Electromagnetism

Christopher R Prior

Fellow and Tutor in Mathematics
Trinity College, Oxford

ASTeC Intense Beams Group
Rutherford Appleton Laboratory

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Reading

- J.D. Jackson: *Classical Electrodynamics*
- H.D. Young and R.A. Freedman: *University Physics (with Modern Physics)*
- P.C. Clemmow: *Electromagnetic Theory*
- *Feynmann Lectures on Physics*
- W.K.H. Panofsky and M.N. Phillips: *Classical Electricity and Magnetism*
- G.L. Pollack and D.R. Stump: *Electromagnetism*

Basic Equations from Vector Calculus

For a scalar function $\varphi(x,y,z,t)$,

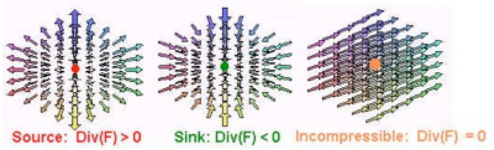
$$\text{gradient}: \quad \nabla\varphi = \left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\right)$$

Gradient is normal to surfaces φ=constant

For a vector $\vec{F} = (F_1,F_2,F_3)$

$$\text{divergence}: \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$\text{curl}: \quad \nabla\wedge\vec{F} = \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$



Basic Vector Calculus

$$\nabla \cdot \vec{F} \wedge \vec{G} = \vec{G} \cdot \nabla \wedge \vec{F} - \vec{F} \cdot \nabla \wedge \vec{G}$$

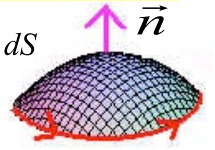
$$\nabla \wedge \nabla \phi = 0, \qquad \nabla \cdot \nabla \wedge \vec{F} = 0$$

$$\nabla \wedge (\nabla \wedge \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

Stokes' Theorem

$$\iint_S \nabla \wedge \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

$$d\vec{S} = \vec{n}\, dS$$



Oriented boundary *C*

Divergence or Gauss' Theorem

$$\iiint_V \nabla \cdot \vec{F}\, dV = \oiint_S \vec{F} \cdot d\vec{S}$$

Closed surface S, volume V, outward pointing normal

What is Electromagnetism?

- The study of Maxwell's equations, devised in 1863 to represent the relationships between electric and magnetic fields in the presence of electric charges and currents, whether steady or rapidly fluctuating, in a vacuum or in matter.
- The equations represent one of the most elegant and concise way to describe the fundamentals of electricity and magnetism. They pull together in a consistent way earlier results known from the work of Gauss, Faraday, Ampère, Biot, Savart and others.
- Remarkably, Maxwell's equations are perfectly consistent with the transformations of special relativity.

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Maxwell's Equations

Relate Electric and Magnetic fields generated by charge and current distributions.

$\vec{E}$ = electric field
$\vec{D}$ = electric displacement
$\vec{H}$ = magnetic field
$\vec{B}$ = magnetic flux density
ρ = electric charge density
$\vec{j}$ = current density
μ_0 = permeability of free space, $4\pi\, 10^{-7}$
ϵ_0 = permittivity of free space, $8.854\, 10^{-12}$
c = speed of light, $2.99792458\, 10^8$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \wedge \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

In vacuum: $\boxed{\vec{D} = \epsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \epsilon_0 \mu_0 c^2 = 1}$

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$

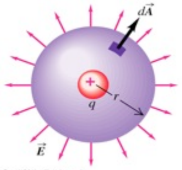
Maxwell's 1st Equation

Equivalent to Gauss' Flux Theorem:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \iff \iiint_V \nabla \cdot \vec{E}\, \mathrm{d}V = \iint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{1}{\epsilon_0} \iiint_V \rho\, \mathrm{d}V = \frac{Q}{\epsilon_0}$$

The flux of electric field out of a closed region is proportional to the total electric charge *Q* enclosed within the surface.

A point charge *q* generates an electric field:



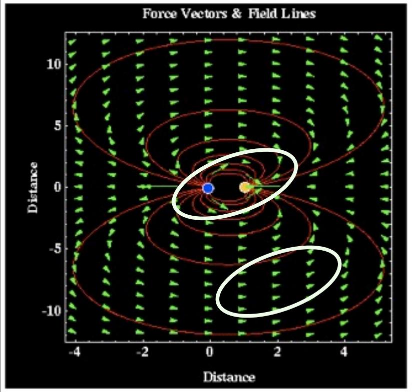
$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

$$\iint_{sphere} \vec{E} \cdot \mathrm{d}\vec{S} = \frac{q}{4\pi\epsilon_0} \iint_{sphere} \frac{\mathrm{d}S}{r^2} = q$$

Area integral gives a measure of the net charge enclosed; divergence of the electric field gives the density of the sources.

$$\boxed{\nabla\cdot\vec{B}=0}$$

Maxwell's 2nd Equation



Gauss' law for magnetism:

$$\nabla\cdot\vec{B}=0 \iff \iint \vec{B}\cdot d\vec{S}=0$$

The net magnetic flux out of any closed surface is zero. Surround a magnetic dipole with a closed surface. The magnetic flux directed inward towards the south pole will equal the flux outward from the north pole.

If there were a magnetic monopole source, this would give a non-zero integral.

Gauss' law for magnetism is then a statement that
There are no magnetic monopoles

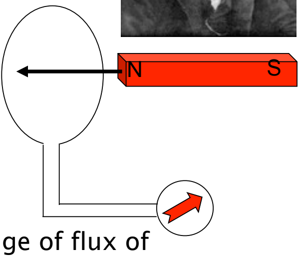
$$\boxed{\nabla\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t}}$$

Maxwell's 3rd Equation

Equivalent to Faraday's Law of Induction:

$$\iint_S \nabla\wedge\vec{E}\cdot d\vec{S}=-\iint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

$$\iff \oint_C \vec{E}\cdot d\vec{l}=-\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}=-\frac{d\Phi}{dt}$$



(for a fixed circuit *C*)

The electromotive force round a circuit $\varepsilon=\oint\vec{E}\cdot d\vec{l}$ is proportional to the rate of change of flux of magnetic field $\Phi=\iint\vec{B}\cdot d\vec{S}$ through the circuit.

