Introduction to Longitudinal Beam Dynamics

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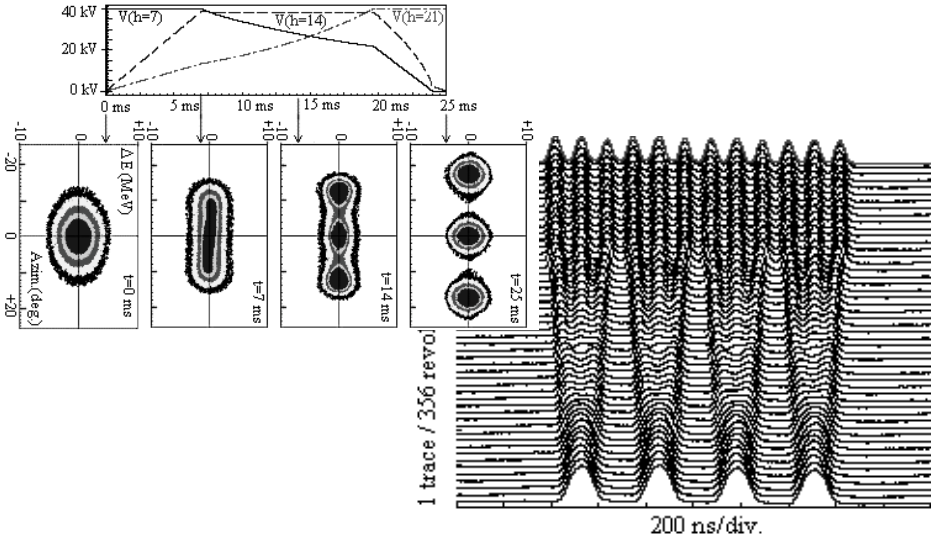
CERN-LHC



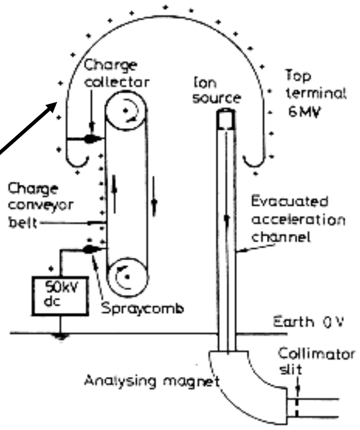
Bibliography : New Books

M. Conte, W.W. Mac Kay **An Introduction to the Physics of particle Accelerators**
(World Scientific, 1991)
P. J. Bryant and K. Johnsen **The Principles of Circular Accelerators and Storage Rings**
(Cambridge University Press, 1993)
D. A. Edwards, M. J. Syphers **An Introduction to the Physics of High Energy Accelerators**
(J. Wiley & sons, Inc, 1993)
H. Wiedemann **Particle Accelerator Physics**
(Springer-Verlag, Berlin, 1993)
M. Reiser **Theory and Design of Charged Particles Beams**
(J. Wiley & sons, 1994)
A. Chao, M. Tigner **Handbook of Accelerator Physics and Engineering**
(World Scientific 1998)
K. Wille **The Physics of Particle Accelerators: An Introduction**
(Oxford University Press, 2000)
E.J.N. Wilson **An introduction to Particle Accelerators**
(Oxford University Press, 2001)

➡ And Joel LeDuff, CAS Proc, CERN 94-01

1.) Electrostatic Machines:
(Tandem –) van de Graaff Accelerator (1930 ...)

creating high voltages by mechanical transport of charges



* Terminal Potential: U ≈ 12 ...28 MV
using high pressure gas to suppress discharge (SF_6)

Problems: * Particle energy limited by high voltage discharges
* high voltage can only be applied once per particle ...
... or twice ?

