Space Charge Effects

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Outline

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If they are at rest the **Coulomb force** exerts a **repulsion**

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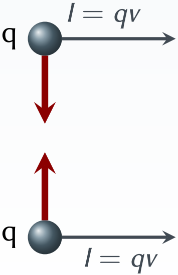
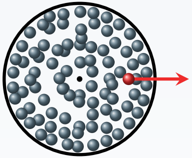
If they are at rest the **Coulomb force** exerts a **repulsion**

If they travel with a velocity $v = \beta c$, they represent two parallel currents $I = qv$ which **attract** each other by the effect of their **magnetic field**.



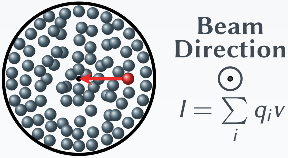
Space charge

Now, consider an unbunched beam of particles (charge q) with a circular cross section.



Charges

The Coulomb repulsion pushes the test particle outward. The induced force is zero in the beam centre and increases toward the edge of the beam.



Parallel currents

The magnetic force is radial and attractive for the test particle in a travelling beam (parallel currents).

Space charge fields

Consider a continuous beam of cylindrical symmetry distribution that moves with a constant velocity $v = \beta c$. It's charge density is:

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For symmetry reason, the electric field has only a **radial component** E_r. Using the integral form of the Gauss' law over a cylinder centred on the beam axis:

$$E_r(r) = \frac{1}{\epsilon_0 r} \int_0^r \rho(r)\, r\, dr \tag{2}$$

Space charge fields

The beam current density is:

$$\mathbf{J}\,(x, y, z) = J(r)\mathbf{u}_z \tag{3}$$

where $\mathbf{u}_z$ is the unitary vector of the beam propagation.
If the particles of the beam have the same longitudinal speed: $\mathbf{v}_z = \beta_z c\,\mathbf{u}_z$, we have:

$$\mathbf{J}\,(r) = \rho(r)\,\beta_z c\,\mathbf{u}_z \tag{4}$$

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For symmetry reason, the magnetic field has only an **azimuthal component** B_θ. Using the integral form of the Ampere's law over a cylinder centred on the beam axis:

$$B_\theta(r) = \frac{\mu_0\,\beta_z c}{r}\int_0^r \rho(r)\,r\,dr \tag{5}$$

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From equations (2) and (5), it comes (as $c = \frac{1}{\sqrt{\epsilon_0\,\mu_0}}$):

$$B_\theta(r) = \frac{\beta_z}{c}E_r(r) \tag{6}$$

Space charge forces

The space charge fields exert a force **F** on a test particle at radius r:

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$$\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 \approx \beta_z^2 \tag{9}$$

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From equations (6) and (8), follows finally:

$$\boxed{F_r = qE_r\left(1 - \beta^2\right) = \frac{qE_r}{\gamma^2}} \tag{10}$$

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- In the above equation, the **1** represents the **electric force** and the $-\beta^2$ the **magnetic force**.
- The electric force is defocusing for the beam; the magnetic force is focusing.
- the ratio of magnetic to electric force, $-\beta^2$, is independent of the beam density distribution.
- For relativistic particles the beam magnetic force almost balance the electric force.
- For non-relativistic particles (like low energy ion beams) the space magnetic force is negligible: **the space charge has a defocusing effect!**

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Space charge forces – Uniform beam density

Example: uniform beam density of radius r_0 and intensity I

$$\rho(r) = \begin{cases} \rho_0 & \text{if } r \leq r_0, \\ 0 & \text{if } r > r_0. \end{cases} \tag{11}$$

The charge per unit length is :

$$\lambda = \rho_0 \pi r_0^2 \tag{12}$$

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So,

$$\rho_0 = \frac{I}{\beta c \pi r_0^2} \tag{14}$$

Space charge forces – Uniform beam density

From equation (2) and (14):

$$E_r(r) = \frac{I}{2\pi\epsilon_0\beta c\, r_0^2}\, r \qquad \text{if } r \leq r_0 \tag{15a}$$

$$E_r(r) = \frac{I}{2\pi\epsilon_0\beta c\, r} \qquad \text{if } r > r_0 \tag{15b}$$

Remarks

- The field is linear inside the beam.
- Outside of the beam, it varies according to $1/r$.

