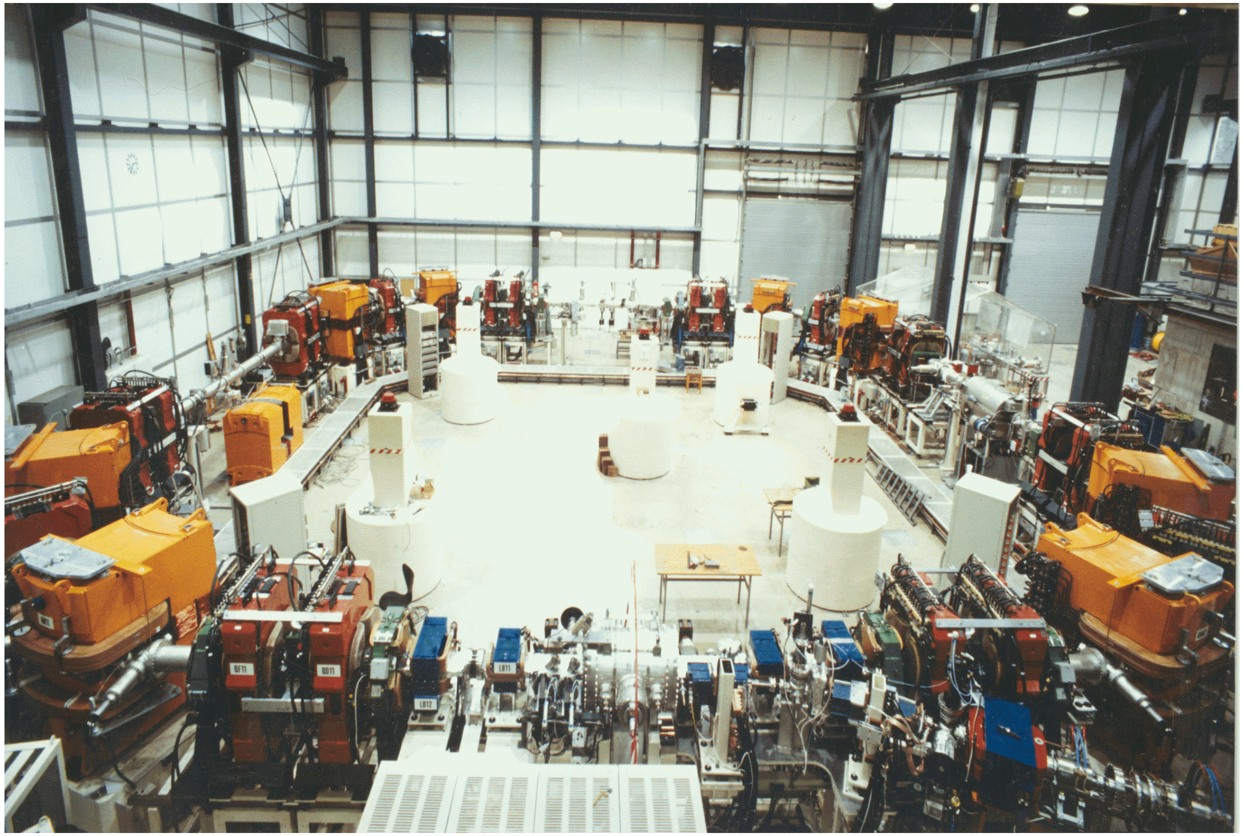
what about the vertical plane:
... Maxwell

$\vec{\nabla} \times \vec{B} = \vec{j} + \dfrac{\partial \vec{E}}{\partial t} = 0 \quad \Rightarrow \quad \dfrac{\partial B_y}{\partial x} = \dfrac{\partial B_x}{\partial y}$

3.) The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \frac{1}{2!} m\,x^2 + \frac{1}{3!} n\,x^3 + \dots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

Split the magnets and optimise them according to their job:

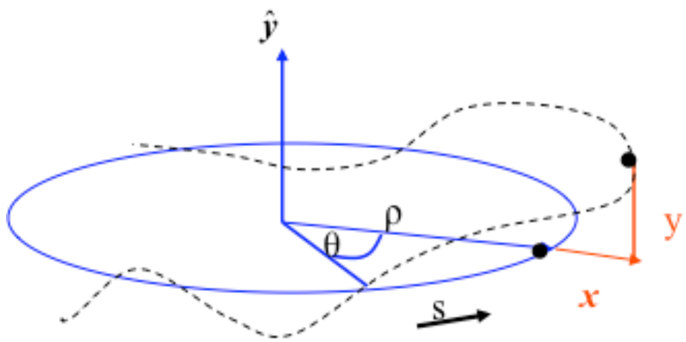
bending, focusing etc

Example:
heavy ion storage ring TSR

* *man sieht nur*
dipole und quads → linear

The Equation of Motion:

general radial acceleration $a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$



general trajectory: $\rho \rightarrow \rho + x$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

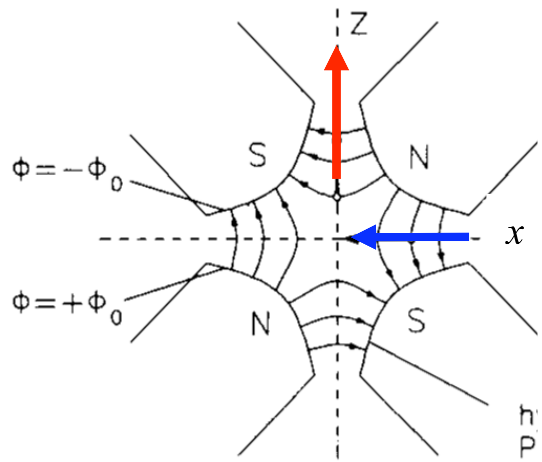
... using $x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds}$

* *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ *no dipoles … in general …*

$k \leftrightarrow -k$ *quadrupole field changes sign*

$$y'' + k\,y = 0$$



4.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$x'' + K\,x = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \longrightarrow \quad \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1, a_2 by boundary conditions:

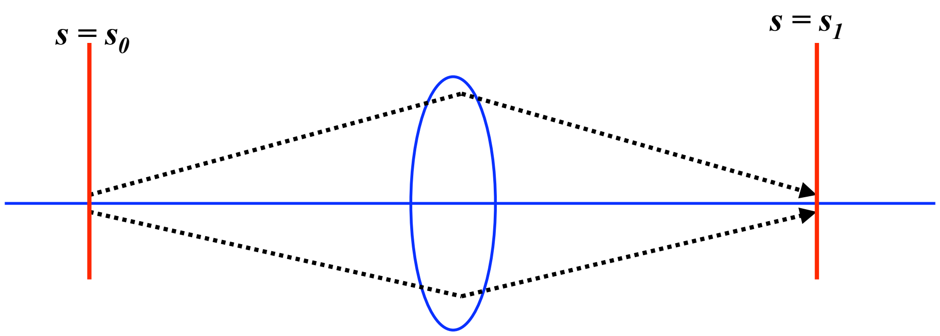
$$s = 0 \quad \longrightarrow \quad \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

Hor. Focusing Quadrupole K > 0:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

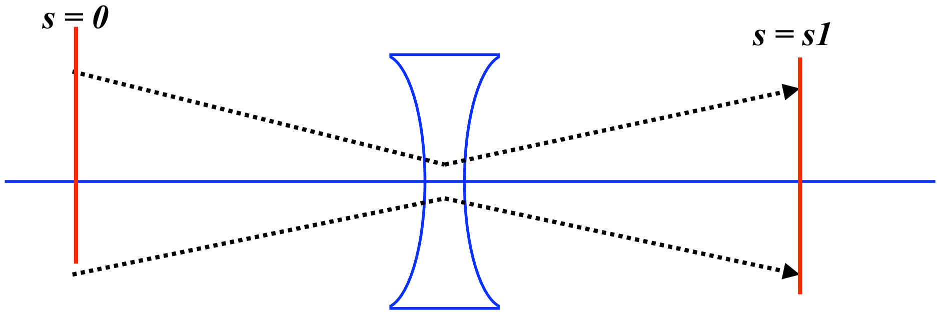


$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



Remember from school:

$$f(s) = \cosh(s) \quad , \quad f'(s) = \sinh(s)$$

Ansatz: $\quad x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space:

$K = 0$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

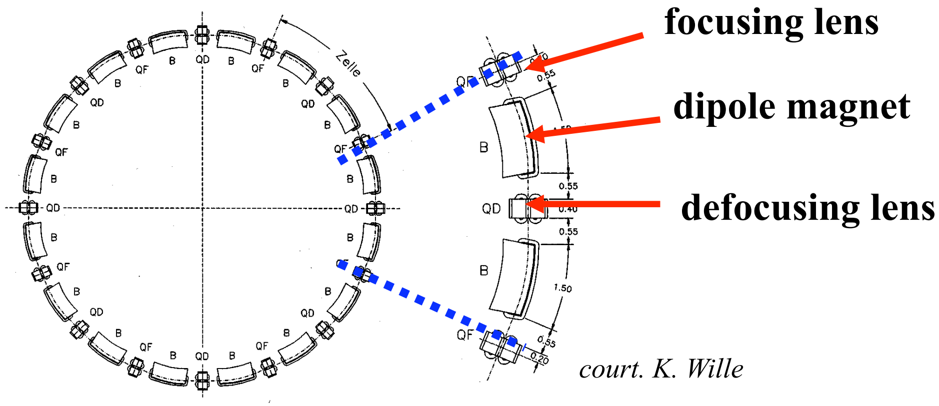
! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled"

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

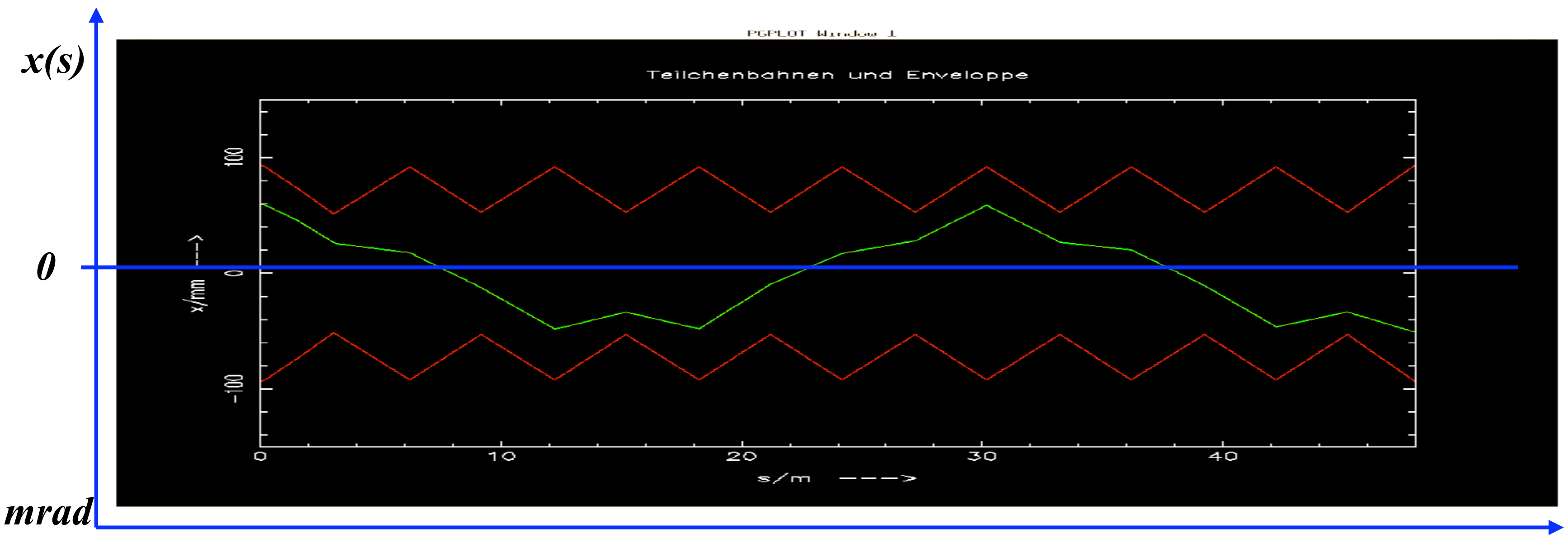
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