Energy Gain

... we have to start again from the basics

Lorentz force $\vec{F} = q*(\vec{E} + \cancel{\vec{v} \times \vec{B}})$

in long. direction the B-field creates no force

v || B

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} = e\vec{E}}$$

acc. force is given by the electr. Field

In relativistic dynamics, energy and momentum satisfy the relation:

$$E^2 = E_0^2 + p^2c^2 \qquad \left(E = E_0 + W\right)$$

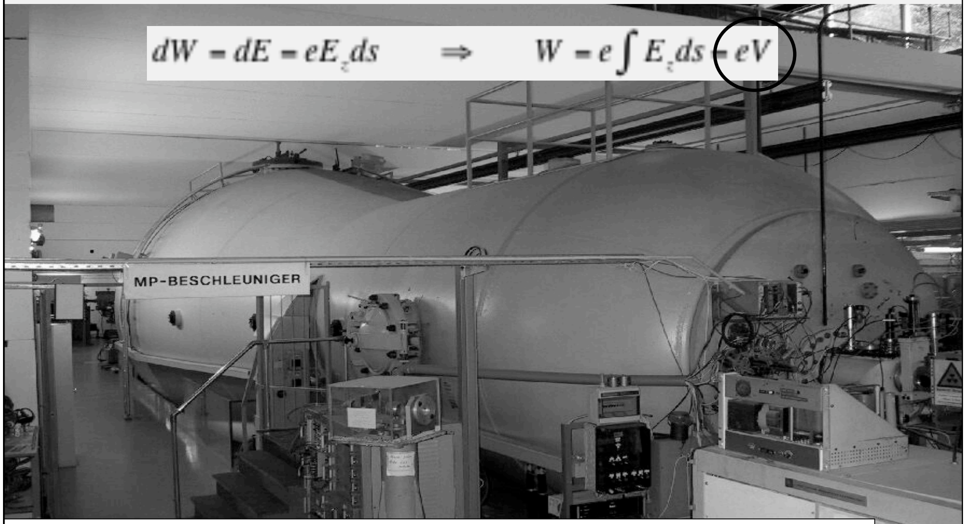
Hence: $dE = \int Fds = vdp$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z ds \quad \Rightarrow \quad W = e\int E_z ds = eV$$

The „Tandem principle“: Apply the accelerating voltage twice ...

... by working with negative ions (e.g. H^-) and stripping the electrons in the centre of the structure



$$dW = dE = eE_z ds \quad \Rightarrow \quad W = e\int E_z ds = \boxed{eV}$$

Electro Static Accelerator:12 MV-Tandem van de Graaff Accelerator at MPI Heidelberg

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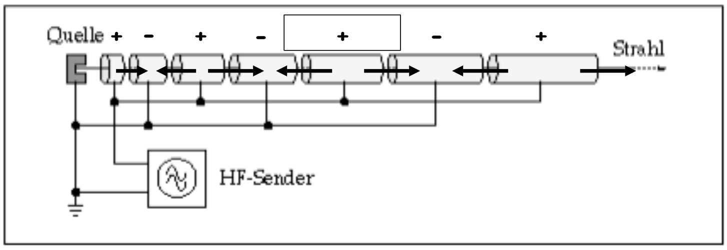
3.) The first RF-Accelerator: „Linac“

1928, Wideroe: how can the acceleration voltage be applied several times to the particle beam

schematic Layout:



Energy gained after n acceleration gaps

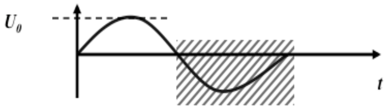
$$E_n = n * q * U_0 * \sin\psi_s$$

n number of gaps between the drift tubes
q charge of the particle
$\mathbf{U_0}$ Peak voltage of the RF System
Ψ_S ***synchronous phase of the particle***

*** the problem of synchronisation ... between the particles and the rf voltage**
*** „voltage has to be flipped“ to get the right sign in the second gap**
→ shield the particle in drift tubes during the negative half wave of the RF voltage

Wideroe-Structure: the drift tubes

shielding of the particles during the negative half wave of the RF



Time span of the negative half wave: $\tau_{RF}/2$

Length of the Drift Tube:

Kinetic Energy of the Particles

$$\left.\begin{array}{l} l_i = v_i * \dfrac{\tau_{rf}}{2} \\[2ex] E_i = \dfrac{1}{2} m v^2 \end{array}\right\} \begin{array}{l} \rightarrow v_i = \sqrt{2E_i/m} \\[2ex] l_i = \dfrac{1}{v_{rf}} * \sqrt{\dfrac{i * q * U_{0*\sin\psi_s}}{2m}} \end{array}$$

valid for non relativistic particles ...