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Similarly, from equation (5) and (13):

$$B_\theta(r) = \mu_0 \frac{I}{2\pi r_0^2}\, r \qquad \text{if } r \leq r_0 \tag{16a}$$

$$B_\theta(r) = \mu_0 \frac{I}{2\pi r} \qquad \text{if } r > r_0 \tag{16b}$$

Space charge forces – Gaussian beam density

Example: gaussian beam density of standard deviation σ_r

$$\rho(r) = \rho_{0g} \exp\left(\frac{-r^2}{2\sigma_r^2}\right) \tag{17}$$

The charge per unit length is:

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The space charge electric field is:

$$E_r(r) = \frac{\rho_{0g}}{\epsilon_0}\frac{\sigma_r^2}{r}\left(1 - \exp\left(\frac{-r^2}{2\sigma_r^2}\right)\right) \tag{19}$$

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Remarks

- The field is **non linear** inside the beam.
- Far from the beam (several σ_r), it varies according to $1/r$.

Space charge expansion in a drift

Consider a particle (charge q, mass m_0) beam of current I, propagating at speed $v = \beta c$ in a drift region, with the following hypothesis:

- the beam has cylindrical symmetry and a radius r_0
- the beam is paraxial ($\beta_r \ll \beta_z$)
- the beam has an emittance equal to 0
- the beam density is uniform

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The second Newton's law for the transverse motion of the beam particles gives:

$$\frac{d(m_0\gamma\beta_r c)}{dt} = m_0\gamma\frac{d^2 r}{dt^2} = qE_r(r) - q\beta c B_\theta(r) \tag{20}$$

Space charge expansion in a drift

Using (15a) for E_r and (16a) for B_θ in (20):

$$m_0\gamma\frac{d^2r}{dt^2} = \frac{qIr}{2\pi\epsilon_0 r_0^2\beta c}(1-\beta^2) \tag{21}$$

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Equation (21) becomes:

$$\boxed{\frac{d^2r}{dz^2} = \frac{qIr}{2\pi\epsilon_0 r_0^2 m_0 c^3\beta^3\gamma^3}} \tag{23}$$

Space charge parameter

The **generalized perveance** K, a dimensionless parameter, is defined by:

$$\boxed{K = \frac{qI}{2\pi\epsilon_0 m_0 c^3 \beta^3 \gamma^3}} \tag{24}$$

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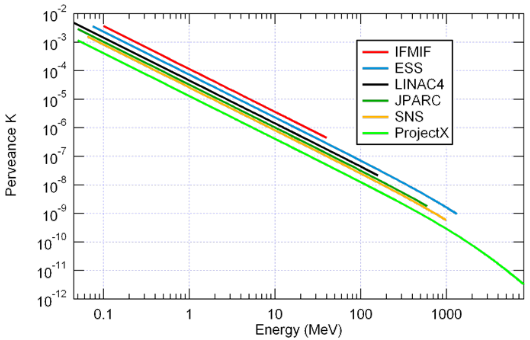
The perveance refers to the magnitude of space charge effect in a beam.



Beam radius equation

The equation for the particle trajectories (23) can be reduced to the form:

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In the case of a laminar beam, the trajectories of all particles are similar and particularly, the particle at $r = r_0$ will always stay at the beam boundary. Considering $r = r_0 = r_{env}$, the equation of the beam radius in a drift space can be written:

$$\boxed{\frac{d^2r_{env}}{dz^2} = \frac{K}{r_{env}}} \tag{26}$$

Outline

RMS quantities

$\langle A \rangle$ represents the mean of the quantity A over the beam particle distribution.

$$\text{RMS size :} \quad \tilde{x} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{27}$$

$$\text{RMS divergence :} \quad \tilde{x'} = \sqrt{\langle x'^2 \rangle - \langle x' \rangle^2} \tag{28}$$

$$\text{RMS emittance :} \quad \tilde{\epsilon}_x = \sqrt{\tilde{x}^2 \tilde{x'}^2 - \langle (x - \langle x \rangle)(x' - \langle x' \rangle) \rangle^2} \tag{29}$$

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$$\beta_x = \frac{\tilde{\epsilon}_x}{\tilde{x}^2} \tag{30}$$

$$\gamma_x = \frac{\tilde{\epsilon}_x}{\tilde{x'}^2} \tag{31}$$

$$\alpha_x = \frac{\tilde{\epsilon}_x}{\langle (x - \langle x \rangle)(x' - \langle x' \rangle) \rangle} \tag{32}$$

RMS envelope equation

Consider a beam moving in the s direction, where individual particles satisfy the equation of motion

$$x'' + \kappa(s)x - F_s = 0 \tag{33}$$

where $\kappa(s)x$ represents a **linear external focusing force** (quadrupole for instance $\kappa = qB/\gamma ma\beta c$) and F_s is a **space charge force term** (in general not linear).