court. K. Wille

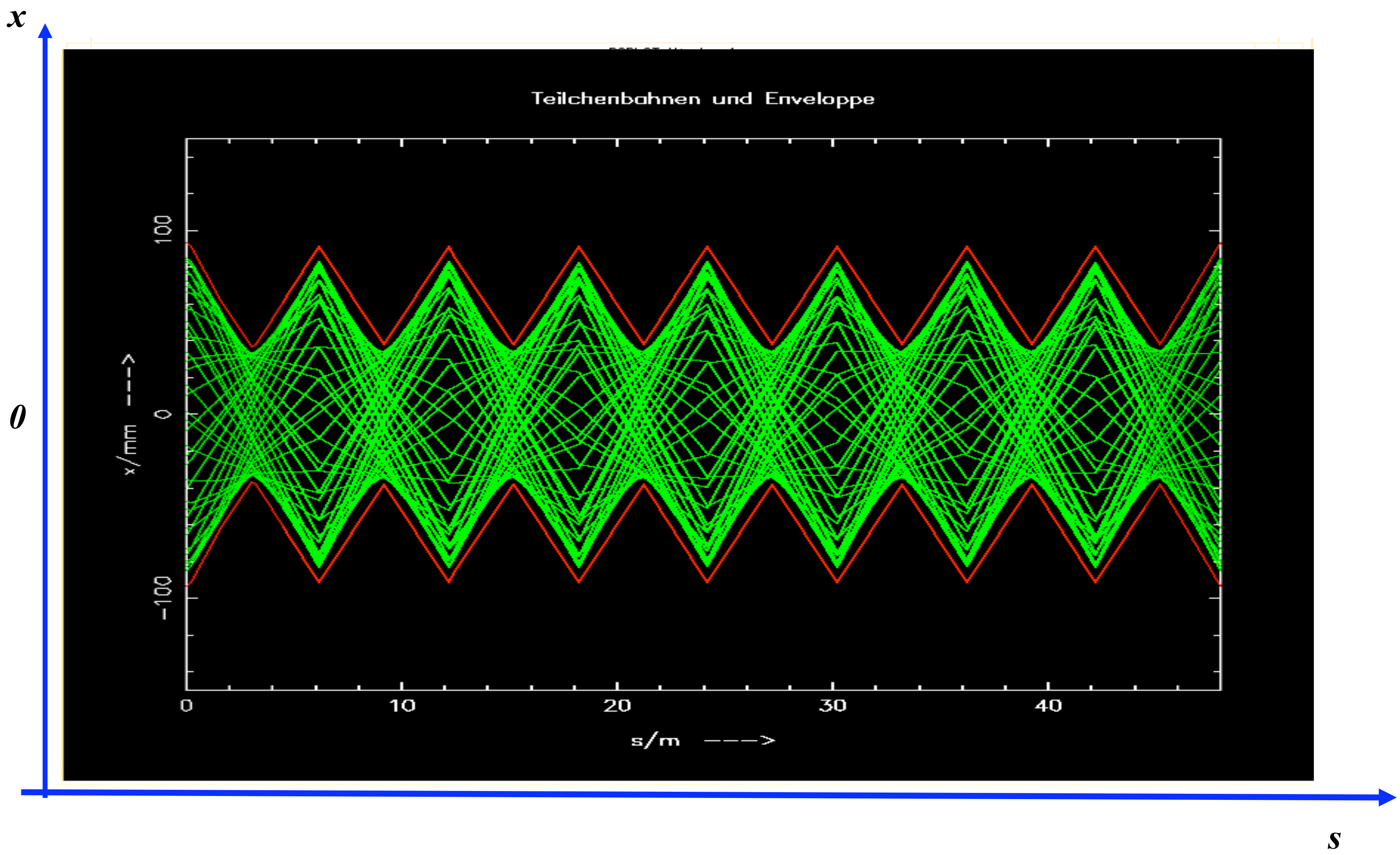
in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values
in a strong
foc. machine:
x ≈ mm, x´ ≤ mrad

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force $\neq$ ***const,***
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

5.) The Beta Function

General solution of Hill´s equation:

(i)

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

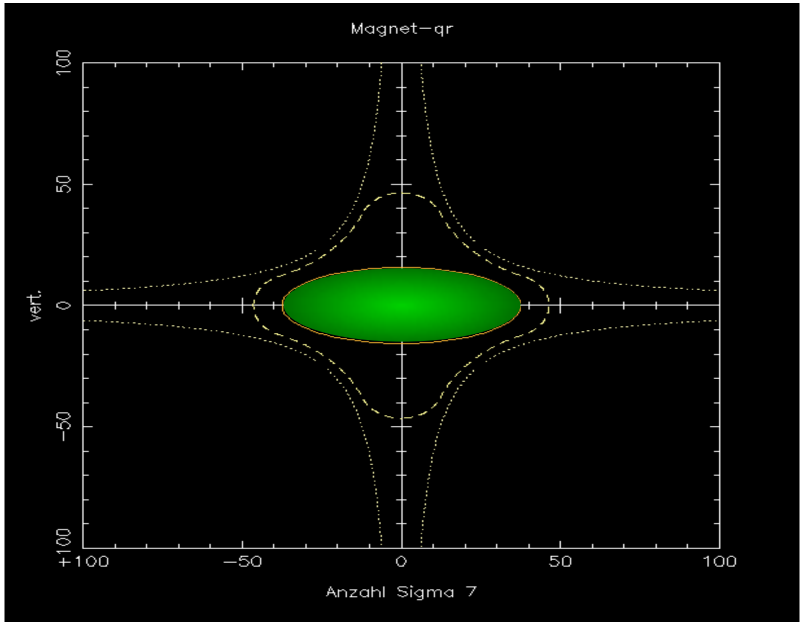
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

The Beta Function

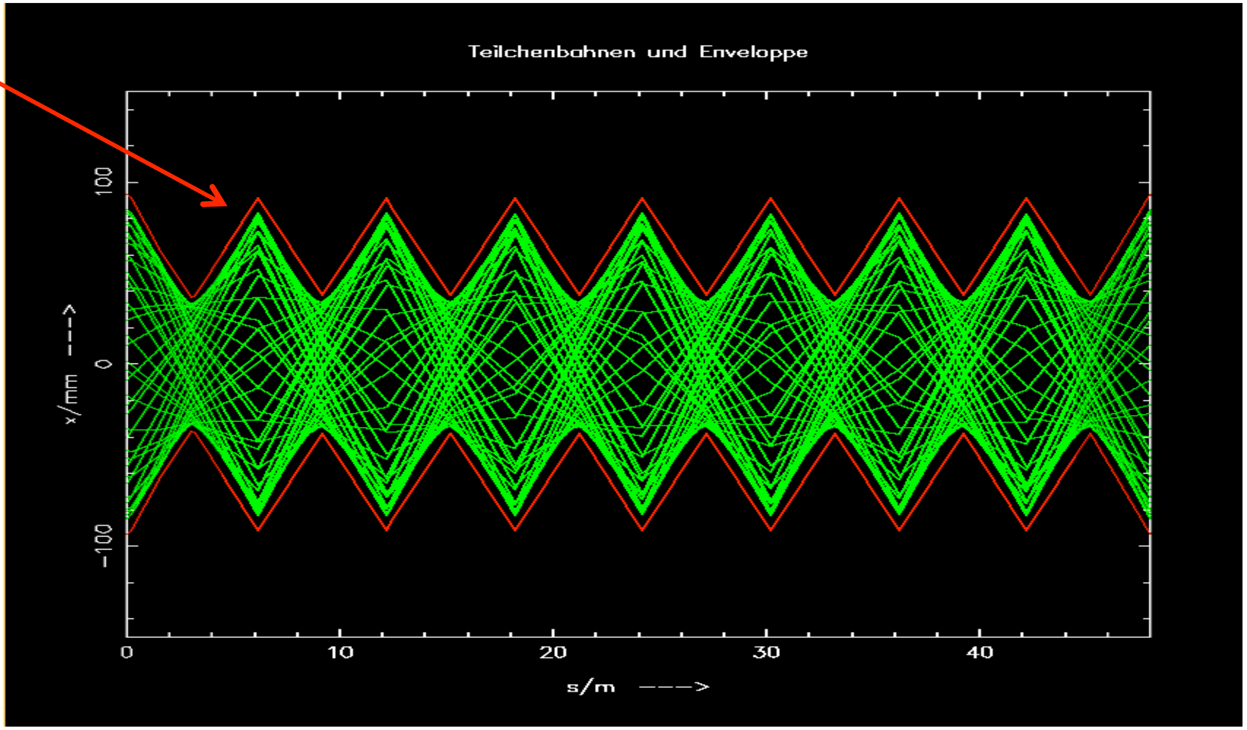


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

6.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

(1) $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi)$$

(2) $$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\right\}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

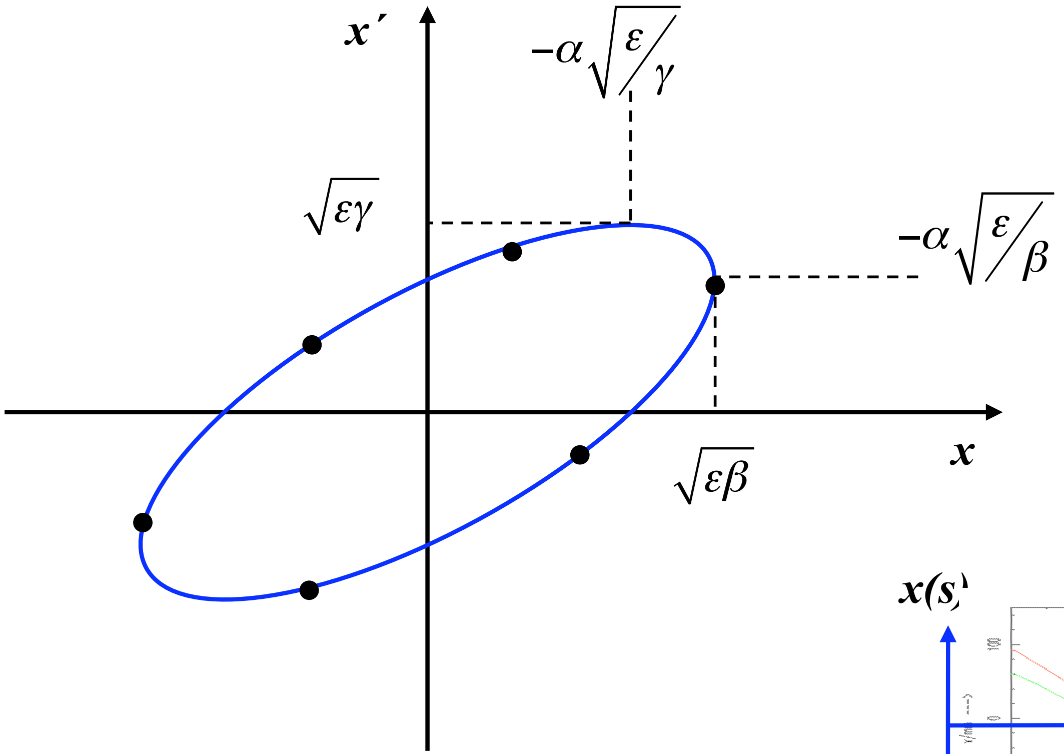
Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

***** ε is a constant of the motion … it is independent of „s“*
***** parametric representation of an ellipse in the x x‘ space*
***** shape and orientation of ellipse are given by α, β, γ*

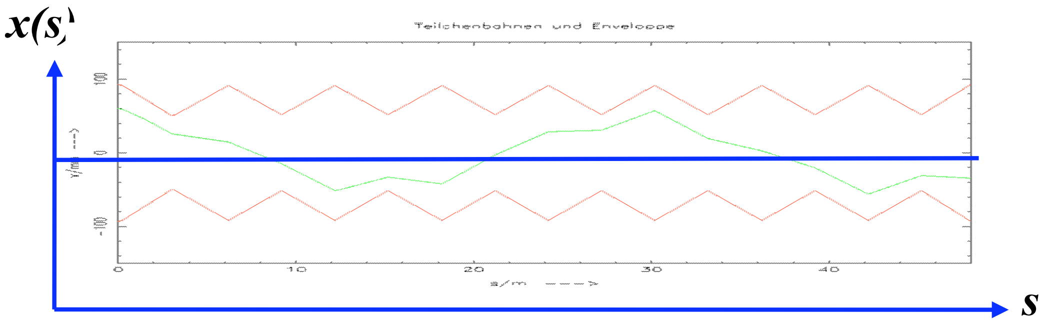
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

$$A = \pi^* \varepsilon = const$$



ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

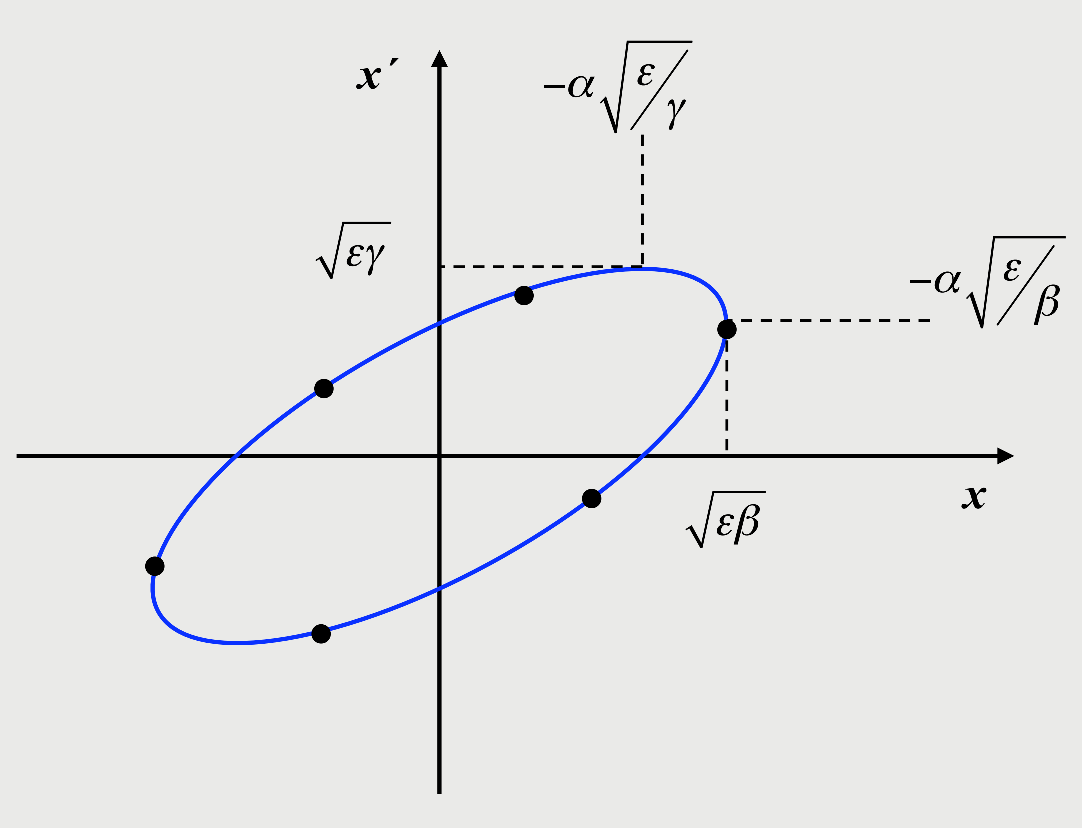
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot x x' + \beta \cdot x'^2$$