Alvarez-Structure: 1946, surround the whole structure by a rf vessel

Energy: ≈ 20 MeV per Nucleon β ≈ 0.04 … 0.6, Particles: Protons/Ions

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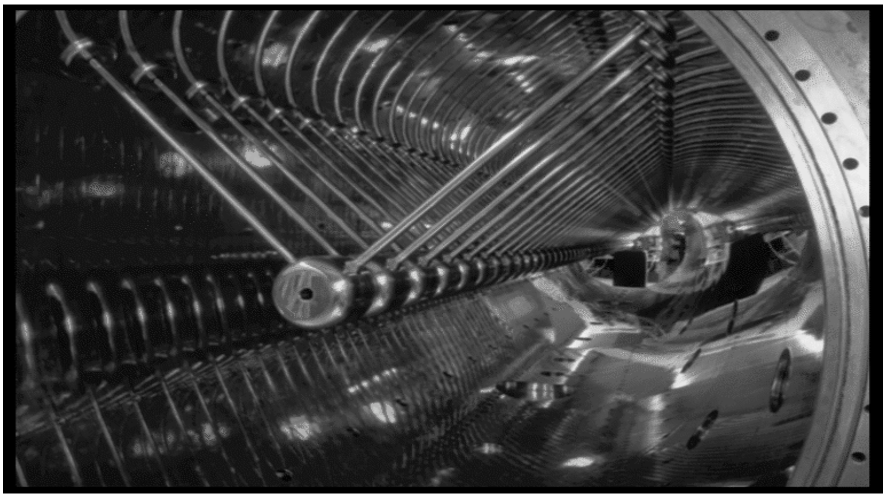
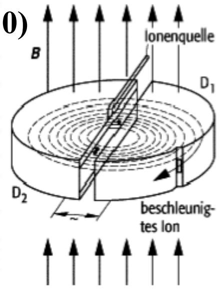
GSI: Unilac, typical Energie ≈ 20 MeV per Nukleon, β ≈ 0.04 … 0.6, Protons/Ions, ν = 110 MHz

Energy Gain per „Gap“: $W = q\, U_0 \sin \omega_{RF} t$



Application: until today THE standard proton / ion pre-accelerator
CERN Linac 4 is being built at the moment

4.) The Cyclotron: (Livingston / Lawrence ~1930)



Idea: B = const, RF = const
Synchronisation particle / RF via orbit

➔ Lorentzforce

$$\vec{F} = q*(\vec{v}\times\vec{B}) = q*v*B$$

circular orbit

$$q*v*B = \frac{m*v^2}{R} \quad \rightarrow \quad B*R = p/q$$

increasing radius for increasing momentum
➔ Spiral Trajectory

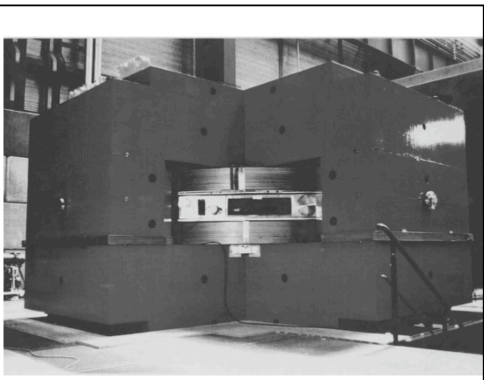
revolution frequency

$$\omega_z = \frac{v}{R} = \frac{q}{m}*B_z$$

the cyclotron (rf-) frequency is independent of the momentum

rf-frequency = h* revolution frequency, h = "harmonic number"

Cyclotron:


Cyclotron SPIRAL at GANIL

exact equation for revolution frequency:

$$\omega_z = \frac{v}{R} = \frac{q}{\boxed{\gamma}*m}*B_z$$

1.) if $v << c \Rightarrow \gamma \cong 1$

2.) γ increases with the energy
$\Rightarrow$ no exact synchronism

Synchrocyclotron

B = constant
$\gamma\ \omega_{RF}$ = constant
ω_{RF} decreases with time

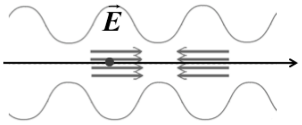
$$\omega_s(t) = \omega_{rf}(t) = \frac{q}{\gamma(t)*m_0}*B$$

keep the synchronisation condition by varying the rf frequency

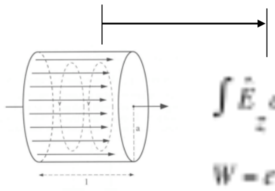
RF Cavities, Acceleration and Energy Gain

$$dW = dE = eE_z ds \quad \Rightarrow \quad W = e\int E_z ds = eV$$

RF acceleration: $V \neq const$



In this case the electric field is oscillating. So it is for the potential. The energy gain will depend on the RF phase experienced by the particle.