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To simplify it is assumed that the beam is centred on axis with no divergence, so : $\langle x\rangle = 0$ and $\langle x'\rangle = 0$ (i.e. $\tilde{x}^2 = \langle x^2\rangle \equiv \overline{x^2}$) .

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$$\frac{d\overline{x^2}}{ds} = 2\overline{xx'} \tag{34}$$

RMS envelope equation

$$\frac{d\overline{xx'}}{ds} = \overline{x'^2} + \overline{xx''}$$
$$= \overline{x'^2} - \kappa(s)\overline{x^2} - \overline{xF_s} \tag{35}$$

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Differentiating two times (27) (using (34)):

$$\tilde{x}'' = \frac{\overline{x''} + \overline{x^2}}{\tilde{x}} - \frac{\overline{xx'}}{\tilde{x}^3} \tag{36}$$

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Using (35) and (29), we have finally the equation of motion of the RMS beam size:

$$\boxed{\tilde{x}'' + \kappa(s)\tilde{x} - \frac{\tilde{\epsilon}_x{}^2}{\tilde{x}^3} - \frac{\overline{xF_s}}{\tilde{x}} = 0} \tag{37}$$

RMS envelope equation – elliptical continuous beam

Example: elliptical continuous beam of uniform density

$$\rho(r) = \begin{cases} \rho_0 & \text{if } \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1, \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

As the distribution is uniform, the semi-axes of the ellipse, r_x and r_y are related to the RMS beam sizes: $r_x = 2\tilde{x}$ and $r_y = 2\tilde{y}$.

$$\tilde{x}'' + \kappa_x(s)\tilde{x} - \frac{\tilde{\epsilon_x}^2}{\tilde{x}^3} - \frac{K}{2(\tilde{x}+\tilde{y})} = 0 \tag{39}$$

$$\tilde{y}'' + \kappa_y(s)\tilde{y} - \frac{\tilde{\epsilon_y}^2}{\tilde{y}^3} - \frac{K}{2(\tilde{x}+\tilde{y})} = 0 \tag{40}$$

Equation of motion of the RMS beam size

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- **Focusing term**
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The equations (39) and (40) are valid for **all density distributions** with elliptical symmetry.
So, it's possible to replace the beam distribution (not known *a priori*) by an **equivalent uniform beam** with the same intensity and second moments.

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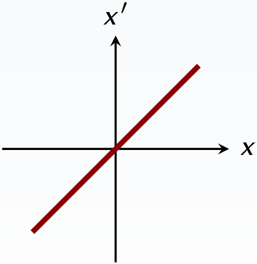
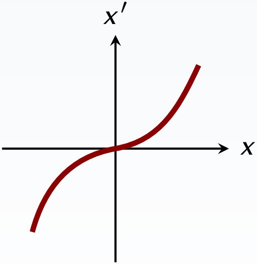


$x' = C\,x$

if $n = 1 \Rightarrow \tilde{\epsilon}_x = 0$



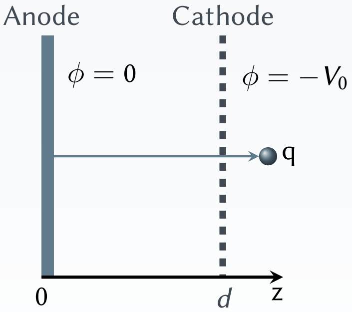
$x' = C\,x^n$

if $n \neq 1 \Rightarrow \tilde{\epsilon}_x \neq 0$

Outline

One dimension ion extraction gap

Consider particles of charge q and mass m_0 that are created at rest at z=0. A potential difference of $-V_0$ is applied between two planar electrodes which are spaced by a distance d.