… solve for x′ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ *via:* $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

$$\longrightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$

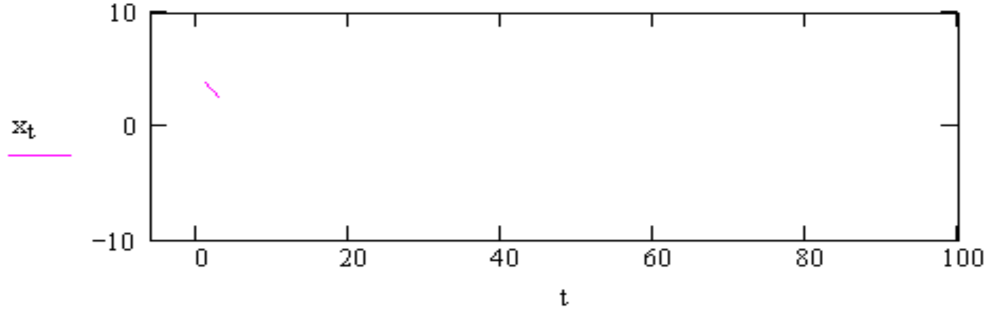


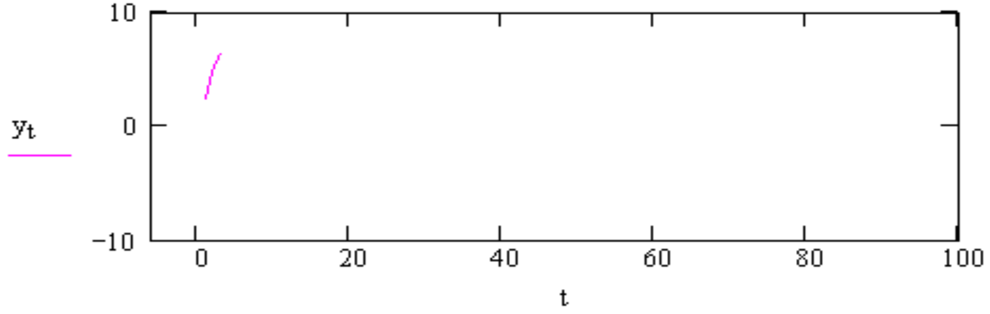
shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

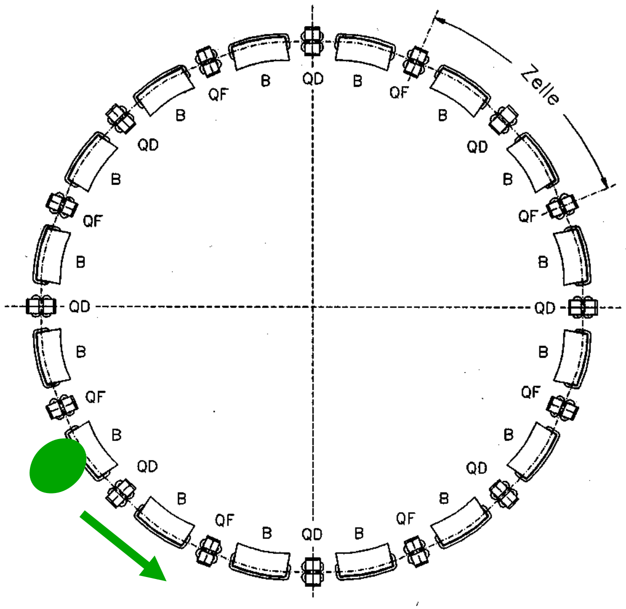
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

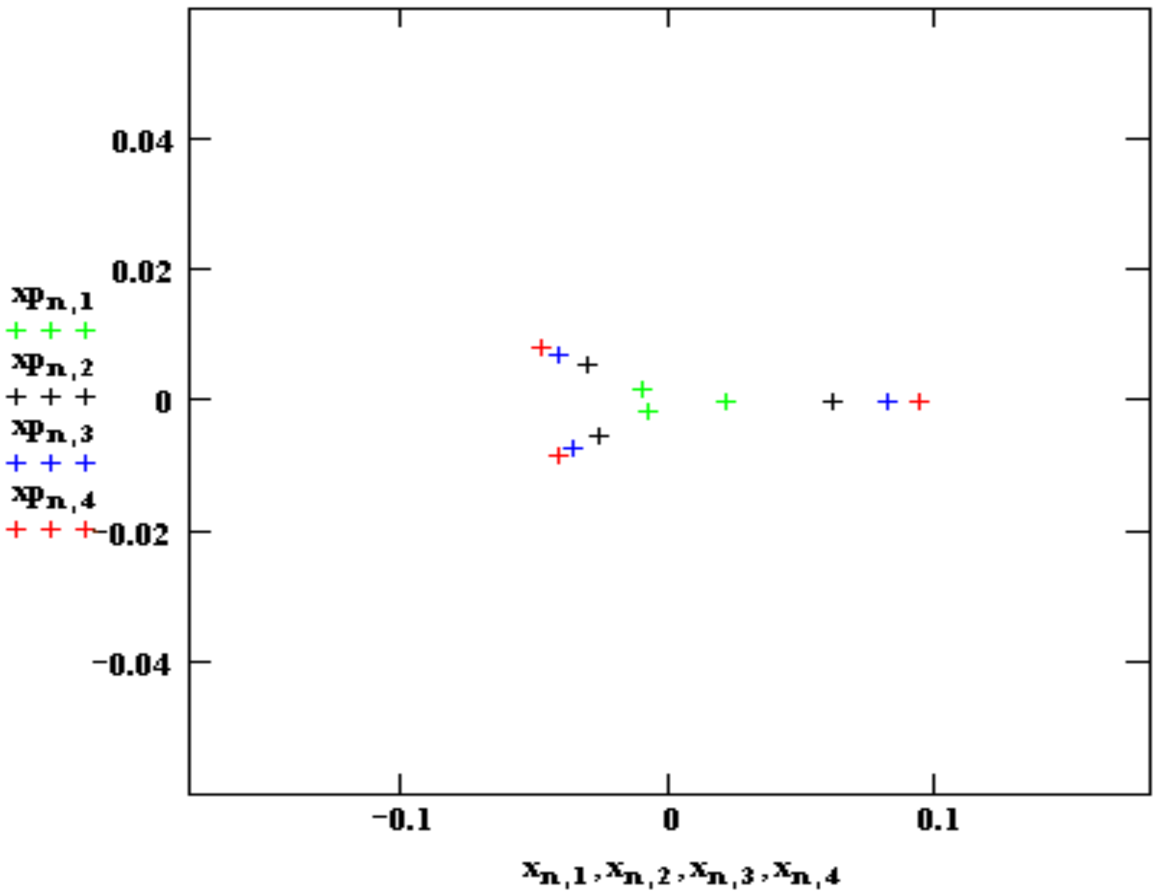






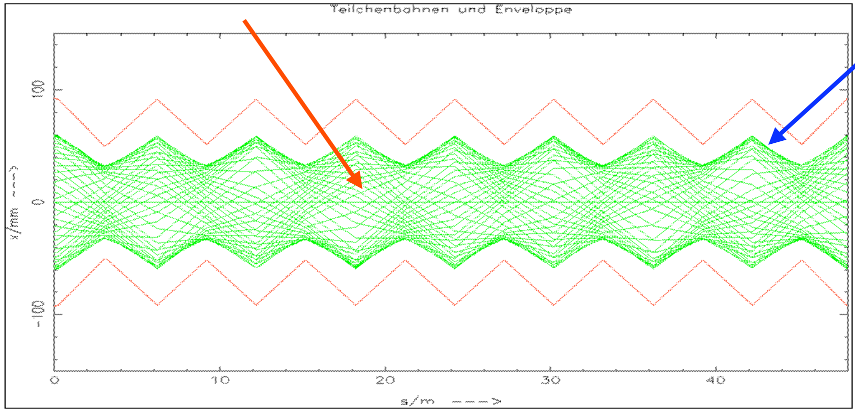
… and now the ellipse: A beam of 4 particles each having a slightly different emittance:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 ^{11} ***per bunch***

Gauß Particle Distribution:

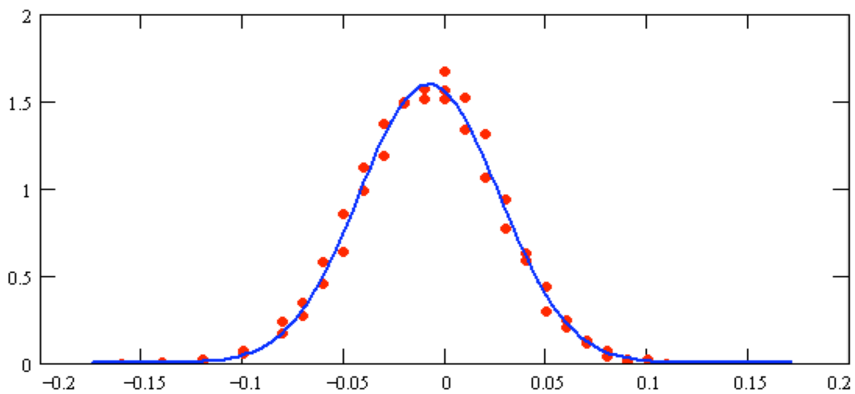
$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

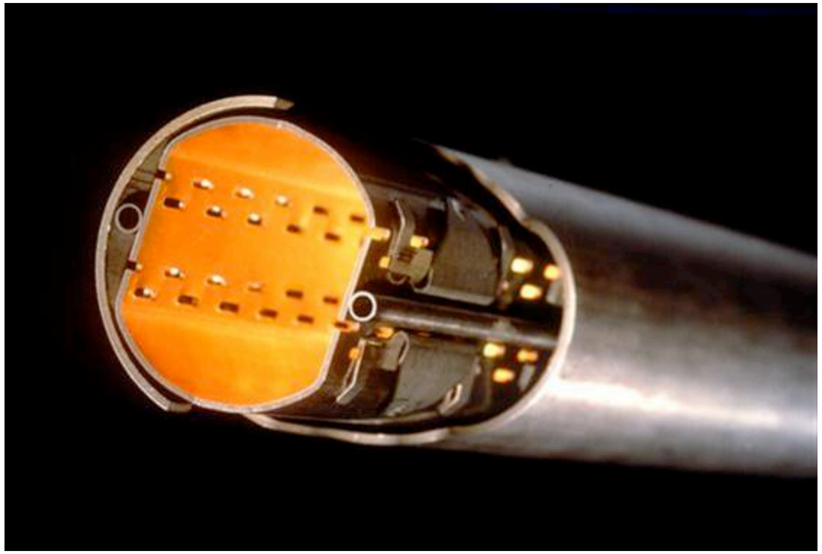
particle at distance** **1 σ from centre
↔ 68.3 % of all beam particles

LHC: $\beta = 180\, m$

$$\varepsilon = 5 * 10^{-10}\, m\, rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}\, m * 180\, m} = 0.3\, mm$$