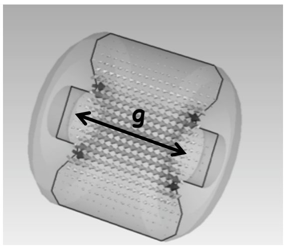


$$\int \hat{E}_z dz = \hat{V} \qquad E_z = \hat{E}_z \cos\omega_{RF} t = \hat{E}_z \cos\Phi(t)$$

$$W = e\hat{V}\cos\Phi$$

Z **Neglecting the transit time in the gap.**

Energy Gain in RF structures: Transit Time Factor



Oscillating field at frequency ω (amplitude is assumed to be constant all along the gap)

$$E_z = E_0 \cos\omega t = \frac{V}{g}\cos\omega t$$

Consider a particle passing through the middle of the gap at time t=0 :

$$z = vt$$

The total energy gain is: $\Delta W = \frac{eV}{g}\int_{-g/2}^{g/2} \cos\omega\frac{z}{v}dz$

$$\Delta W = eV\frac{\sin\theta/2}{\theta/2} = eVT$$

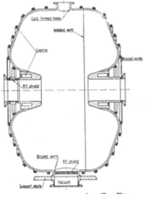
$T = \frac{\sin\theta/2}{\theta/2}$ transit time factor (0 < T < 1)

$\theta = \frac{\omega g}{v}$ *transit angle*

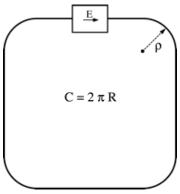
ideal case: $T = \frac{\sin\theta/2}{\theta/2} \rightarrow 1 \quad \Leftrightarrow \quad \theta/2 \rightarrow 0$

el. static accelertors $\omega \rightarrow 0$

minimise acc. gap $g \rightarrow 0$



The Synchrotron (Mac Millan, Veksler, 1945)



The synchrotron: Ring Accelerator of const. R where the increase in momentum (i.e. B-field) is automatically synchronised with the correct synchronous phase of the particle in the rf cavities

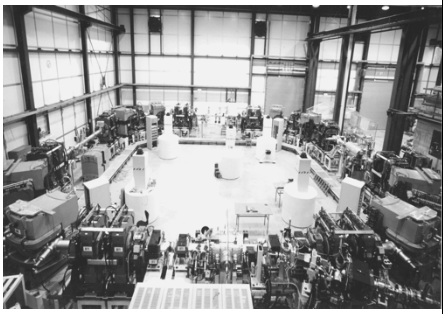
$e\hat{V}\sin\Phi$ → **Energy gain per turn**

$\Phi = \Phi_s = cte$ → **Synchronous particle**

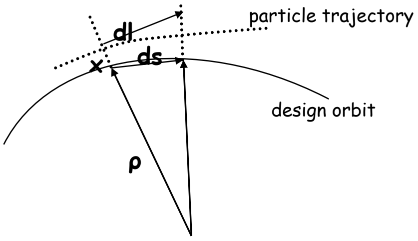
$\omega_{RF} = h\omega_r$ → **RF synchronism**

$\rho = cte \quad R = cte$ → **Constant orbit**

$B\rho = {}^{P}\!/_{e} \Rightarrow B$ → **Variable magnetic field**



Momentum Compaction Factor: α_p



particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember: $x_{\Delta E}(s) = D(s)\frac{\Delta p}{p}$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

* The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.

Definition: $\frac{\delta l_{\varepsilon}}{L} = \alpha_p \frac{\Delta p}{p}$

$$\rightarrow \alpha_p = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume: $\frac{1}{\rho} = const.$

$$\int_{dipoles} D(s)\, ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

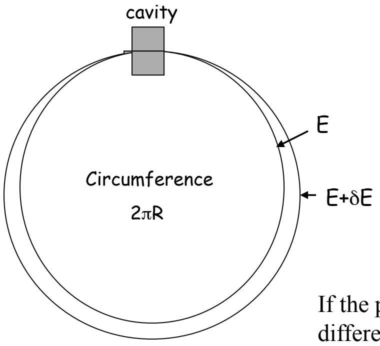
$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \frac{\delta T}{T} = \frac{\delta l_{\varepsilon}}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

This is the “momentum compaction” generated by the bending field.