Faraday's Law is the basis for electric generators. It also forms the basis for inductors and transformers.

$$\boxed{\nabla\wedge\vec{B}=\mu_0\vec{j}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$$

Maxwell's 4th Equation

Ampère

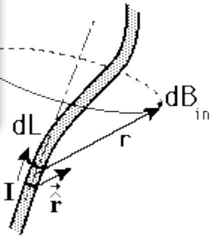
Biot

Originates from Ampère's (Circuital) Law : $\boxed{\nabla\wedge\vec{B}=\mu_0\vec{j}}$

$$\oint_C \vec{B}\cdot d\vec{l}=\iint_S \nabla\wedge\vec{B}\cdot d\vec{S}=\mu_0\iint_S \vec{j}\cdot d\vec{S}=\mu_0 I$$

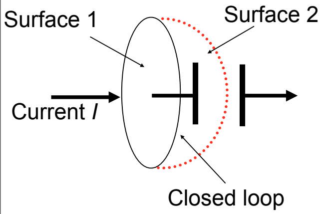
Satisfied by the field for a steady line current (Biot-Savart Law, 1820):

$$\boxed{\vec{B}=\frac{\mu_0 I}{4\pi}\oint\frac{d\vec{l}\wedge\vec{r}}{r^3}}$$



For a straight line current $B_\theta=\frac{\mu_0 I}{2\pi r}$

Displacement Current



- Apply Ampère to surface 1 (flat disk): line integral of $B=\mu_0 I$
- Applied to surface 2, line integral is zero since no current penetrates the deformed surface.
- In capacitor, $E=\frac{Q}{\varepsilon_0 A}$, so $I=\frac{dQ}{dt}=\varepsilon_0 A\frac{dE}{dt}$
- Displacement current density is $\vec{j}_d=\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\boxed{\nabla\wedge\vec{B}=\mu_0(\vec{j}+\vec{j}_d)=\mu_0\vec{j}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}}$$

Consistent with Charge Conservation

Charge conservation:

Total current flowing out of a region equals the rate of decrease of charge within the volume.

$$\iint \vec{j}\cdot \mathrm{d}\vec{S} = -\frac{\mathrm{d}}{\mathrm{d}t}\iiint \rho\,\mathrm{dV}$$

$$\Longleftrightarrow \iiint \nabla\cdot\vec{j}\,\mathrm{dV} = -\iiint \frac{\partial\rho}{\partial t}\,\mathrm{dV}$$

$$\Longleftrightarrow \nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

From Maxwell's equations:

Take divergence of Ampère's equation (incl. displacement current)

$$\nabla\wedge\vec{B} = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\Longrightarrow \nabla\cdot\nabla\wedge\vec{B} = \mu_0\nabla\cdot\vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\cdot\vec{E})$$

$$\Longrightarrow 0 = \nabla\cdot\vec{j} + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right)$$

$$\Longrightarrow 0 = \nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t}$$

Charge conservation is implicit in Maxwell's Equations

Maxwell's Equations in Vacuum

In vacuum:

$$\vec{D} = \epsilon_0\vec{E},\quad \vec{B} = \mu_0\vec{H},\quad \epsilon_0\mu_0 c^2 = 1$$

Source-free equations:

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\wedge\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

Source equations:

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla\wedge\vec{B} - \frac{1}{c^2}\frac{\partial\vec{B}}{\partial t} = \mu_0\vec{j}$$

Equivalent integral form (useful for simple geometries):

$$\iint \vec{E}\cdot\mathrm{d}\vec{S} = \frac{1}{\epsilon_0}\iiint \rho\,\mathrm{dV}$$

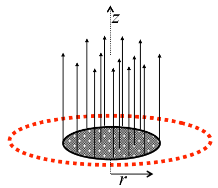
$$\iint \vec{B}\cdot\mathrm{d}\vec{S} = 0$$

$$\oint \vec{E}\cdot\mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{B}\cdot\mathrm{d}\vec{S} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

$$\oint \vec{B}\cdot\mathrm{d}\vec{l} = \mu_0\iint \vec{j}\,\mathrm{d}\vec{S} + \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{E}\cdot\mathrm{d}\vec{S}$$

Example: Calculate E from B

$$\boxed{\oint \vec{E}\cdot\mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{B}\cdot\mathrm{d}\vec{S}}$$



$$B_z = \begin{cases} B_0\sin\omega t & r < r_0 \\ 0 & r > r_0 \end{cases}$$

$$r < r_0 \quad 2\pi r E_\theta = -\frac{d}{dt}\pi r^2 B_0\sin\omega t = -\pi r^2 B_0\omega\cos\omega t$$

$$\Rightarrow \quad E_\theta = -\frac{1}{2}B_0\omega r\cos\omega t$$

$$r > r_0 \quad 2\pi r E_\theta = -\frac{d}{dt}\pi r_0^2 B_0\sin\omega t = -\pi r_0^2 B_0\omega\cos\omega t$$

$$\Rightarrow \quad E_\theta = -\frac{\omega r_0^2 B_0}{2r}\cos\omega t$$

Also from $\nabla\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\nabla\wedge\vec{B} = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{dt}$ then gives current density necessary to sustain the fields

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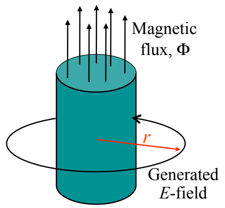
The Betatron