Extraction gap

One dimension ion extraction gap

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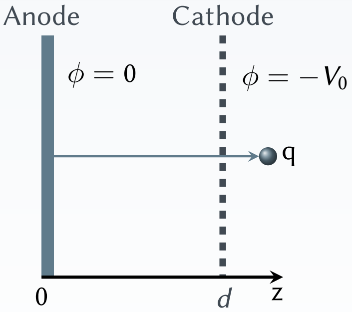
Hypothesis

- Particle motion is non-relativistic ($qV_0 \ll m_0 c^2$).
- The source on the left-hand boundary can supply an unlimited flux of particles.
- The transverse dimension of the gap is large compared with d.
- Particles flow continuously.



Extraction gap

Child–Langmuir law

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Using (45) and (46)

$$\rho(z) = \frac{J_0}{\sqrt{2q\phi(z)/m_0}} \tag{47}$$

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If we introduce the dimensionless variables $\zeta = z/d$ and $\Phi = -\phi/V_0$

$$\frac{d^2\Phi(z)}{d\zeta^2} = -\frac{\alpha}{\phi^{1/2}} \tag{50}$$

with

$$\alpha = \frac{J_0 d^2}{\epsilon_0 V_0\sqrt{2qV_0/m_0}} \tag{51}$$

Child–Langmuir law

Three boundary conditions are needed to integrate (50): $\Phi(0) = 0$, $\Phi(1) = 1$ and $d\Phi(0)/d\zeta = 0$. Multiplying both side of equation (50) by $\Phi' = d\Phi/d\zeta$ we can integrate and obtain

$$(\Phi')^2 = 4\alpha\sqrt{\Phi(\zeta)} \tag{52}$$

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In (53), the condition $\Phi(1) = 1$ implies $\alpha = \frac{4}{9}$. Substituting in (51):

$$\boxed{J_0 = \frac{4}{9}\epsilon_0\left(\frac{2q}{m_0}\right)^{1/2}\frac{V_0^{3/2}}{d^2}} \tag{55}$$

Child–Langmuir law

- The Child–Langmuir law represents **the maximum current density** that can be achieved in the diode by increasing the ion supply by the anode. It's a **space-charge limitation**.
- For a given gap voltage and geometry, the current density is proportional to $\sqrt{q/m_0}$.

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- The possible current density of electrons is **$\sim$43 times higher** than that of protons.
- For ions:

$$J_0 = 5.44 \times 10^{-8} \sqrt{Z/A} V_0^{3/2} / d^2$$

- For electrons:

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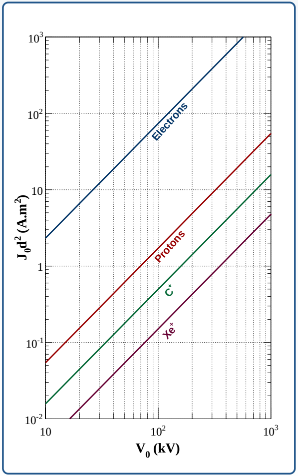
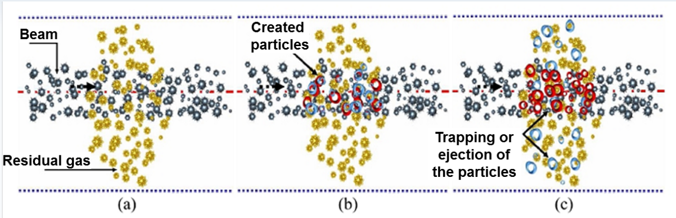


J_0d^2 versus V_0

Outline

The space charge compensation (SCC) principle



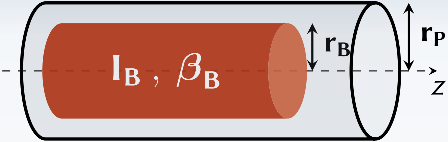
Example

We consider a proton beam propagating through a H_2 residual gas. It induces a production of pairs e^-/H_2^+ by ionization.

$$p + H_2 \rightarrow p + e^- + H_2^+$$

We assume that $n_{gas}/n_{beam} \ll 1$, with n_{gas} and n_{beam} the gas and beam density.