If the particle is shifted in momentum it will have also a different velocity. As a result of both effects **the revolution frequency changes:**

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

p=particle momentum
R=synchrotron physical radius
f_r=revolution frequency

Dispersion Effects in a Synchrotron

$$\eta = \frac{p}{f_r}\frac{df_r}{dp} \qquad f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

$$p = mv = \beta\gamma\frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \left(1-\beta^2\right)^{-1}\frac{d\beta}{\beta}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \longrightarrow \eta = \frac{1}{\gamma^2} - \alpha$$

η=0 at the transition energy $\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$

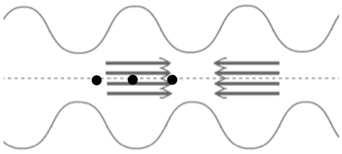
14.) The Acceleration for Δp/p≠0

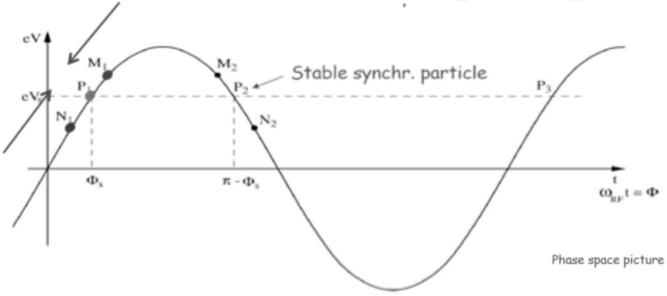
"Phase Focusing" below transition

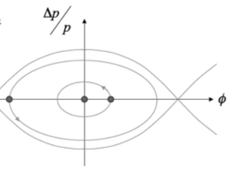
ideal particle ●

particle with Δp/p > 0 ● **faster**

particle with Δp/p < 0 ● **slower**



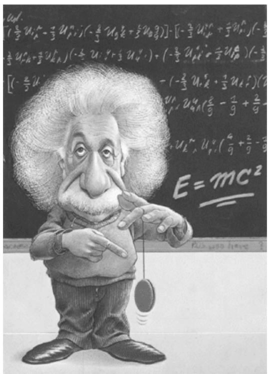


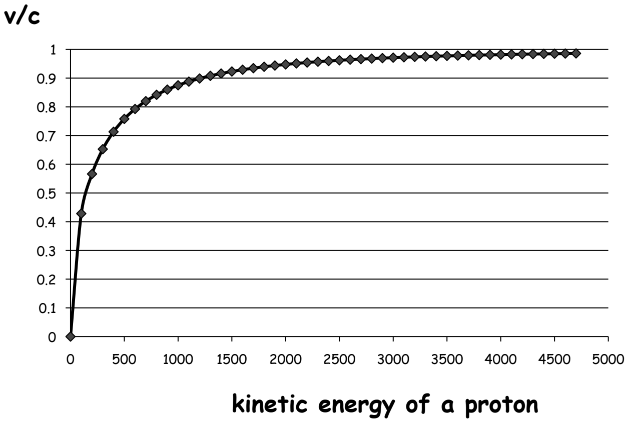


Focussing effect in the longitudinal direction keeping the particles close together ... forming a "bunch"

... so sorry, here we need help from Albert:

$$\gamma = \frac{E_{total}}{mc^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \longrightarrow \frac{v}{c} = \sqrt{1-\frac{mc^2}{E^2}}$$





... some when the particles do not get faster anymore

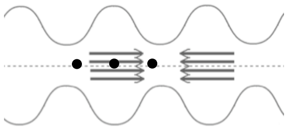
.... but heavier !

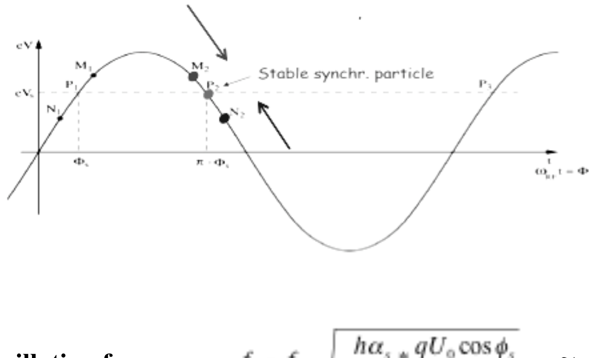
15.) The Acceleration for Δp/p≠0
“Phase Focusing” above transition