Particles accelerated by the rotational electric field generated by a time-varying magnetic field



$$\oint \vec{E}\cdot\mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{B}\cdot\mathrm{d}\vec{S}$$

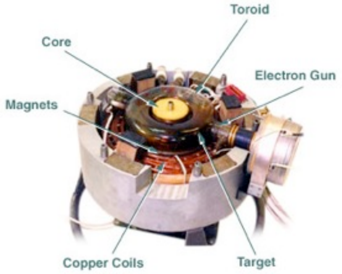
$$\Longrightarrow \quad 2\pi r E_\theta = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

For circular motion at a constant radius:

$$-\frac{mv^2}{r} = evB \quad\Longrightarrow\quad B = -\frac{p}{er}$$

$$\Longrightarrow \frac{\partial}{\partial t}B(r,t) = -\frac{1}{er}\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{E}{r} = \frac{1}{2\pi r^2}\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

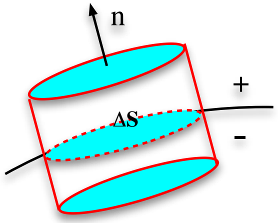
$$\Longrightarrow B(r,t) = \frac{1}{2}\frac{1}{\pi r^2}\iint B\,\mathrm{d}S$$



B-field on orbit needs to be one half the average *B* over the circle. This imposes a limit on the energy that can be achieved. Nevertheless the constant radius principle is attractive for high energy circular accelerators.

Boundary Conditions I

Maxwell's equations involving divergence can be integrated over a small "pillbox" across the boundary surface



$$\nabla\cdot\vec{B} = 0 \implies \iiint \nabla\cdot\vec{B}\,\mathrm{d}V = \iint \vec{B}\cdot\,\mathrm{d}\vec{S} = 0$$

$$\implies \left(\vec{n}\cdot\vec{B}^{+} - \vec{n}\cdot\vec{B}^{-}\right)\Delta S = 0$$

$$\implies \boxed{\left[\vec{n}\cdot\vec{B}\right]_{-}^{+} = 0}$$

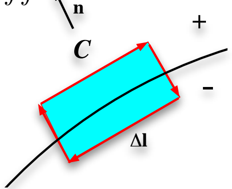
$$\nabla\cdot\vec{D} = \rho \implies \iiint \nabla\cdot\vec{D}\,\mathrm{d}V = \iint \vec{D}\cdot\,\mathrm{d}\vec{S} = \iiint \rho\,\mathrm{d}V$$

$$\implies \left(\vec{n}\cdot\vec{D}^{+} - \vec{n}\cdot\vec{D}^{-}\right)\Delta S = \sigma\Delta S$$

$$\implies \boxed{\left[\vec{n}\cdot\vec{D}\right]_{-}^{+} = \sigma} \quad \text{where } \sigma \text{ is the surface charge density}$$

Boundary Conditions II

Maxwell's equations involving curl can be integrated over a closed contour close to, and straddling, the boundary surface



$$\nabla\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t} \implies \iint \nabla\wedge\vec{E}\cdot\,\mathrm{d}\vec{S} = \oint \vec{E}\cdot\,\mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{B}\cdot\,\mathrm{d}\vec{S}$$

$$\implies \left(\vec{E}_{\parallel}^{+} - \vec{E}_{\parallel}^{-}\right)\Delta l \to 0$$

$$\implies \boxed{\left[\vec{n}\wedge\vec{E}\right]_{-}^{+} = 0}$$

$$\nabla\wedge\vec{H} = \vec{j} + \frac{\partial\vec{D}}{\partial t} \implies \iint \nabla\wedge\vec{H}\cdot\,\mathrm{d}\vec{S} = \oint \vec{H}\cdot\,\mathrm{d}\vec{l} = \iint \vec{j}\cdot\,\mathrm{d}\vec{S} + \frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{D}\cdot\,\mathrm{d}\vec{S}$$

$$\implies \left(\vec{H}_{\parallel}^{+} - \vec{H}_{\parallel}^{-}\right)\Delta l \to \vec{K}\Delta l$$

$$\implies \boxed{\left[\vec{n}\wedge\vec{H}\right]_{-}^{+} = \vec{K}} \quad \text{where } \vec{K} \text{ is the surface current density}$$

Lorentz Force Law

- Thought of as a supplement to Maxwell's equations but actually implicit in relativistic formulation, gives force on a charged particle moving in an electromagnetic field:

$$\vec{f} = q\left(\vec{E} + \vec{v}\wedge\vec{B}\right)$$

- For continuous distributions, use force density

$$\vec{f}_d = \rho\vec{E} + \vec{j}\wedge\vec{B}$$

- Relativistic equation of motion
  - 4-vector form: $F = \frac{dP}{d\tau} \implies \gamma\left(\frac{\vec{v}\cdot\vec{f}}{c}, \vec{f}\right) = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$
  - 3-vector component:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\vec{v}) = \vec{f} = q\left(\vec{E} + \vec{v}\wedge\vec{B}\right)}$$

Energy component:

$$\boxed{\vec{v}\cdot\vec{f} = \frac{\mathrm{d}E}{\mathrm{d}t} = m_0c^2\frac{\mathrm{d}\gamma}{\mathrm{d}t}}$$

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Motion of Charged Particles in Constant Magnetic Fields

$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\vec{v}) &= \vec{f} = q\left(\vec{E} + \vec{v}\wedge\vec{B}\right) = q\vec{v}\wedge\vec{B} \\ \frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma c^2) &= \vec{v}\cdot\vec{f} = q\vec{v}\cdot\vec{v}\wedge\vec{B} = 0 \end{aligned}}$$