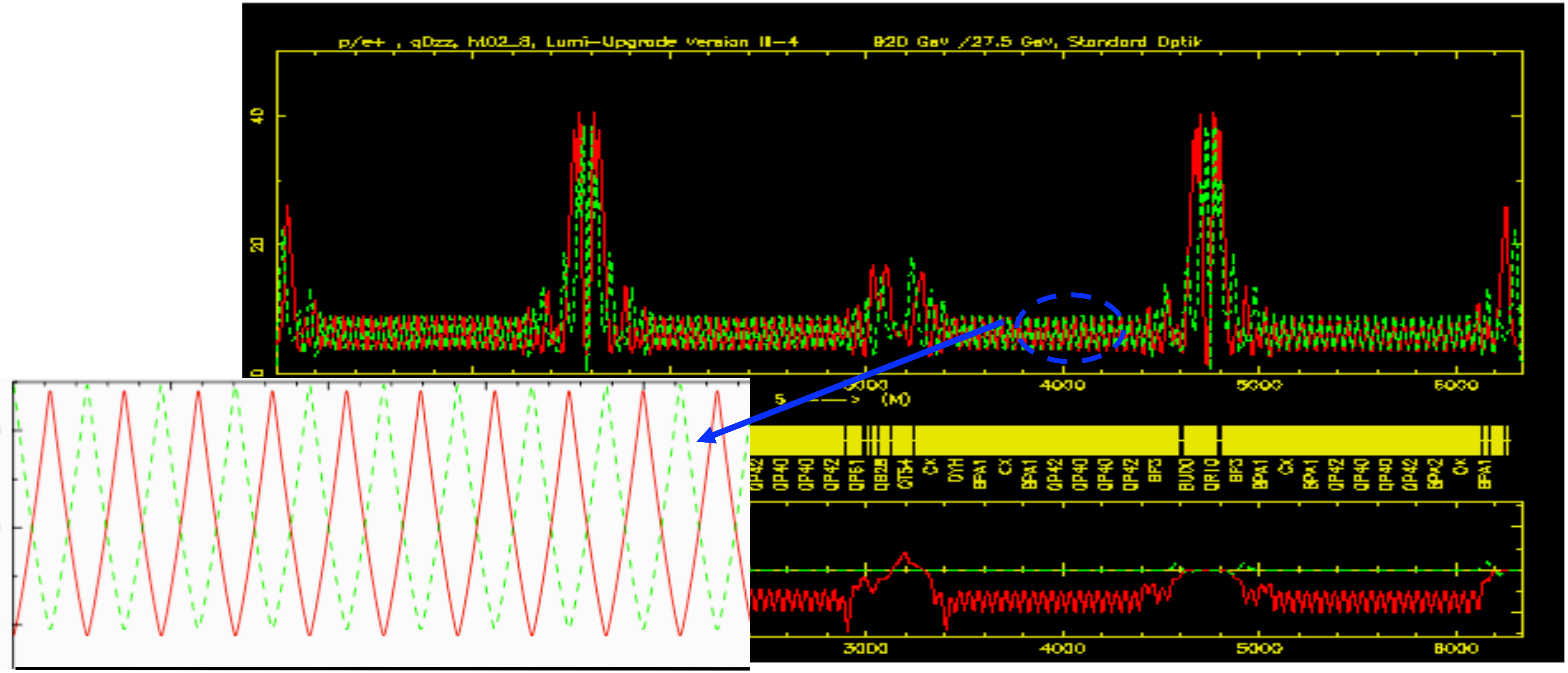




aperture requirements: $r_0 = 12 * \sigma$

The FoDo-Lattice

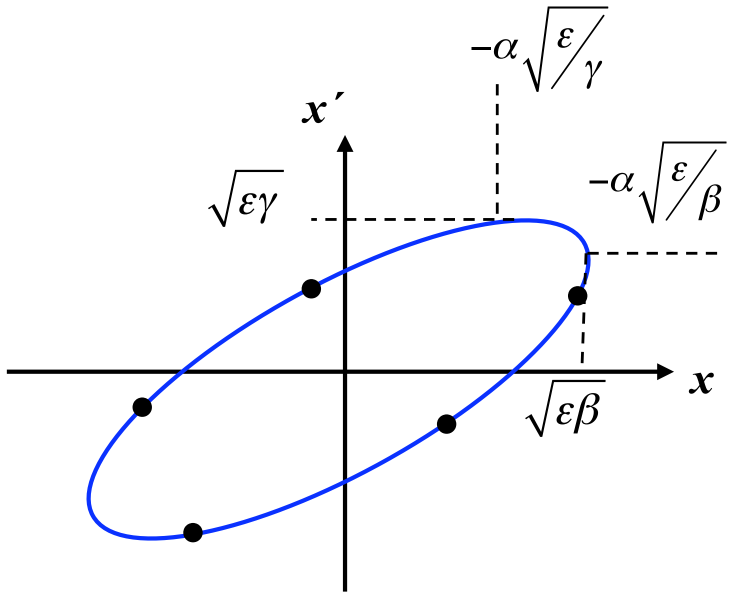
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

7.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

phase space = diagram of the two canonical variables
position & momentum
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$$q = position = x$$
$$p = momentum = \gamma m v = mc\gamma\beta_x$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\, dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int p\, dq = mc \int \gamma \beta_x dx$$

$$\int p\, dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

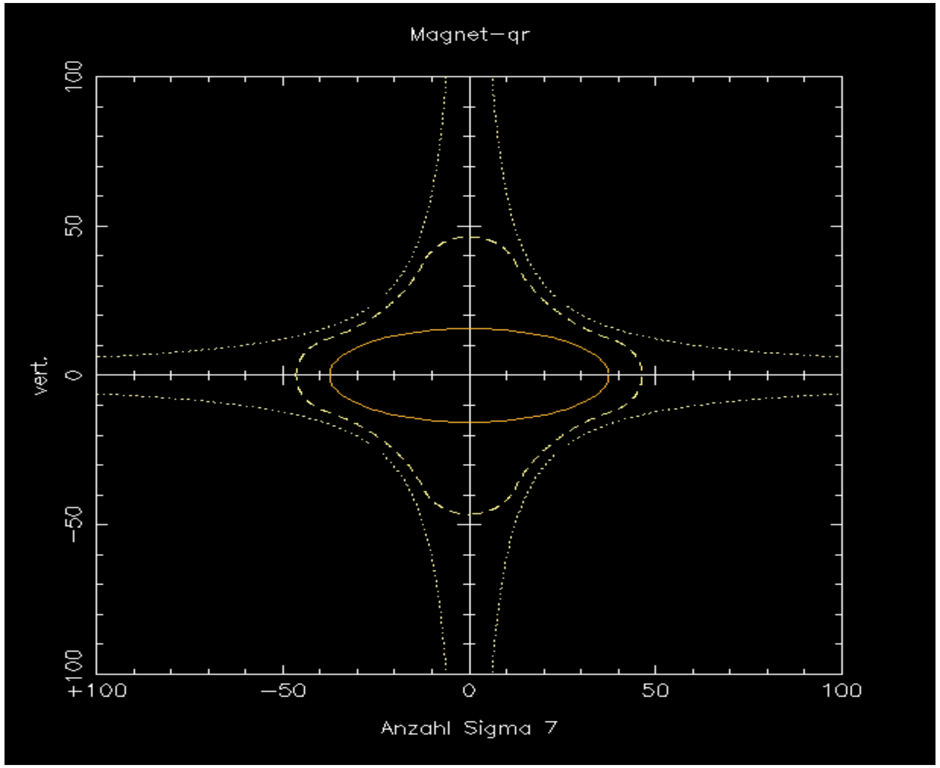
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$

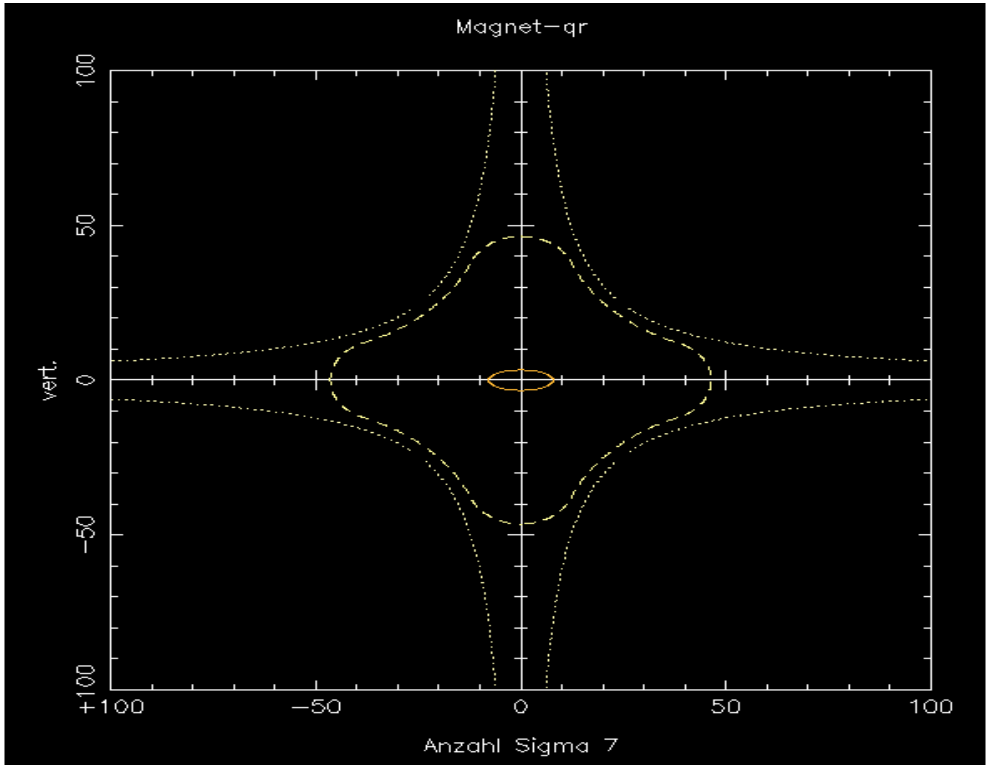
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\ (40GeV) = 1.2 * 10^{-7}$

$\varepsilon\ (920GeV) = 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

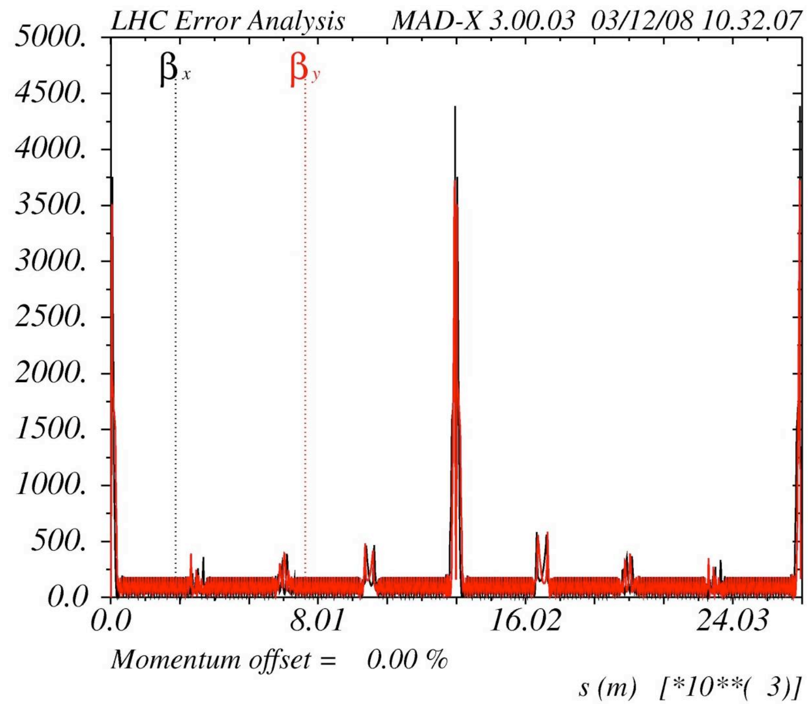
Nota bene:

1.) *A proton machine … or an electron linac … needs the highest aperture at injection energy !!!*
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ *in both planes.*

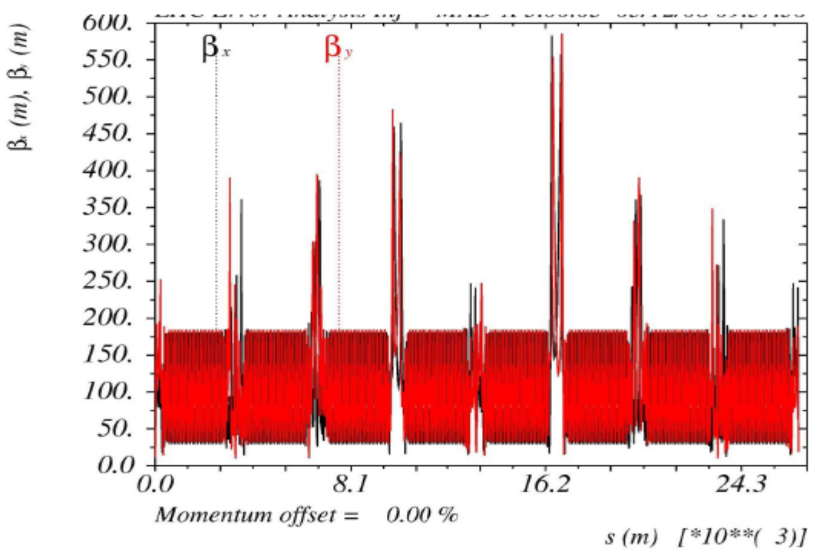
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) *At lowest energy the machine will have the major aperture problems,*
→ *here we have to minimise* $\hat{\beta}$

3.) *we need different beam optics adopted to the energy:*
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

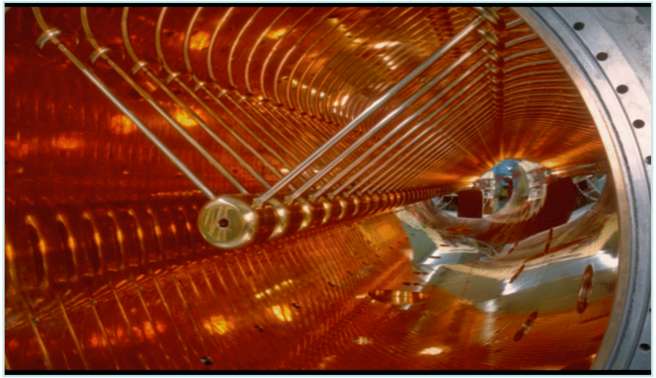


LHC injection optics at 450 GeV

8.) Problem: panta rhei ... RF Acceleration

(Heraklit: 540-480 v. Chr.)