ideal particle ●

particle with Δp/p > 0 ● **heavier**

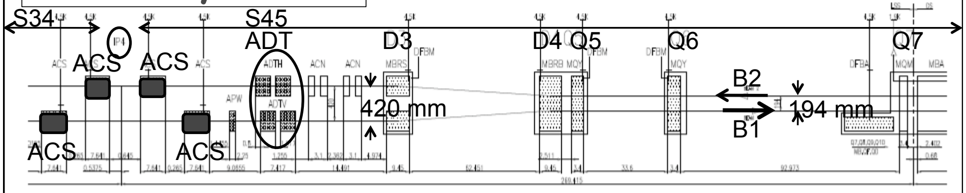
particle with Δp/p < 0 ● **lighter**

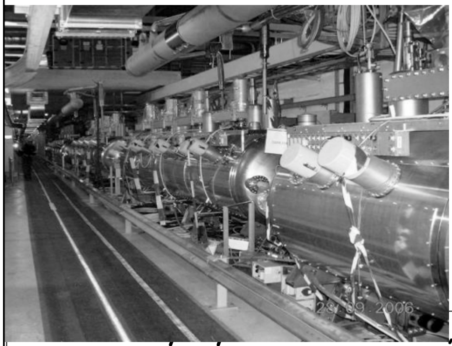




oscillation frequency: $f_s = f_{rev}\sqrt{-\frac{h\alpha_s}{2\pi} * \frac{qU_0\cos\phi_s}{E_s}}$ **≈ some Hz**

The RF system: IR4





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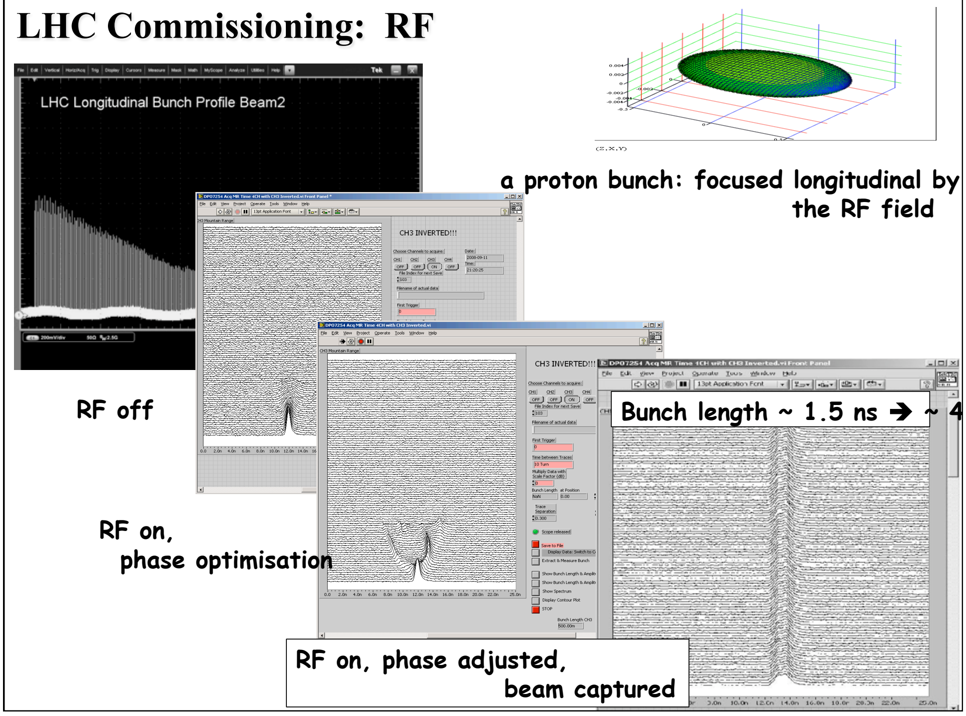
MV/beam
Nb on Cu cavities @4.5 K (=LEP2)
Beam pipe diam.=300mm

Bunch length (4σ)	*ns*	*1.06*
Energy spread (2σ)	*10^{-3}*	*0.22*
Synchr. rad. loss/turn *Synchr. rad. power*	*keV* *kW*	*7* *3.6*
RF frequency	*M Hz*	*400*
Harmonic number		*35640*
RF voltage/beam	*MV*	*16*
Energy gain/turn	*keV*	*485*
Synchrotron frequency	*Hz*	*23.0*

LHC Commissioning: RF

a proton bunch: focused longitudinal by the RF field



RF off

RF on,
phase optimisation

RF on, phase adjusted,
beam captured

Bunch length ~ 1.5 ns ➔ ~ 4

... and how do we accelerate now ???
with the dipole magnets !