1. From energy equation, γ is constant

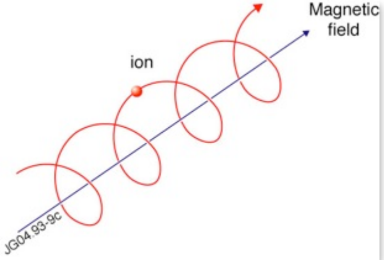
   No acceleration with a magnetic field

2. From momentum equation,

$$\vec{B}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\vec{v}) = 0 = \gamma\frac{\mathrm{d}}{\mathrm{d}t}(\vec{B}\cdot\vec{v}) \quad \implies \vec{v}_{\parallel} \text{ is constant}$$

$$\boxed{\begin{aligned} &|\vec{v}| \text{ constant and } |\vec{v}_{\parallel}| \text{ constant} \\ &\implies \quad |\vec{v}_{\perp}| \text{ also constant} \end{aligned}}$$

Motion in Constant Magnetic Field



$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\vec{v}) = q\vec{v}\wedge\vec{B}$$

$$\implies \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m_0\gamma}\vec{v}\wedge\vec{B}$$

$$\implies \frac{v_\perp^2}{\rho} = \frac{q}{m_0\gamma}v_\perp B$$

$$\implies \text{circular motion with radius } \rho = \frac{m_0\gamma v_\perp}{qB}$$

$$\text{at an angular frequency } \omega = \frac{v_\perp}{\rho} = \frac{qB}{m_0\gamma} = \frac{qB}{m}$$

Constant magnetic field gives uniform spiral about B with constant energy.

$$B\rho = \frac{m_0\gamma v}{q} = \frac{p}{q}$$

Magnetic Rigidity

Motion in Constant Electric Field

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\vec{v}) = \vec{f} = q(\vec{E}+\vec{v}\wedge\vec{B}) \implies \boxed{\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\vec{v}) = q\vec{E}}$$

Solution is $\gamma\vec{v} = \frac{q\vec{E}}{m_0}t$

Then $\gamma^2 = 1 + \left(\frac{\gamma\vec{v}}{c}\right)^2 \implies \gamma = \sqrt{1+\left(\frac{q\vec{E}t}{m_0c}\right)^2}$

If $\vec{E} = (E,0,0)$, $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(\gamma v)}{\gamma} \implies x = x_0 + \frac{m_0c^2}{qE}\left[\sqrt{1+\left(\frac{qEt}{m_0c}\right)^2} - 1\right]$

$$\approx x_0 + \frac{1}{2}\left(\frac{qE}{m_0}\right)t^2 \quad \text{for} \quad qE \ll m_0c$$

Energy gain is $\boxed{m_0c^2(\gamma-1) = qE(x-x_0)}$

Constant E-field gives uniform acceleration in straight line

Relativistic Transformations of E and B

- According to observer O in frame F, particle has velocity ***v***, fields are ***E*** and ***B*** and Lorentz force is $\vec{f} = q\left(\vec{E}+\vec{v}\times\vec{B}\right)$
- In Frame F′, particle is at rest and force is $\vec{f}' = q'\vec{E}'$
- Assume measurements give same charge and force, so
  $$q' = q \quad\text{and}\quad \vec{E}' = \vec{E}+\vec{v}\times\vec{B}$$
- Point charge q at rest in F: $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}, \quad \vec{B} = 0$
- See a current in F′, giving a field $\vec{B}' = -\frac{\mu_0 q}{4\pi}\frac{\vec{v}\times\vec{r}}{r^3} = -\frac{1}{c^2}\vec{v}\times\vec{E}$
- Suggests $\vec{B}' = \vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}$

Relativistic Transformations of E and B

- According to observer O in frame F, particle has velocity ***v***, fields are ***E*** and ***B*** and Lorentz force is $\vec{f} = q\left(\vec{E}+\vec{v}\times\vec{B}\right)$

$$\vec{E}'_\perp = \gamma\left(\vec{E}_\perp+\vec{v}\times\vec{B}\right), \qquad \vec{E}'_\parallel = \vec{E}_\parallel$$

$$\vec{B}'_\perp = \gamma\left(\vec{B}_\perp-\frac{\vec{v}\times\vec{E}}{c^2}\right), \qquad \vec{B}_\parallel = \vec{B}_\parallel$$

- Suggests $\vec{B}' = \vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}$

Potentials

- Magnetic vector potential

$$\nabla\cdot\vec B = 0 \iff \exists\vec A \quad \text{such that} \quad \vec B = \nabla\wedge\vec A$$

- Electric scalar potential

$$\nabla\wedge\vec E = -\frac{\partial\vec B}{\partial t} \iff \nabla\wedge\left(\vec E + \frac{\partial\vec A}{\partial t}\right) = 0$$

$$\iff \exists\phi \quad \text{such that} \quad \vec E = -\nabla\phi - \frac{\partial\vec A}{\partial t}$$

- Lorentz gauge $\phi\to\phi+f(t), \quad \vec A\to\vec A+\nabla\chi$

$$\text{Use freedom to set} \quad \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\vec A = 0$$

Electromagnetic 4-Vectors

- Lorentz gauge

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\vec A = 0 = \underset{\substack{\uparrow \\ \text{4-gradient } \nabla_4}}{\left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)}\cdot\underset{\substack{\uparrow \\ \text{4-potential } \Phi}}{\left(\frac{1}{c}\phi, \vec A\right)} = \nabla_4\cdot\Phi$$

- Current 4-vector

3D: $\vec j = \rho\vec v$

4D: $J = \rho_0 V = \rho_0\gamma(c,\vec v) = (c\rho,\vec j), \quad \text{where} \quad \rho = \rho_0\gamma$

- Continuity equation

$$\nabla_4\cdot J = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)\cdot(c\rho,\vec j) = \frac{\partial\rho}{\partial t} + \nabla\cdot\vec j = 0$$