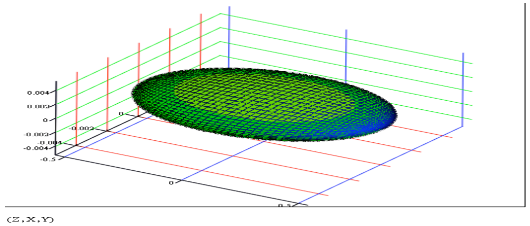
The „ $\Delta p / p \neq 0$" Problem



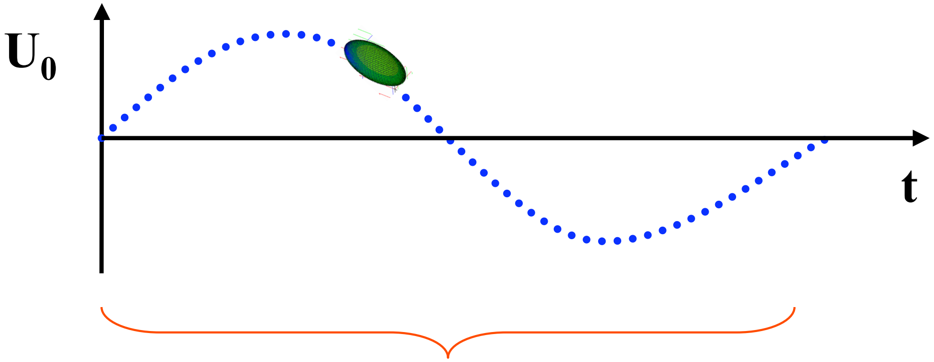
drift tube structure (GSI Unilac)

Energy Gain per „Gap":

$$W = q\, U_0 \sin \omega_{RF} t$$



Bunch length of Electrons ≈ 1cm



$$\lambda = 60\ cm$$

$$\left.\begin{aligned} f_{rf} &= 500\, MHz \\ c &= \lambda\, f \end{aligned}\right\} \quad \lambda = 60\ cm$$

$$\sin(90^o) = 1$$

$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

9.) Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

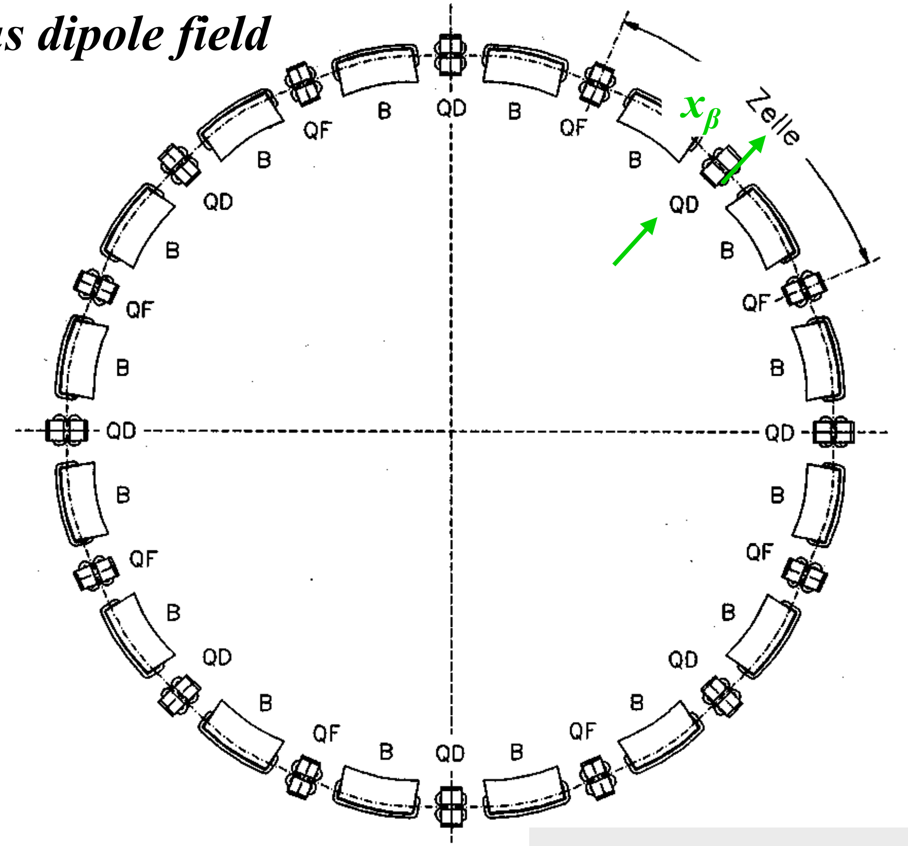
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



it for $\Delta p/p > 0$

$$: D(s)\cdot\frac{\Delta p}{p}$$

$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

Amplitude of Orbit Oscillation contribution due to Dispersion ≈ beam size

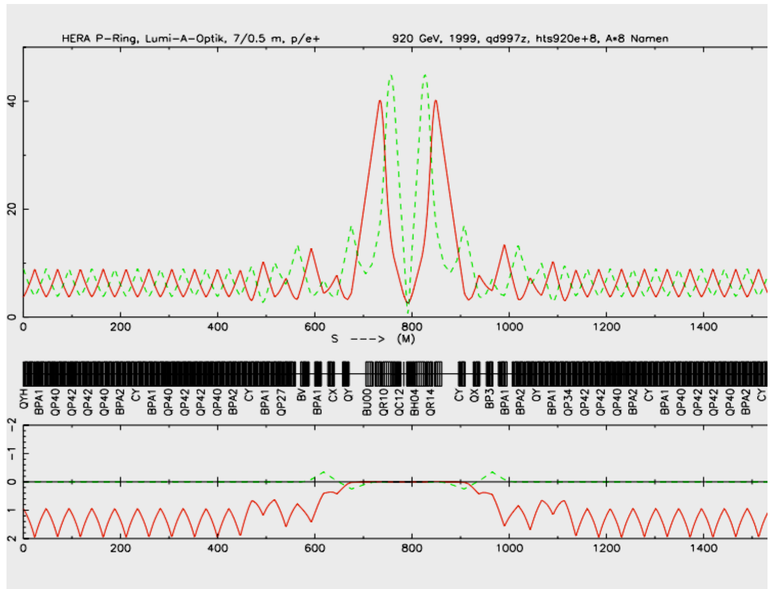
→Dispersion must vanish at the collision point

Example

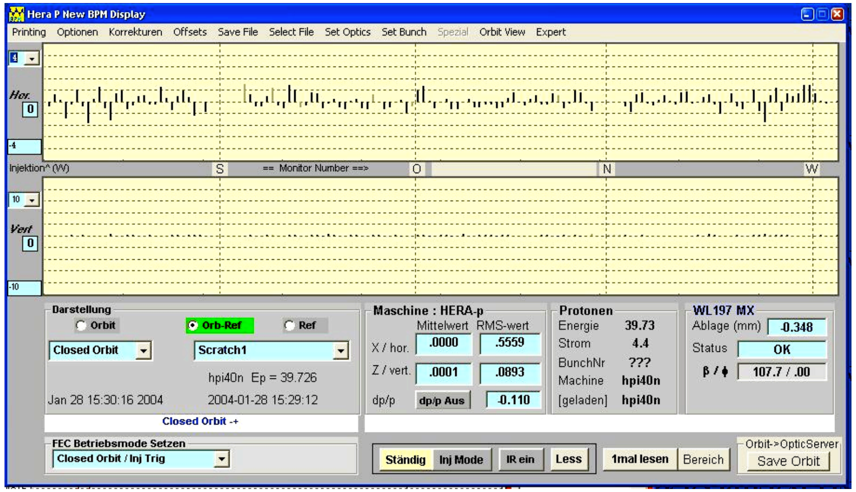
$$x_\beta = 1 \ldots 2\ mm$$

$$D(s) \approx 1 \ldots 2\ m$$

$$\Delta p / p \approx 1\cdot 10^{-3}$$



Dispersion is visible



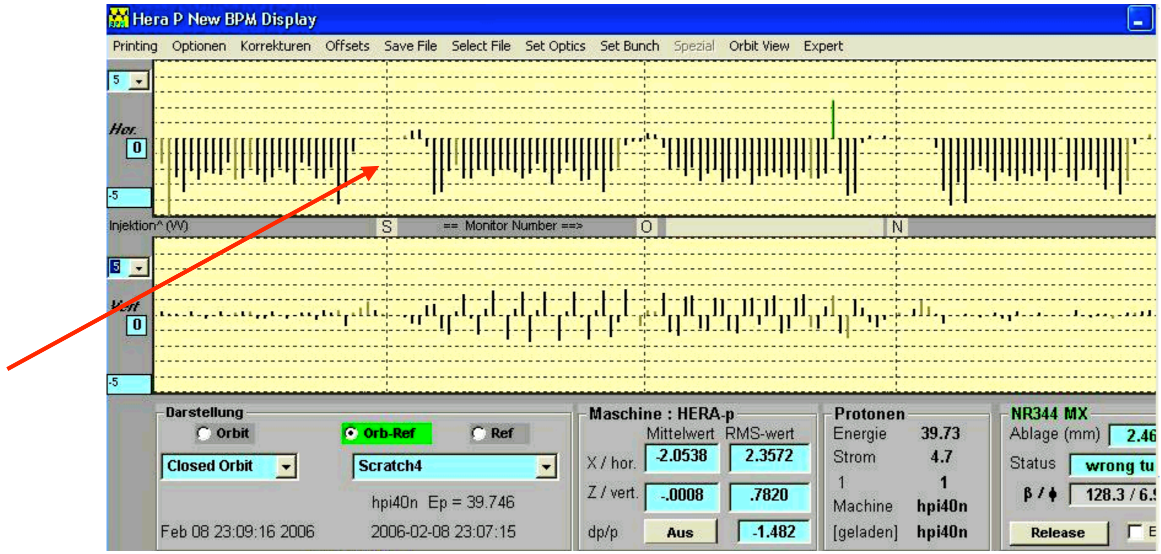
HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$