Space charge compensation degree



For a uniform cylindrical beam the electric field is (equations (15a) & (15b)):

$$E_r(r) = \frac{Ir}{2\pi\epsilon_0\beta c\, r_0^2} \qquad \text{if } r \leq r_0$$

$$E_r(r) = \frac{I}{2\pi\epsilon_0\beta c\, r} \qquad \text{if } r > r_0$$

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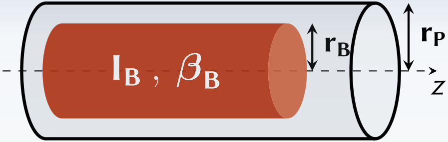

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By integrating these equations with the boundary condition, $\phi(r_p) = 0$

$$\phi(r) = \frac{I_B}{4\pi\epsilon_0\beta_B c}\left(1 + 2\ln\frac{r_P}{r_B} - \frac{r}{r_B^2}\right) \qquad \text{if } r \leq r_B$$

$$\phi(r) = \frac{I_B}{2\pi\epsilon_0\beta_B c}\ln\frac{r_P}{r} \qquad \text{if } r_B \leq r \leq r_P$$

Space charge compensation degree

The potential well (i.e. potential on the beam axis, $r = 0$) created by a uniform beam, without space charge compensation, is given by:

$$\boxed{\phi_0 = \frac{I_B}{4\pi\varepsilon_0\beta_B c}\left(1 + 2\ln\left(\frac{r_P}{r_B}\right)\right)} \tag{58}$$

where I_B and β_B are respectively the intensity and the reduced speed of the beam.

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where I_B and β_B are respectively the intensity and the reduced speed of the beam.
A more elaborated expression of the potential on axis that takes into account the collisional effects in the beam as well as the space charge compensation can be found in reference [Soloshenko, 1999].

Soloshenko, I. (1999).
Space charge compensation of technological ion beams.
Plasma Science, IEEE Transactions on, 27(4):1097 –1100.

Space charge compensation degree

If ϕ_c and ϕ_0 are respectively the potential wells (i.e. potential on the beam axis) of the compensated and uncompensated beam, the **space charge compensation degree** is then given by:

$$\boxed{\eta = 1 - \frac{\phi_c}{\phi_0}} \tag{59}$$

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The space charge compensation degree for the 75 keV – 130 mA proton beam of the LEDA has been measured [Ferdinand et al., 1997]:

$$\boxed{95\% < \eta < 99\%}$$

Ferdinand, R., Sherman, J., Stevens Jr., R. R., and Zaugg, T. (1997).
Space-charge neutralization measurement of a 75-kev, 130-ma hydrogen-ion beam.
In *Proceedings of PAC'97*, Vancouver, Canada.

Space charge compensation transient time

The characteristic **space charge compensation transient time**, τ, can be determined by considering the time it takes for a particle of the beam to produce a neutralizing particle on the residual gas.

$$\boxed{\tau = \frac{1}{\sigma_{ionis.} n_g \beta_B c}} \tag{60}$$

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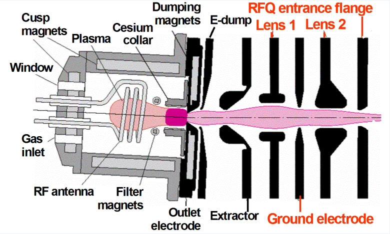
The space charge compensation transient time transient time for a 95 keV proton beam propagating in H_2 gas with a pressure of 5×10^{-5} hPa is:

$$\boxed{\boldsymbol{\tau = 15\ \mu s}}$$

The SNS electrostatic LEBT *(Courtesy of M. Stockli, SNS)*



[Keller et al., 2002]

- Two lenses, 12 cm long LEBT.
- Lens 2 is split into four quadrants to steer, chop, and blank the beam.
- Routinely produces the 38 mA LINAC beam current required for 1-1.4 MW operations.
- **Uncompensated space charge regime.**

Keller, R., Thomae, R., Stockli, M., and Welton, R. (2002).
Design, operational experiences and beam results obtained with the sns h[sup -] ion source and lebt at berkeley lab.
AIP Conference Proceedings, 639(1):47–60.

Magnetostatic LEBT – Uncompensated zones

Electric fields applied in the beam line **remove** the neutralising particles from the beam. Locally, the beam is **uncompensated**.

Example: RFQ injection region.