Relativistic Transformation of Potentials

- 4-potential vector: $\Phi = \left(\frac{1}{c}\phi, \vec A\right)$

- Lorentz transformation

$$\begin{pmatrix} \frac{1}{c}\phi' \\ A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c}\phi \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\implies \begin{aligned} \phi' &= \gamma(\phi - vA_x) \\ A'_x &= \gamma\left(A_x - \frac{v\phi}{c^2}\right), A'_y = A_y, A'_z = A_z \end{aligned}$$

Electromagnetic Energy

- Rate of doing work on unit volume of a system is

$$-\vec v\cdot\vec f = -\vec v\cdot\left(\rho\vec E + \vec j\wedge\vec B\right) = -\rho\vec v\cdot\vec E = -\vec j\cdot\vec E$$

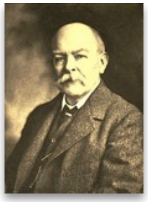
- Substitute for $\vec j$ from Maxwell's equations and re-arrange:

$$\begin{aligned} -\vec j\cdot\vec E &= -\left(\nabla\wedge\vec H - \frac{\partial\vec D}{\partial t}\right)\cdot\vec E \\ &= \nabla\cdot\vec E\wedge\vec H - \vec H\cdot\nabla\wedge\vec E + \vec E\cdot\frac{\partial\vec D}{\partial t} \\ &= \nabla\cdot\vec S + \vec H\cdot\frac{\partial\vec B}{\partial t} + \vec E\cdot\frac{\partial\vec D}{\partial t} \quad \text{where} \quad \underset{\uparrow}{\vec S} = \vec E\wedge\vec H \end{aligned}$$

- For linear, non-dispersive media where $\vec B = \mu\vec H, \vec D = \epsilon\vec E$

$$-\vec j\cdot\vec E = \nabla\cdot\vec S + \frac{1}{2}\frac{\partial}{\partial t}(\vec E\cdot\vec D + \vec B\cdot\vec H)$$

Poynting vector



Energy Conservation

$$-\vec{j}\cdot\vec{E} == \frac{\partial}{\partial t}\left\{\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})\right\}+\nabla\cdot\vec{S}$$

- Integrated over a volume, this represents an **energy conservation law**:
  - the rate of doing work on a system equals the rate of increase of stored electromagnetic energy+ rate of energy flow across boundary.

$$\frac{\mathrm{d}W}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\iiint\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})\,\mathrm{d}V+\iint\vec{E}\wedge\vec{H}\cdot\mathrm{d}\vec{S}$$

electric + magnetic energy densities of the fields

Poynting vector gives flux of e/m energy across boundaries

Review of Waves

- 1D wave equation is $\frac{\partial^2 u}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$ with general solution

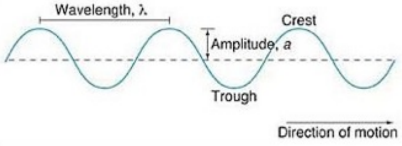
$$u(x,t)=f(vt-x)+g(vt+x)$$

- Simple plane wave: → ←

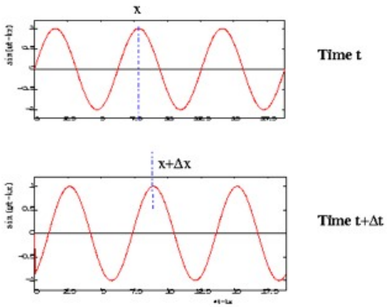
$$\text{1D}:\quad \sin(\omega t-kx)\qquad \text{3D}:\quad \sin\left(\omega t-\vec{k}\cdot\vec{x}\right)$$

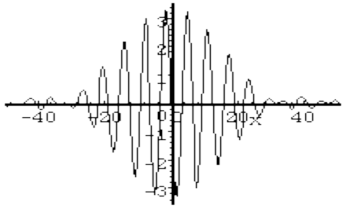
Wavelength is $\lambda=\frac{2\pi}{|\vec{k}|}$

Frequency is $\nu=\frac{\omega}{2\pi}$



Phase and group velocities





$$\int_{-\infty}^{\infty}A(k)e^{i[\omega(k)t-kx]}dk$$

Plane wave $\sin(\omega t-kx)$ has constant phase $\omega t-kx=\frac{1}{2}\pi$ at peaks

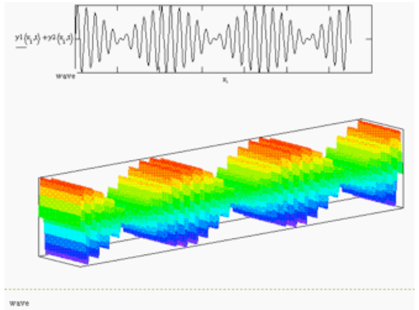
$$\omega\Delta t-k\Delta x=0$$

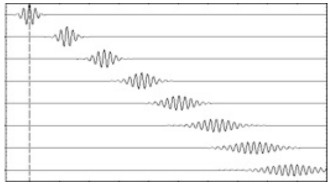
$$\Longleftrightarrow\quad v_p=\frac{\Delta x}{\Delta t}=\frac{\omega}{k}$$

Superposition of plane waves. While shape is relatively undistorted, pulse travels with the Group Velocity

$$v_g=\frac{d\omega}{dk}$$

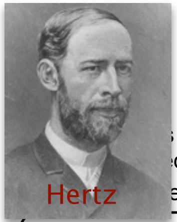
Wave Packet Structure





- Phase velocities of individual plane waves making up the wave packet are different,
- The wave packet will then disperse with time

Electromagnetic waves



s equations predict the existence of electromagnetic waves, later d by Hertz.

es, no currents:

$$\nabla\wedge\vec{H} = \frac{\partial\vec{D}}{\partial t}, \qquad \nabla\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{D} = 0, \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\wedge(\nabla\wedge\vec{E}) = -\nabla\wedge\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\wedge\vec{B}) = -\mu\frac{\partial^2\vec{D}}{\partial t^2} = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

3D wave equation:

$$\nabla^2\vec{E} = \frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial y^2} + \frac{\partial^2\vec{E}}{\partial z^2} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla\wedge(\nabla\wedge\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}$$

Similarly for $\vec{H}$.