Energy ramping is simply obtained by varying the B field:

$$p = eB\rho \quad \rightarrow \quad \frac{dp}{dt} = e\rho\,\dot{B} \quad \rightarrow (\Delta p)_{turn} = e\rho\,\dot{B}T_r = \frac{2\pi\, e\rho\, R\dot{B}}{v}$$

Energy Gain per turn: $E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$

$$\Delta E_{turn} = \Delta W_{turn} = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s$$

- **The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.**
- **Each synchronous particle satifies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.**

The Synchrotron: Frequency Change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency : $\longrightarrow \quad \omega_r = \frac{\omega_{RF}}{h} = \omega(B, R_s)$

hence : $$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{e}{m} < B(t) > \quad \Rightarrow \quad \frac{f_{RF}(t)}{h} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{r}{R_s}B(t)$$

Since $E^2 = (m_0c^2)^2 + p^2c^2$

the RF frequency must follow the variation of the **B** field with the law :

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0c^2/ecr)^2 + B(t)^2}\right\}^{1/2}$$

and as soon as $B > \frac{m_0c^2}{ecr}$ $$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} = const$$

Longitudinal Dynamics: synchrotron motion

We have to follow two coupled variables:

- the energy gained by the particle
- and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient and elegant to follow other particles with respect to that particle.

We will introduce the following relative variables:

revolution frequency : $\Delta f_r = f_r - f_{rs}$

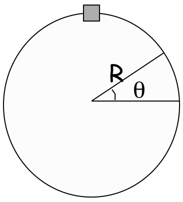
particle RF phase : $\Delta\phi = \phi - \phi_s$

particle momentum : $\Delta p = p - p_s$

particle energy : $\Delta E = E - E_s$

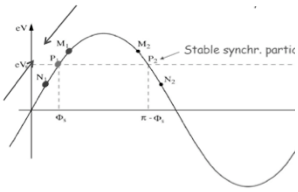
azimuth angle : $\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = hf_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

For a given particle with respect to the reference one:



$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$ and $E^2 = E_0^2 + p^2c^2$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets: $\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi} \longrightarrow \frac{d(\Delta\phi)}{dt} = -\frac{h\eta}{p_s v_s}\Delta E$

keep in mind !!

Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = \frac{1}{T_r} * e\hat{V}\sin\phi = \frac{\omega_r}{2\pi} * e\hat{V}\sin\phi$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}_s\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\,\dot{T}_{rs} + T_{rs}\Delta\dot{E} = \frac{d}{dt}(T_{rs}\Delta E)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This rather formidable looking differential equation simplifies a lot if we consider ...

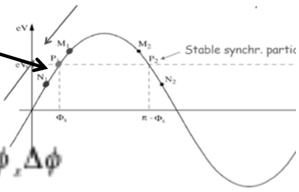
R_s, p_s, ω_s, η as constant (or slowly varying with time).

Small Amplitude Oscillations

Let’s assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi}+\frac{\Omega_s^2}{\cos\phi_s}(\sin\phi-\sin\phi_s)=0$$ with $$\Omega_s^2=\frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:



$$\sin\phi-\sin\phi_s=\sin(\phi_s+\Delta\phi)-\sin\phi_s\cong\cos\phi_s\Delta\phi$$

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi}+\Omega_s^2\Delta\phi=0$$ stable for $\Omega_s^2>0$ and Ω_s real

Small Amplitude Oscillations: stable phase

We get a harmonic oscillation of the particle phase

$$\ddot{\phi}+\Omega_s^2\Delta\phi=0$$

with the oscillation frequency

$$\Omega_s=\sqrt{\frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}}$$

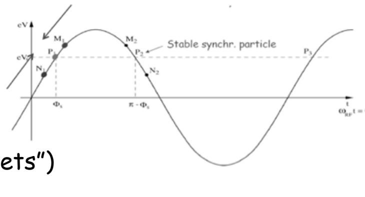
The oscillation is stable for $\Omega_s^2>0$ and Ω_s real