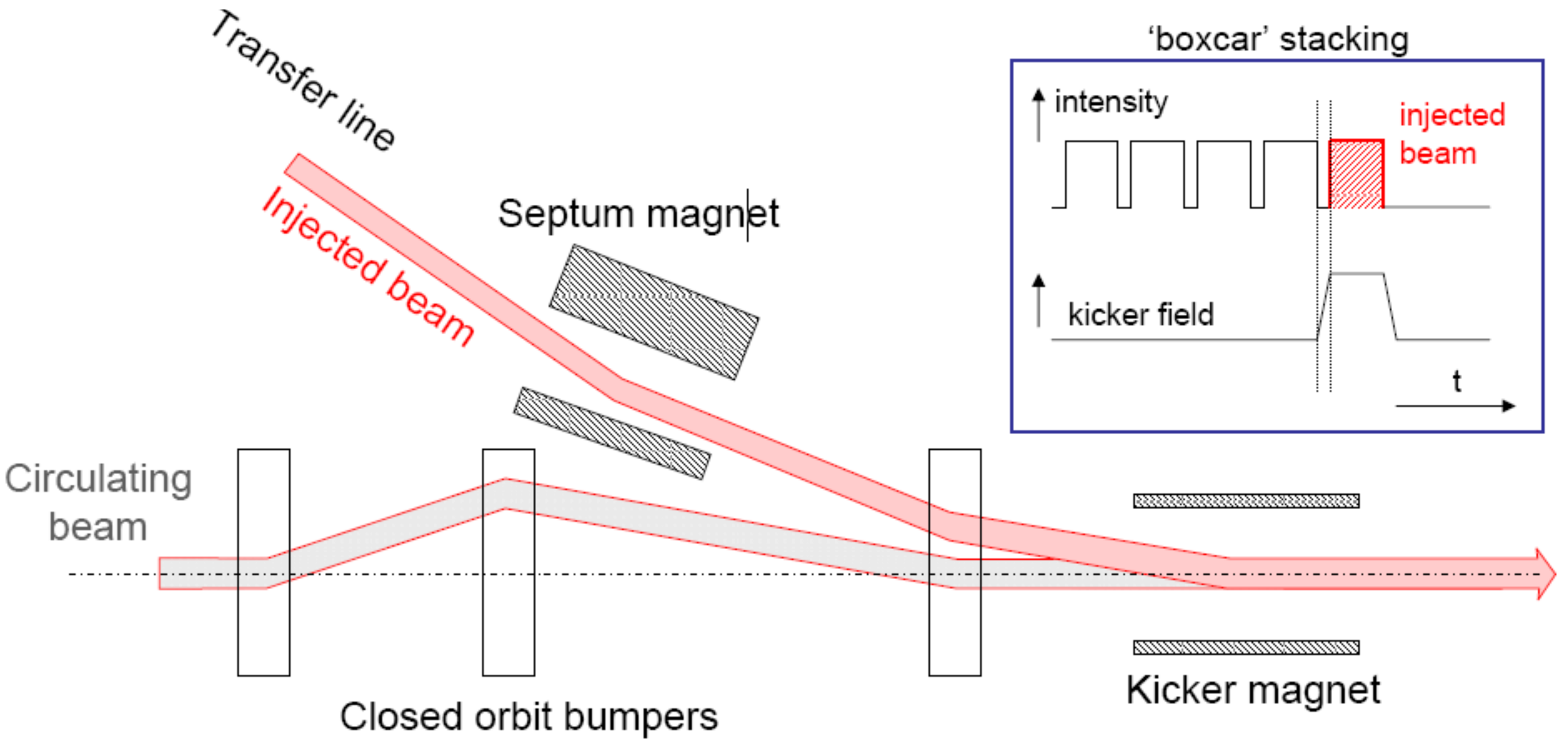
Attention: at the Interaction Points we require D=D′= 0

10.) Injection Schemes & Phase Space:

Standard Proton Beam ... single turn Injection
Electron Beam .... "off axis" Injection
Ion Beam .... "multi turn" injection

Single Turn Injection

Example: LHC, HERA-P



Transferlines & Injection: *Orbit Errors*

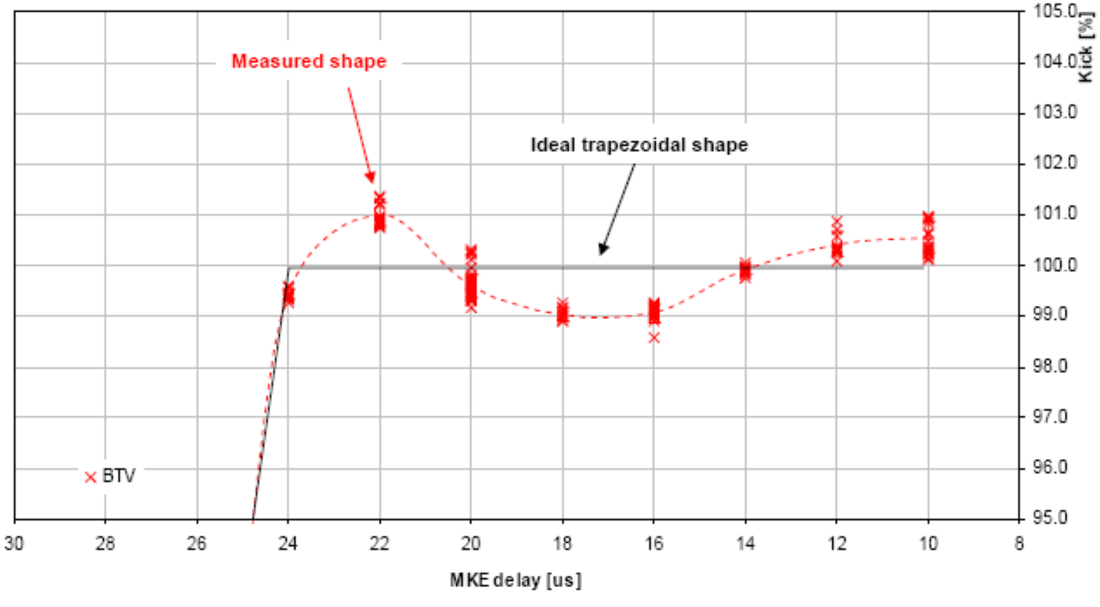
** quadrupole strengths --> "beta beat" Δβ / β*
** alignment of magnets --> orbit distortion in transferline & storage ring*
** septum & kicker pulses --> orbit distortion & emittance dilution in storage ring*

Example: Error in position Δa:

$$\varepsilon_{new} = \varepsilon_0 * (1 + \frac{\Delta a^2}{2})$$

Δa =0.5 σ

$$\rightarrow \quad \varepsilon_{new} = 1.125 * \varepsilon_0$$



Kicker "plateau" at the end of the PS - SPS transferline measured via injection - oscillations

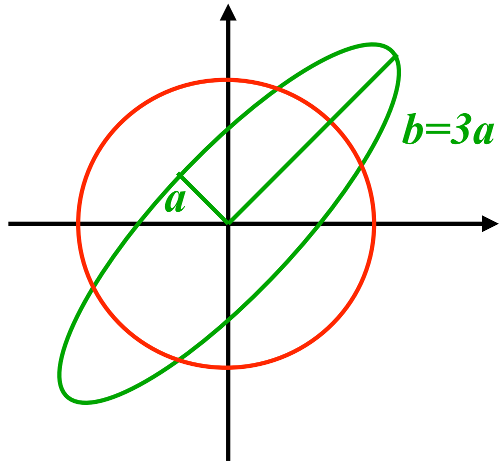
Transferlines & Injection: Optics Errors

Normalised Phasespace:

$$x \rightarrow \frac{x}{\sqrt{\beta}}$$

$$x' \rightarrow \frac{\alpha}{\sqrt{\beta}} x + \sqrt{\beta} x'$$

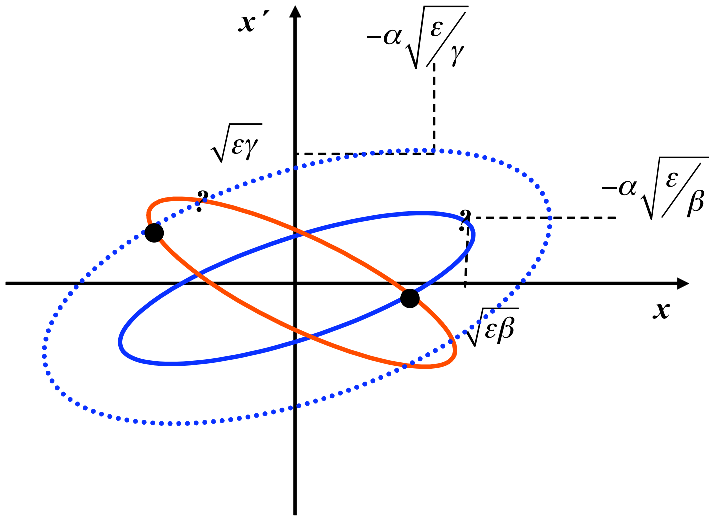
Ellipse → circle



Mismatch of Beam Optics

$$\lambda = \sqrt{b/a}$$

$$\varepsilon_{new} = \frac{1}{2} \varepsilon_0 (\lambda^2 + \frac{1}{\lambda^2})$$



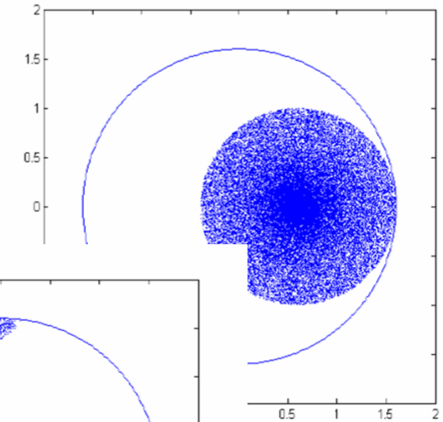
Example: b = 3a

$$\lambda = \sqrt{3}$$

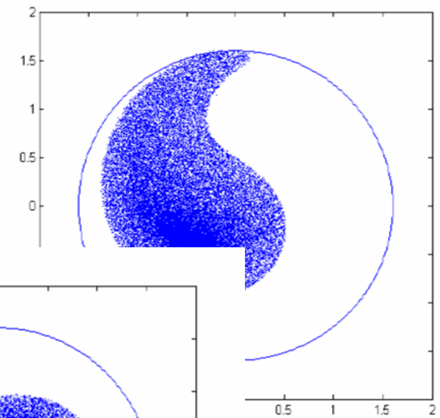
$$\rightarrow \quad \varepsilon_{new} = 1.67 * \varepsilon_0$$

see also: Edwards / Syphers,
K. Brown,
Chao, Tigner

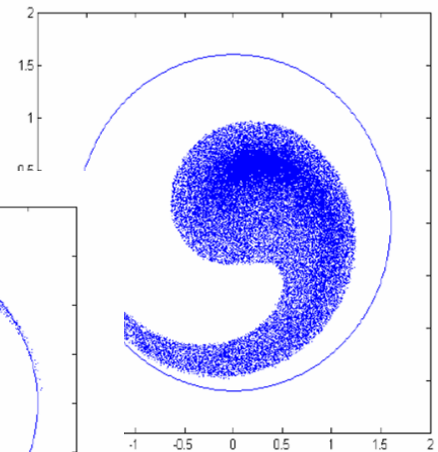
Filamentation



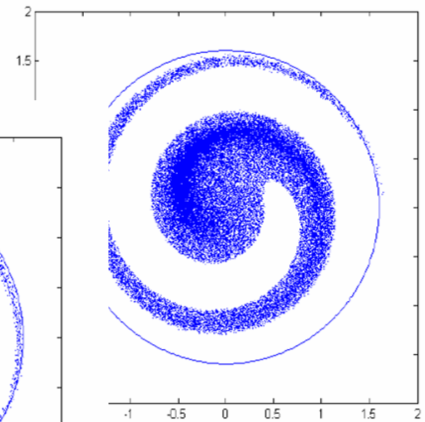
Injection errors ***(position or angle) dilute the beam emittance***

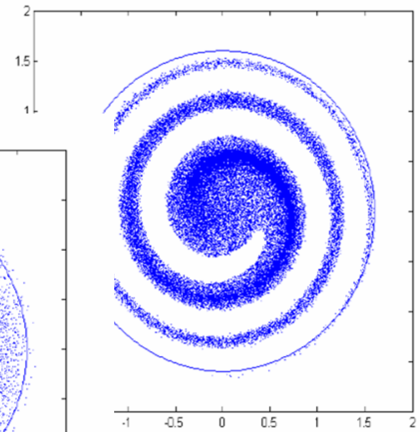


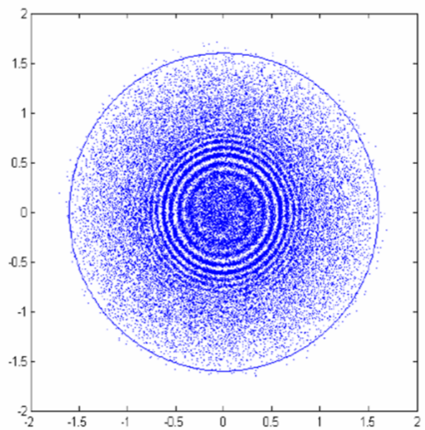
Non-linear effects ***(e.g. magnetic field multipoles) introduce distort the harmonic oscillation and lead to*** ***amplitude dependent effects*** ***into particle motion.***



Over many turns, ***a phase-space oscillation is transformed into an emittance increase.***