Electromagnetic waves travelling with speed $\boxed{\frac{1}{\sqrt{\epsilon\mu}}}$

Nature of Electromagnetic Waves

- A general plane wave with angular frequency ω travelling in the direction of the wave vector $\vec{k}$ has the form

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}, \qquad \vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

- Phase $\omega t - \vec{k}\cdot\vec{x} = 2\pi\times$ number of waves and so is a Lorentz invariant.
- Apply Maxwell's equations:

$$\nabla \leftrightarrow -i\vec{k}, \qquad \frac{\partial}{\partial t} \leftrightarrow i\omega$$

$$\nabla\cdot\vec{E} = 0 = \nabla\cdot\vec{B} \leftrightarrow \vec{k}\cdot\vec{E} = 0 = \vec{k}\cdot\vec{B}$$

$$\nabla\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t} \leftrightarrow \vec{k}\wedge\vec{E} = \omega\vec{B}$$

- Waves are transverse to the direction of propagation; $\vec{E}, \vec{B}$ and $\vec{k}$ are mutually perpendicular

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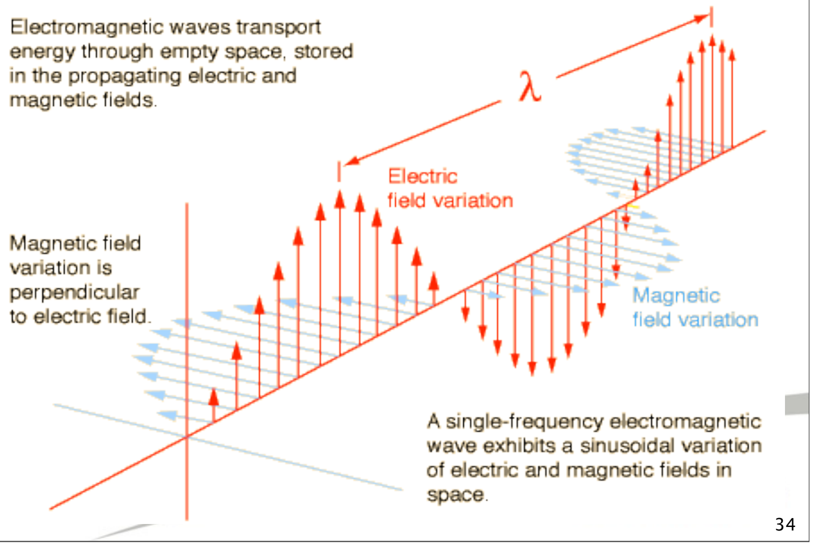
Plane Electromagnetic Wave

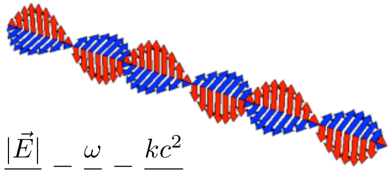


Electromagnetic waves transport energy through empty space, stored in the propagating electric and magnetic fields.

Magnetic field variation is perpendicular to electric field.

A single-frequency electromagnetic wave exhibits a sinusoidal variation of electric and magnetic fields in space.

Plane Electromagnetic Waves



$$\nabla\wedge\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Longleftrightarrow \vec{k}\wedge\vec{B} = -\frac{\omega}{c^2}\vec{E}$$

$$\text{Combined with} \quad \vec{k}\wedge\vec{E} = \omega\vec{B} \quad \Longrightarrow \frac{|\vec{E}|}{|\vec{B}|} = \frac{\omega}{k} = \frac{kc^2}{\omega}$$

$$\Longrightarrow \text{ speed of electromagnetic waves in vacuum is } \frac{\omega}{k} = c$$

$$\text{Wavelength} \quad \lambda = \frac{2\pi}{|\vec{k}|}$$

$$\text{Frequency} \quad \nu = \frac{\omega}{2\pi}$$

Reminder: The fact that $\omega t - \vec{k}\cdot\vec{x}$ is an invariant tells us that

$$\Lambda = \left(\frac{\omega}{c}, \vec{k}\right)$$

is a Lorentz 4-vector, the 4-Frequency vector. Deduce frequency transforms as

$$\omega' = \gamma(\omega - \vec{v}\cdot\vec{k}) = \omega\sqrt{\frac{c-v}{c+v}}$$

Waves in a Conducting Medium

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}, \quad \vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

- (Ohm's Law) For a medium of conductivity σ, $\vec{j} = \sigma\vec{E}$
- Modified Maxwell: $\nabla \wedge \vec{H} = \vec{j} + \epsilon\frac{\partial \vec{E}}{\partial t} = \sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t}$

$$-i\vec{k} \wedge \vec{H} = \sigma\vec{E} + i\omega\epsilon\vec{E}$$

conduction current

displacement current

- Put $D = \frac{\sigma}{\omega\epsilon}$

Dissipation factor

Copper: $\sigma = 5.8\times10^{7}, \varepsilon = \varepsilon_0 \quad \Rightarrow \quad D = 10^{12}$

Teflon: $\sigma = 3\times10^{-8}, \varepsilon = 2.1\varepsilon_0 \quad \Rightarrow \quad D = 2.57\times10^{-4}$