$\gamma < \gamma_{tr}$	$\eta > 0$	$0 < \phi_s < \pi/2$
$\gamma > \gamma_{tr}$	$\eta < 0$	$\pi/2 < \phi_s < \pi$

remember

$$\eta=\frac{1}{\gamma^2}-\alpha$$

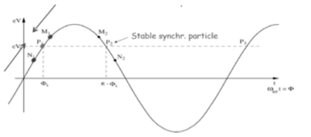
And we will find this situation
“h”-times in the machine



LHC:
35640 Possible Bunch Positions (“buckets”)
2808 Bunches

(small) ... Synchrotron Oscillations in Energy and Phase

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$



Ansatz: $\Delta\phi = \Delta\phi_{\max} * \cos(\Omega_s t)$

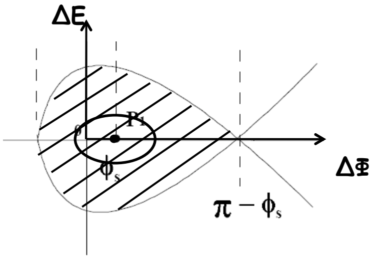
calculate the derivative and put it into ... *"keep in mind"* $\frac{d(\Delta\phi)}{dt} = -\frac{h\eta}{p_s v_s}\Delta E$

to get an expression for ΔE:

$$\Delta E = \Delta E_{\max} * \sin(\Omega_s t) \qquad \text{with} \qquad \Delta E_{\max} = \frac{\Omega_s}{\omega_s} * \frac{p_s v_s}{h\eta_s} * \Delta\Phi_{\max}$$

which defines an ellipse in phase space ΔΦ, ΔE:

$$\left(\frac{\Delta\Phi}{\Delta\Phi_{\max}}\right)^2 + \left(\frac{\Delta E}{\Delta E_{\max}}\right)^2 = 1$$



Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

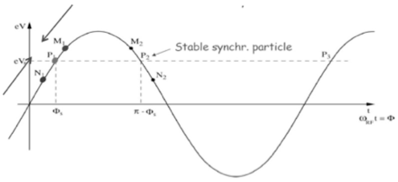
which for small amplitudes reduces to:

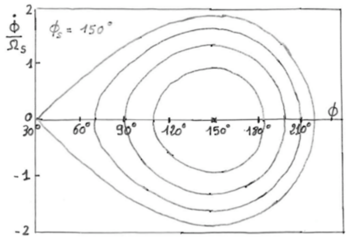
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{\left(\Delta\phi\right)^2}{2} = I$$

Similar equations exist for the second variable : ΔE∝dφ/dt

Large Amplitude Oscillations

When ϕ reaches π-ϕ$_s$ the force goes to zero and beyond it becomes non restoring. Hence π-ϕ$_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\phi}{\Omega_s}$, $\Delta\phi$) is shown as closed trajectories.





Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s\right)$$

Second value ϕ$_m$ where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m \sin\phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extremum when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2\left\{2 + (2\phi_s - \pi)\tan\phi_s\right\}$$

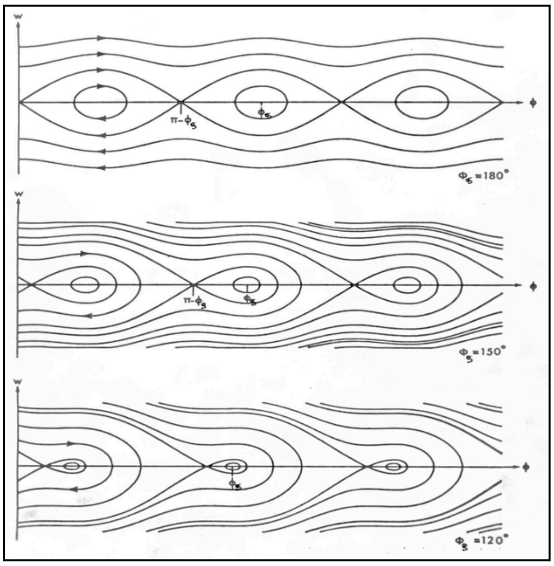
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \mp\beta\left\{-\frac{e\hat{V}}{\pi h \eta E_s}G(\phi_s)\right\}^{\frac{1}{2}}$$

$$G(\phi_s) = \left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right]$$

This "RF acceptance" depends strongly on ϕ$_s$ and plays an important role for the electron capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.