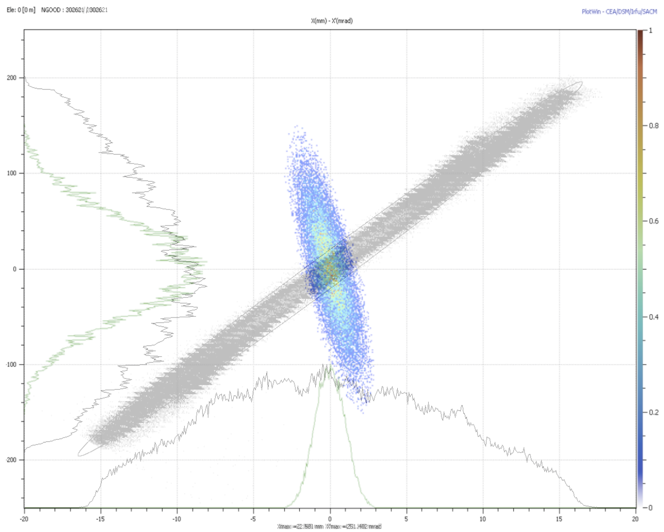




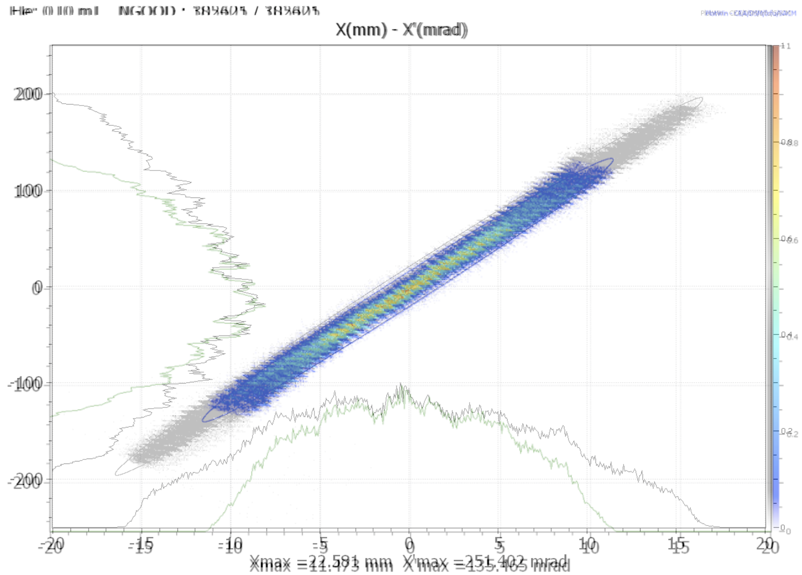


Example: Linac 4 source

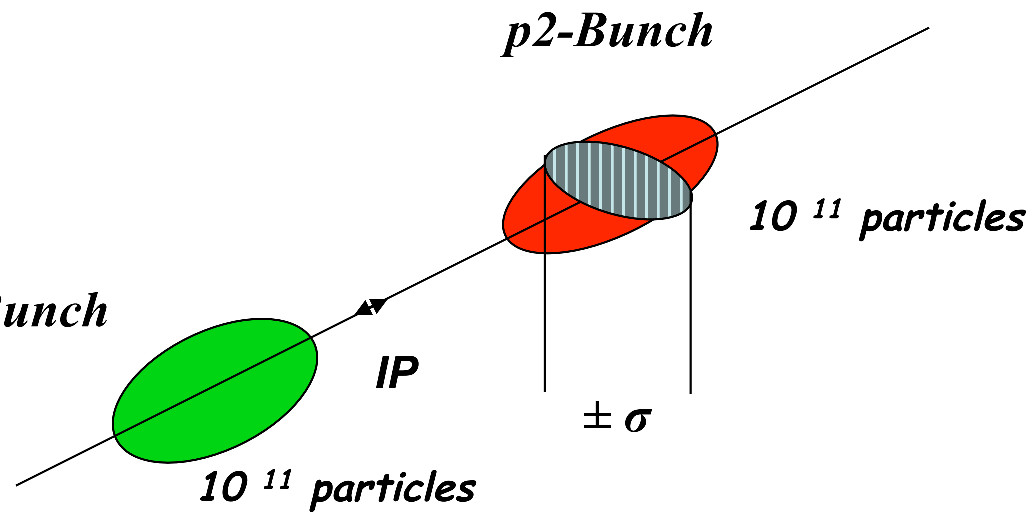
horizontal phase space
of beam from particle source
and required phase space configuration
at RFQ entrance



horizontal phase space
of beam from particle source
and phase space configuration
after optics match in the LEBT
at RFQ entrance



11.) Luminosity



Example: Luminosity run at LHC

$$\beta^*_{x,y} = 0.55\ m \qquad f_0 = 11.245\ kHz$$

$$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m \qquad n_b = 2808$$

$$\sigma_{x,y} = 17\ \mu m$$

$$I_p = 584\ mA$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$$L = 1.0 * 10^{34}\ \frac{1}{cm^2 s}$$

Mini-β Insertions: Betafunctions

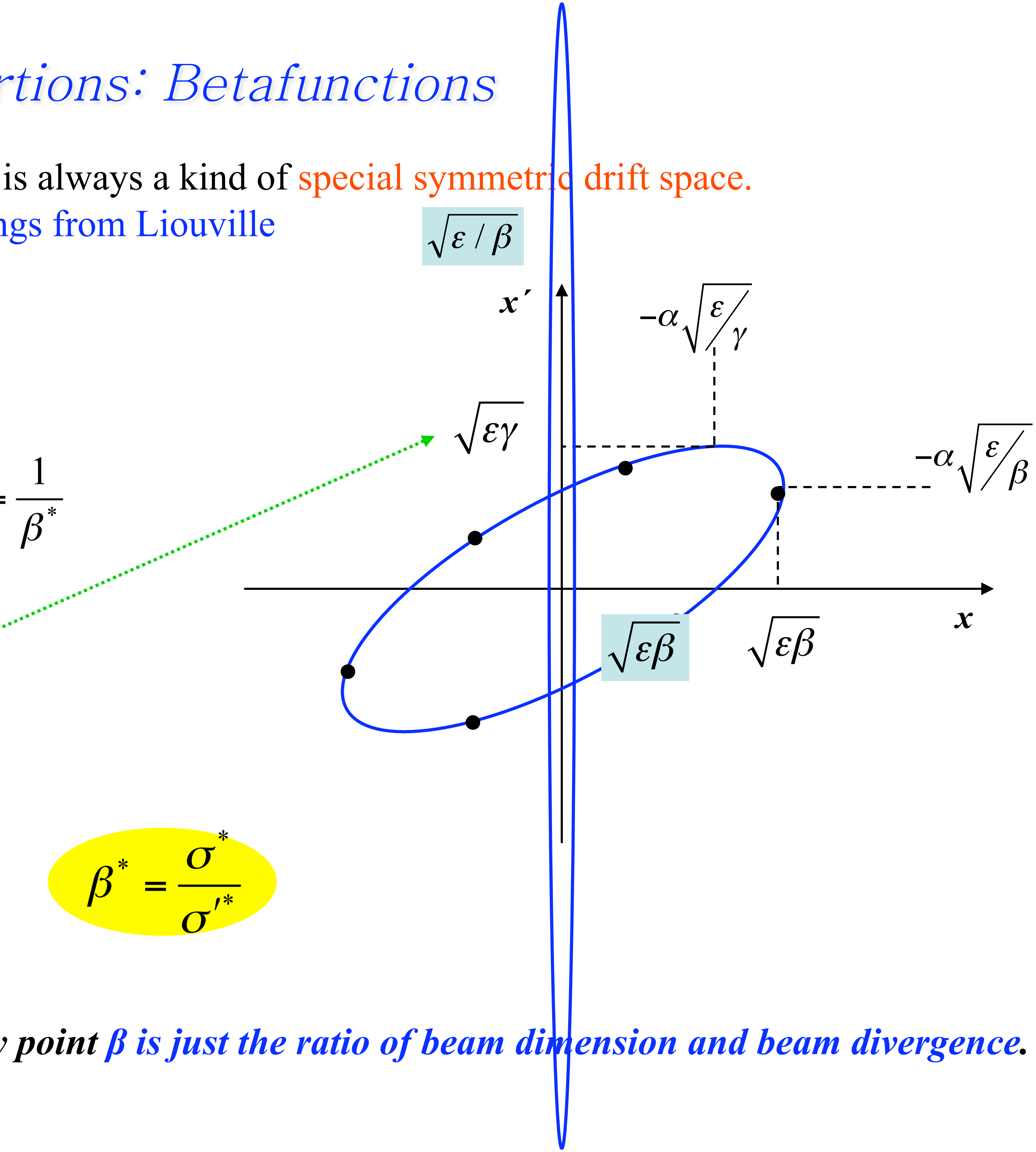
A mini-β insertion is always a kind of special symmetric drift space.

→greetings from Liouville

$$\alpha^* = 0$$

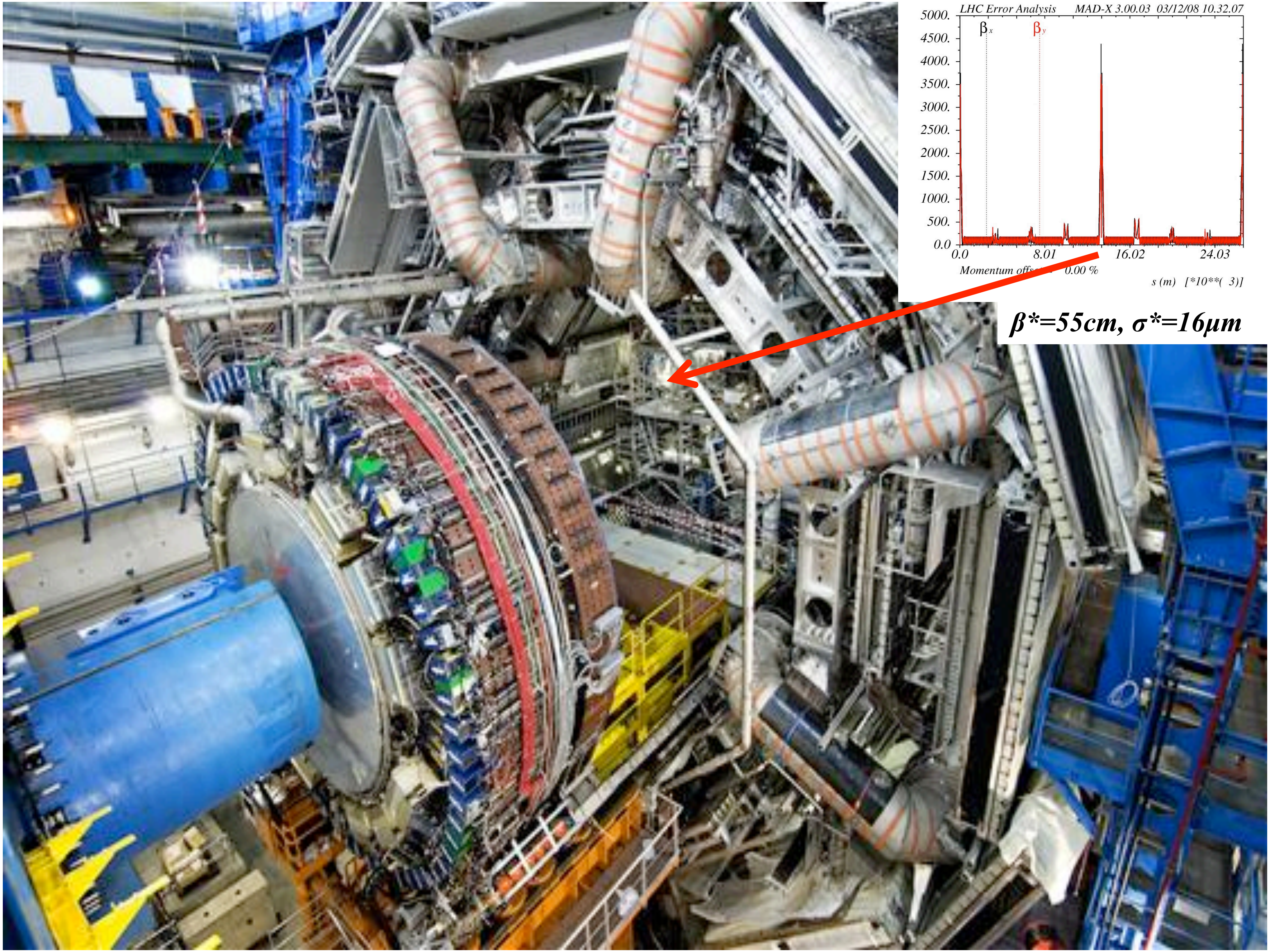
$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$



$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.