Attenuation in a Good Conductor

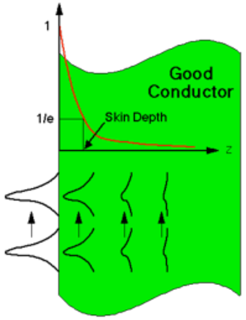
$$-i\vec{k} \wedge \vec{H} = \sigma\vec{E} + i\omega\epsilon\vec{E} \iff \vec{k} \wedge \vec{H} = i\sigma\vec{E} - \omega\epsilon\vec{E} = (i\sigma - \omega\epsilon)\vec{E}$$

Combine with $\quad \nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \vec{k} \wedge \vec{E} = \omega\mu\vec{H}$

$$\implies \quad \vec{k} \wedge (\vec{k} \wedge \vec{E}) = \omega\mu\vec{k} \wedge \vec{H} = \omega\mu(i\sigma - \omega\epsilon)\vec{E}$$

$$\implies \quad (\vec{k}\cdot\vec{E})\vec{k} - k^2\vec{E} = \omega\mu(i\sigma - \omega\epsilon)\vec{E}$$

$$\implies \quad \boxed{k^2 = \omega\mu(-i\sigma + \omega\epsilon)} \text{ since } \vec{k}\cdot\vec{E} = 0$$



For a good conductor, $D \gg 1$, $\sigma \gg \omega\epsilon$, $k^2 \approx -i\omega\mu\sigma$

$$\implies k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1-i) = \frac{1}{\delta}(1-i) \text{ where } \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \text{ is the skin-depth}$$

$$\text{Wave-form is:} \quad e^{i(\omega t - kx)} = e^{i(\omega t - (1-i)x/\delta)} = e^{-x/\delta}e^{i(\omega t - x/\delta)}$$

Charge Density in a Conducting Material

- Inside a conductor (Ohm's law) $\vec{j} = \sigma\vec{E}$
- Continuity equation is

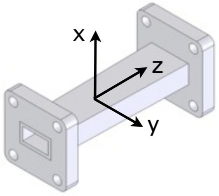
$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0 \iff \frac{\partial\rho}{\partial t} + \sigma\nabla\cdot\vec{E} = 0$$

$$\iff \frac{\partial\rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0.$$

- Solution is $\boxed{\rho = \rho_0\, e^{-\sigma t/\epsilon}}$
- Charge density decays exponentially with time. For a very good conductor, charge flows instantly to the surface to form a surface current density and (for time varying fields) a surface current. Inside a perfect conductor:

$$(\sigma \to \infty) \quad \vec{E} = \vec{H} = 0$$

Maxwell's Equations in a Uniform Perfectly Conducting Guide



Hollow metallic cylinder with perfectly conducting boundary surfaces

Maxwell's equations with time dependence $e^{i\omega t}$ are:

$$\underbrace{\left.\begin{aligned} \nabla \wedge \vec{E} &= -\frac{\partial \vec{B}}{\partial t} = -i\omega\mu\vec{H} \\ \nabla \wedge \vec{H} &= \frac{\partial \vec{D}}{\partial t} = i\omega\epsilon\vec{E} \end{aligned}\right\} \implies \begin{aligned} \nabla^2\vec{E} &= \nabla(\nabla\cdot\vec{E}) - \nabla\wedge\nabla\wedge\vec{E} \\ &= i\omega\mu\nabla\wedge\vec{H} \\ &= -\omega^2\epsilon\mu\vec{E} \end{aligned}}_{(\nabla^2 + \omega^2\epsilon\mu)\left\{\begin{array}{c}\vec{E}\\ \vec{H}\end{array}\right\} = 0}$$

Assume $\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{(i\omega t - \gamma z)}$

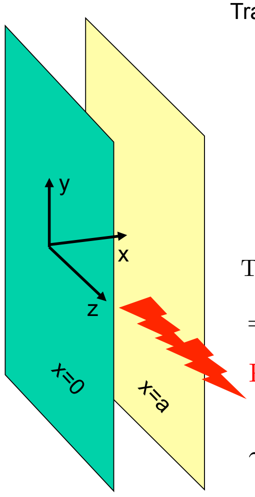
$\vec{H}(x,y,z,t) = \vec{H}(x,y)e^{(i\omega t - \gamma z)}$

γ is the propagation constant

$$\text{Then } \left[\nabla_t^2 + (\omega^2\epsilon\mu + \gamma^2)\right]\left\{\begin{array}{c}\vec{E}\\ \vec{H}\end{array}\right\} = 0$$

Can solve for the fields completely in terms of E_z and H_z

A simple model: “Parallel Plate Waveguide”



Transport between two infinite conducting plates (TE_{01} mode):

$$\vec{E} = (0,1,0)E(x)\mathrm{e}^{\mathrm{i}\omega t - \gamma z} \quad \text{where } E \text{ satisfies}$$

$$\nabla_t^2 E = \frac{\mathrm{d}^2 E}{\mathrm{d}x^2} = -K^2 E, \qquad K^2 = \omega^2 \epsilon\mu + \gamma^2.$$

$$\text{with solution } E = A\cos Kx \text{ or } A\sin Kx$$

$$\text{To satisfy boundary conditions: } E = 0 \text{ on } x = 0 \text{ and } x = a.$$

$$\Longrightarrow \qquad E = A\sin Kx, \quad \text{with } K = K_n \equiv \frac{n\pi}{a}, \quad n \text{ integer}$$

Propagation constant is

$$\gamma = \sqrt{K_n^2 - \omega^2\epsilon\mu} = \frac{n\pi}{a}\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}, \qquad \omega_c = \frac{K_n}{\sqrt{\epsilon\mu}}$$

Cut-off Frequency, ω_c

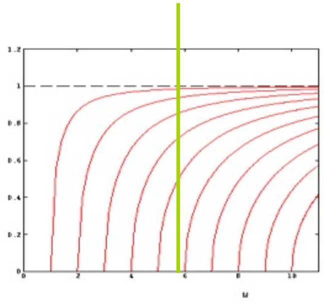
$$\gamma = \frac{n\pi}{a}\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}, \qquad E = \sin\frac{n\pi x}{a}e^{i\omega t - \gamma z}, \qquad \omega_c = \frac{n\pi}{a\sqrt{\epsilon\mu}}$$

- $\omega<\omega_c$ gives real solution for γ, so attenuation only. No wave propagates: cut-off modes.
- $\omega>\omega_c$ gives purely imaginary solution for γ, and a wave propagates without attenuation.

$$\gamma = ik, \quad k = \sqrt{\epsilon\mu}(\omega^2 - \omega_c^2)^{\frac{1}{2}} = \omega\sqrt{\epsilon\mu}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{\frac{1}{2}}$$

- For a given frequency ω only a finite number of modes can propagate.

$$\omega > \omega_c = \frac{n\pi}{a\sqrt{\epsilon\mu}} \qquad \Longrightarrow n < \frac{a\omega}{\pi}\sqrt{\epsilon\mu}$$



For given frequency, convenient to choose *a* s.t. only *n=1* mode occurs.

Phase and Group Velocities in the Simple Wave-Guide

- Wave number $k = \sqrt{\epsilon\mu}(\omega^2 - \omega_c^2)^{\frac{1}{2}} < \omega\sqrt{\epsilon\mu}$
- Wavelength $\lambda = \frac{2\pi}{k} > \frac{2\pi}{\omega\sqrt{\epsilon\mu}},$
  - free-space wavelength
- Phase velocity $v_p = \frac{\omega}{k} > \frac{1}{\sqrt{\epsilon\mu}}$
  - larger than free-space velocity
- Group velocity $k^2 = \epsilon\mu(\omega^2 - \omega_c^2) \implies v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{k}{\omega\epsilon\mu} < \frac{1}{\sqrt{\epsilon\mu}}$
  - smaller than free-space velocity

Calculation of Wave Properties

- If $a = 3\,\mathrm{cm}$, cut-off frequency of lowest order mode is

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2a\sqrt{\epsilon\mu}} \approx \frac{3\times10^8}{2\times0.03} \approx 5\,\mathrm{GHz} \qquad \left(\omega_c = \frac{n\pi}{a\sqrt{\epsilon\mu}}\right)$$

- At 7 GHz, only the n=1 mode propagates and

$$k = \sqrt{\epsilon\mu}(\omega^2 - \omega_c^2)^{\frac{1}{2}} \approx 2\pi(7^2 - 5^2)^{\frac{1}{2}} \times 10^9/3\times10^8 = 103\,\mathrm{m}^{-1}$$

$$\lambda = \frac{2\pi}{k} \approx 6\,\mathrm{cm}$$

$$v_p = \frac{\omega}{k} = 4.3\times10^8\,\mathrm{ms}^{-1} > c$$

$$v_g = \frac{k}{\omega\epsilon\mu} = 2.1\times10^8\,\mathrm{ms}^{-1} < c$$

Flow of EM Energy along the Simple Wave-Guide

- Fields ($\omega > \omega_c$) are:

$$E_x = E_z = 0, \quad E_y = A \sin\frac{n\pi x}{a}\cos(\omega t = kz)$$

$$H_x = -\frac{k}{\omega\mu}E_y, \quad H_y = 0, \quad H_z = -\frac{n\pi}{a\omega\mu}\cos\frac{n\pi x}{a}\sin(\omega t - kz)$$

- Time averaged energies: $\langle \sin^2\omega t\rangle = \langle\cos^2\omega t\rangle = \frac{1}{2}, \ \langle \sin\omega t\cos\omega t\rangle = 0$

$$\text{Electric energy:} \qquad W_e = \frac{1}{4}\epsilon\int_0^a |\vec{E}|^2\,\mathrm{d}x = \frac{1}{8}\epsilon A^2 a$$

$$\text{Magnetic energy:} \qquad W_m = \frac{1}{4}\mu\int_0^a |\vec{H}|^2\,\mathrm{d}x = \frac{1}{8}\mu A^2 a\left\{\left(\frac{n\pi}{a\omega\mu}\right)^2 + \left(\frac{k}{\omega\mu}\right)^2\right\}$$

$$= W_e \quad \text{since } k^2 + \frac{n^2\pi^2}{a^2} = \omega^2\epsilon\mu$$

Flow of E/M Energy

- Poynting vector: $\vec{S} = \vec{E}\wedge\vec{H} = \left(E_y H_z, 0, -E_y H_x\right)$
- Time averaged: $\langle\vec{S}\rangle = \frac{1}{2}(0,0,1)\frac{kA^2}{\omega\mu}\sin^2\frac{n\pi x}{a}$
- Integrate over x: $\langle S_z\rangle = \frac{1}{4}\frac{kA^2}{\omega\mu}a$
- So energy is transported at a rate:

$$\frac{\langle S_z\rangle}{W_e + W_m} = \frac{k}{\omega\epsilon\mu} = v_g$$

Total e/m energy density

$$W = \frac{1}{4}\epsilon A^2 a$$

Flow of E/M Energy

- Poynting vector: $\vec{S} = \vec{E}\wedge\vec{H} = \left(E_y H_z, 0, -E_y H_x\right)$
- Time averaged: $\langle\vec{S}\rangle = \frac{1}{2}(0,0,1)\frac{kA^2}{\omega\mu}\sin^2\frac{n\pi x}{a}$
- Integrate over x: $\langle S_z\rangle = \frac{1}{4}\frac{kA^2}{\omega\mu}a$
- So energy is transported at a rate:

$$\frac{\langle S_z\rangle}{W_e + W_m} = \frac{k}{\omega\epsilon\mu} = v_g$$

Total e/m energy density

$$W = \frac{1}{4}\epsilon A^2 a$$

Electromagnetic energy is transported down the waveguide with the group velocity

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