LHC Error Analysis
MAD-X 3.00.03 03/12/08 10.32.07
βx
βy
5000.
4500.
4000.
3500.
3000.
2500.
2000.
1500.
1000.
500.
0.0
0.0
8.01
16.02
24.03
s (m) [*10**(3)]
β*=55cm, σ*=16μm

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APPENDIX: Multiturn Injection & "Phase Space Stacking"

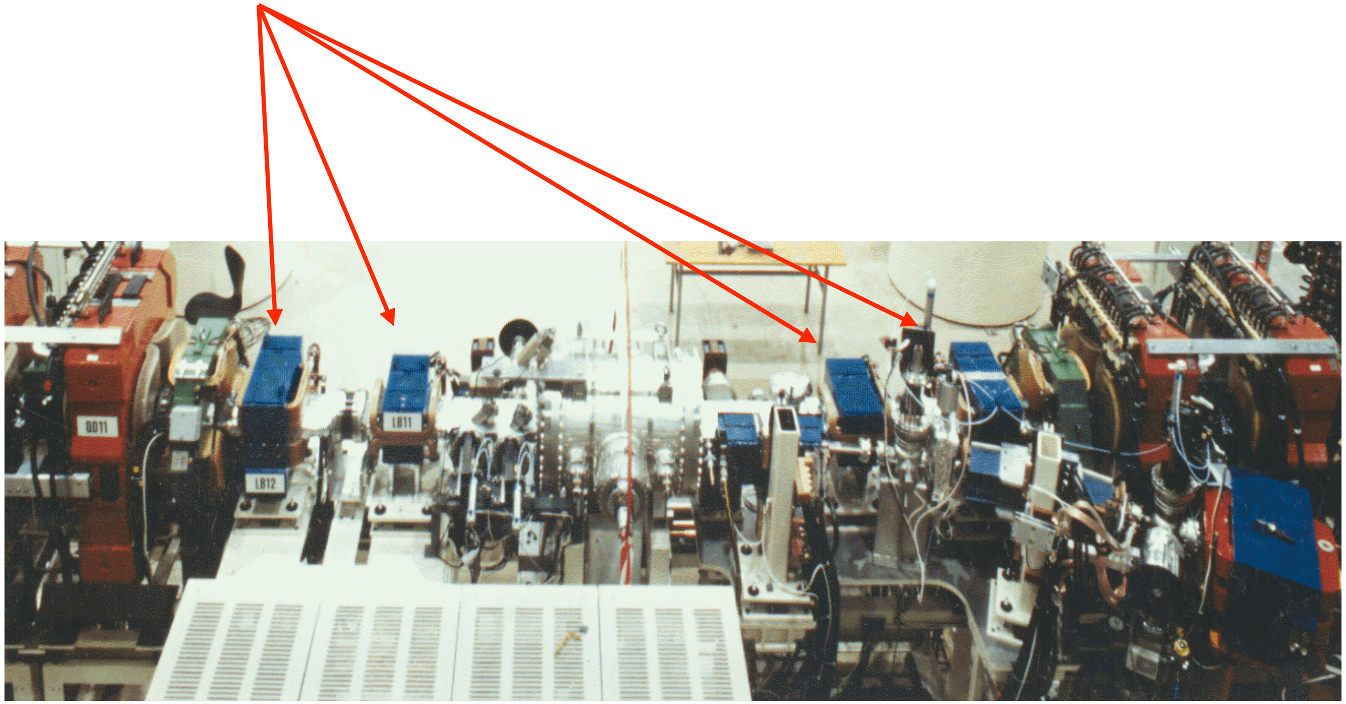
For hadrons the beam density at injection can either limited by space charge effects or by the injector capacity

If we cannot increase charge density, we can sometimes fill the horizontal phase space to increase injected intensity.

On the condition that the acceptance of receiving machine is larger than delivered beam emittance

Elements used

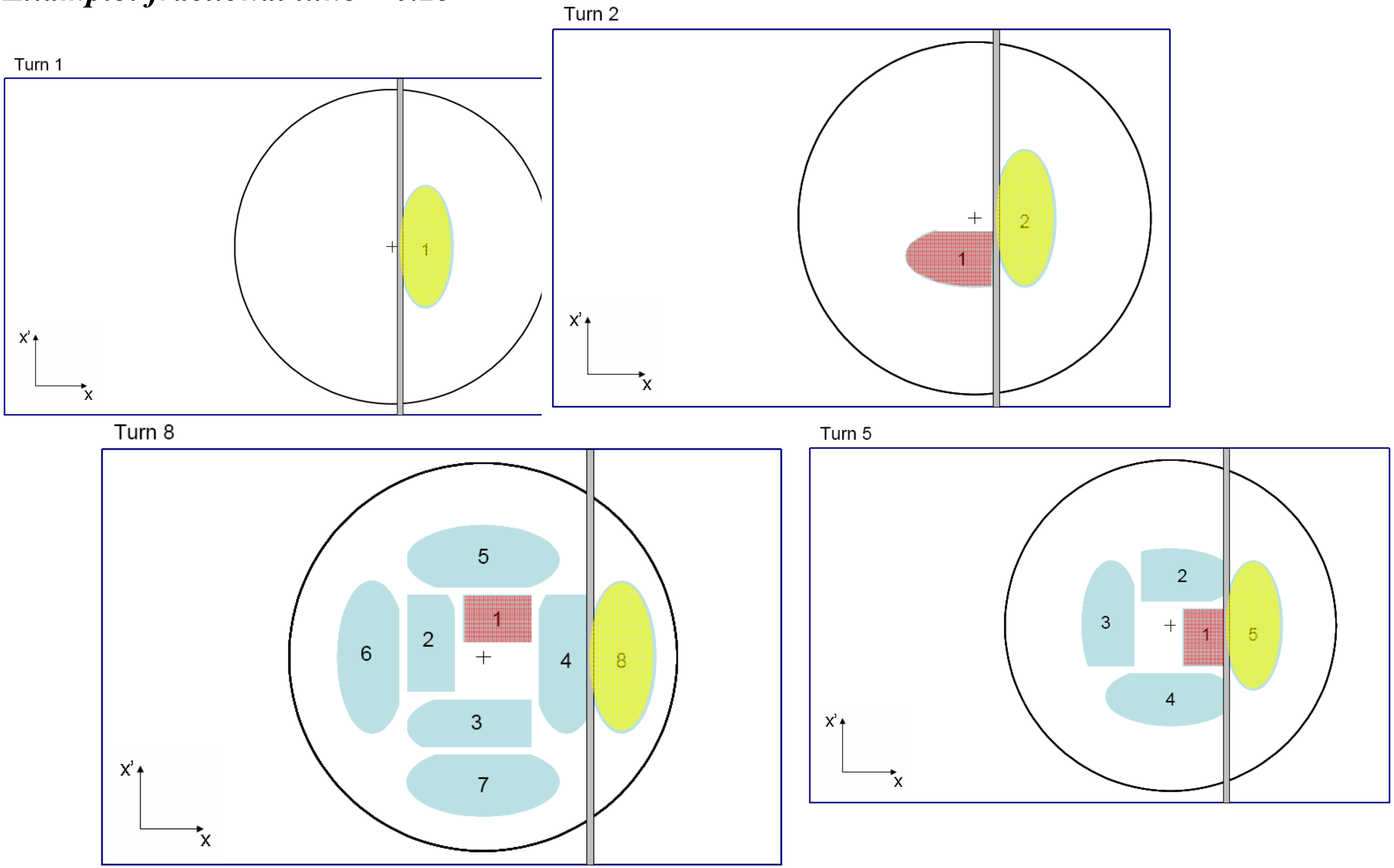
- ***Septum***
- ***3 or 4 fast kicker magnets to create a closed local beam bump***



Multiturn Injection, "Phase Space Stacking"

... and how it looks in phase space

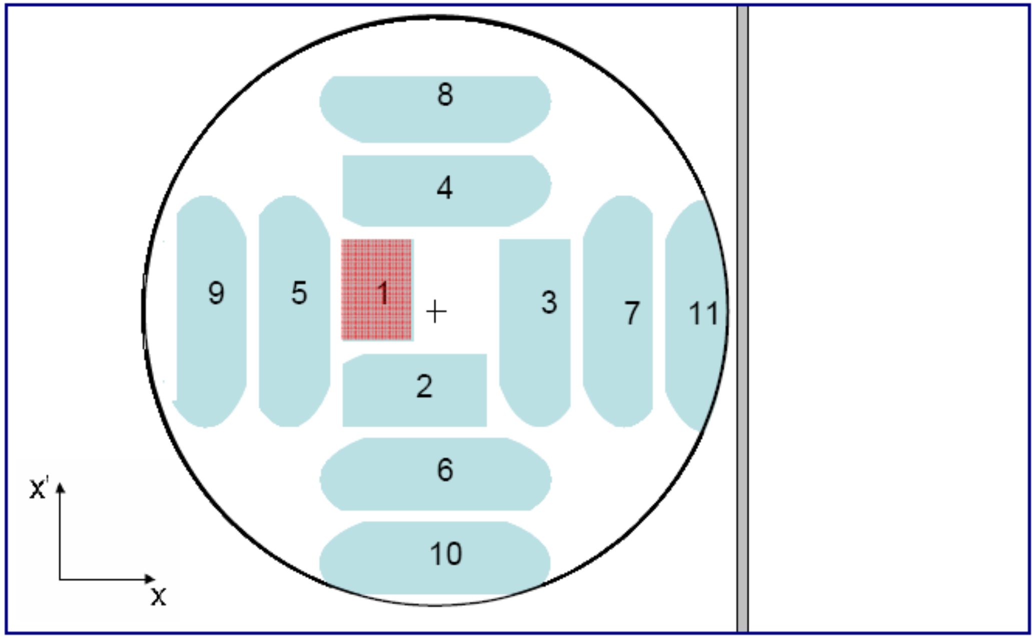
Example: fractional tune ≈ 0.25



Multiturn Injection, "Phase Space Stacking"

... and how it looks in phase space

Turn 11



Nota bene: accurate tune control ... Q_x
accurate bump control ... in steps !
thin septum (electrostatic ...)

filamentation fills smeers out the phase space
often combined with (electron-) cooling techniques

APPENDIX: The equation of motion:

Linear approximation:

* *ideal particle* → *design orbit*

* *any other particle* → *coordinates x, y small quantities*
$x,y \ll \rho$

→ *magnetic guide field: only linear terms in x & y of B have to be taken into account*

Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{eg''}{dx^3} + \dots$$

normalise to momentum
$p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g * x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \dots$$

Equation of Motion:

Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

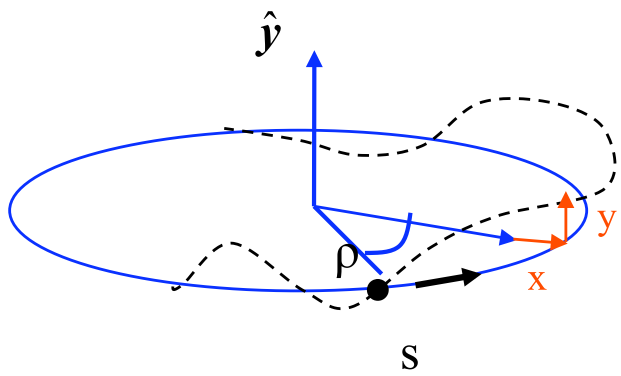
$$F = mv^2 / \rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F} = \boldsymbol{m}\frac{\boldsymbol{d}^2}{\boldsymbol{dt}^2}(\boldsymbol{x}+\rho) - \frac{\boldsymbol{mv}^2}{\boldsymbol{x}+\rho} = \boldsymbol{e}\,\boldsymbol{B}_y\,\boldsymbol{v}$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

1 2



1 $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ *… as ρ = const*

2 ***remember:*** *x ≈ mm , ρ ≈ m …* → ***develop for small x***

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}(1-\frac{x}{\rho})$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \quad \ldots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x} \qquad m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \Big| \; : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: t → s

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

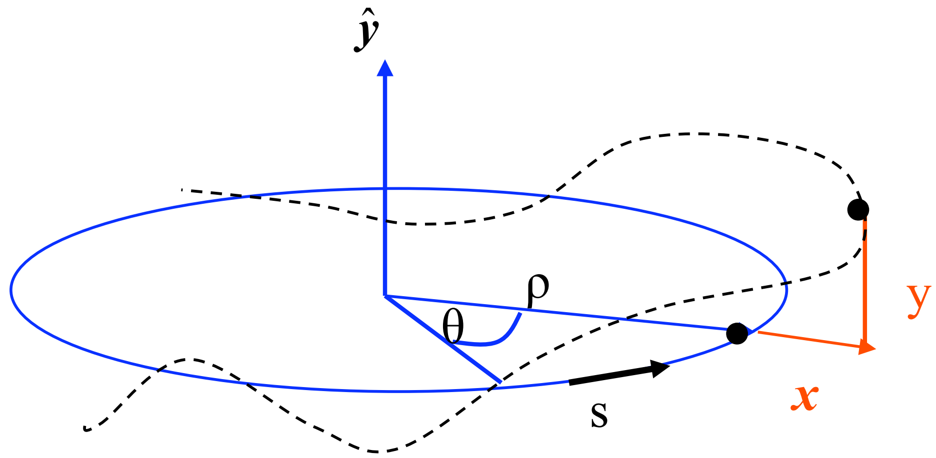
$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}\,v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad \Big| \; : v^2$$

$$x'' - \frac{1}{\rho}(1 - \frac{x}{\rho}) = \frac{e\ B_0}{mv} + \frac{e\ x\,g}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\,g}{p/e}$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\,x$$



$$x'' + x\,(\frac{1}{\rho^2} - k) = 0$$

***** *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ ***no dipoles … in general …***

$k \quad \leftrightarrow \quad -k$ ***quadrupole field changes sign***

$$y'' + k\ y = 0$$