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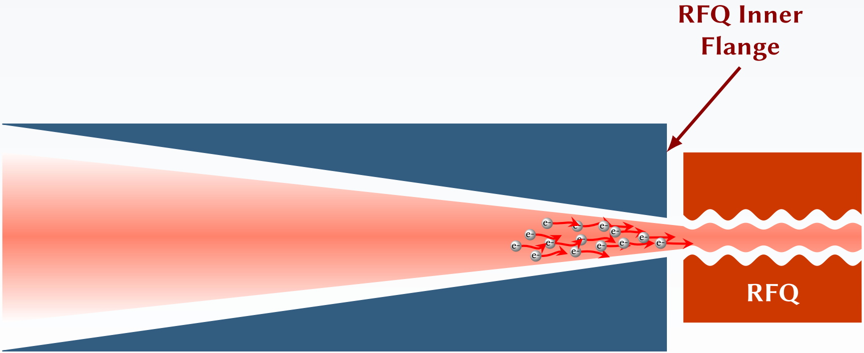
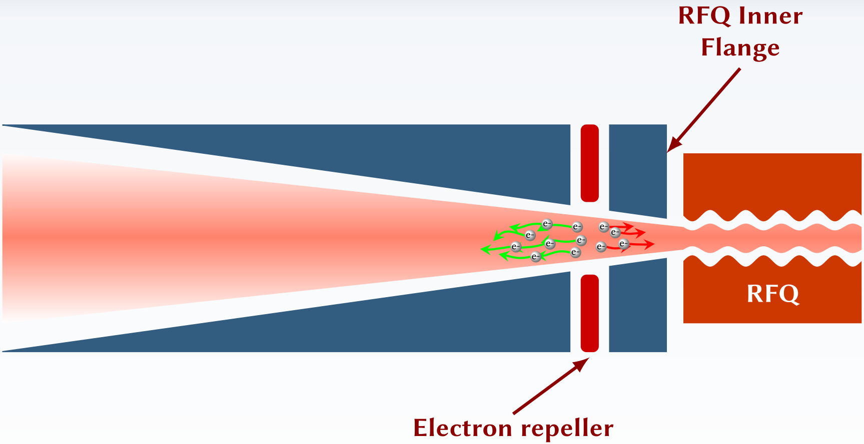


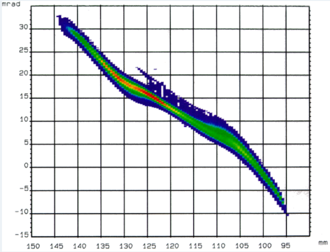
Magnetostatic LEBT – Role of electron repellers



A RFQ injection cone with an electron repeller

Space charge compensation – Measurements

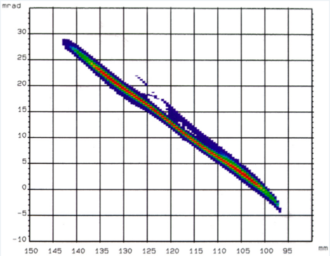
SILHI beam of 75 mA @ 95 keV. [Gobin et al., 1999]



Without ^{84}Kr injection

Pressure: 2.4×10^{-5} hPa.

ϵ_{RMS} = 0.335 π mm.mrad



With ^{84}Kr injection

Pressure: 4.6×10^{-5} hPa.

ϵ_{RMS} = 0.116 π mm.mrad

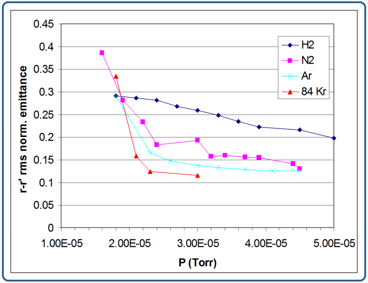
Gobin, R., Beauvais, P., Ferdinand, R., Leroy, P., Celona, L., Ciavola, G., and Gammino, S. (1999).

Improvement of beam emittance of the CEA high intensity proton source SILHI.

Review of Scientific Instruments, 70(6):2652–2654.

Space charge compensation – Measurements

[Gobin et al., 1999]



- In all the cases considered, a **decrease of beam emittance** has been observed with the **increase of beam line** pressure.
- Using ^{84}Kr gas addition a **decrease of a factor three** in beam emittance has been achieved.

Beam losses by charge exchange !

The gas injection in the beam line leads to beam losses by charge exchange.

Example

Kr pressure of 4×10^{-5} hPa, 2 m LEBT, H^+ beam @ 100 keV $\Rightarrow$ **loss rate of 2.4%**.

Outline

Codes for ion beam transport with space charge

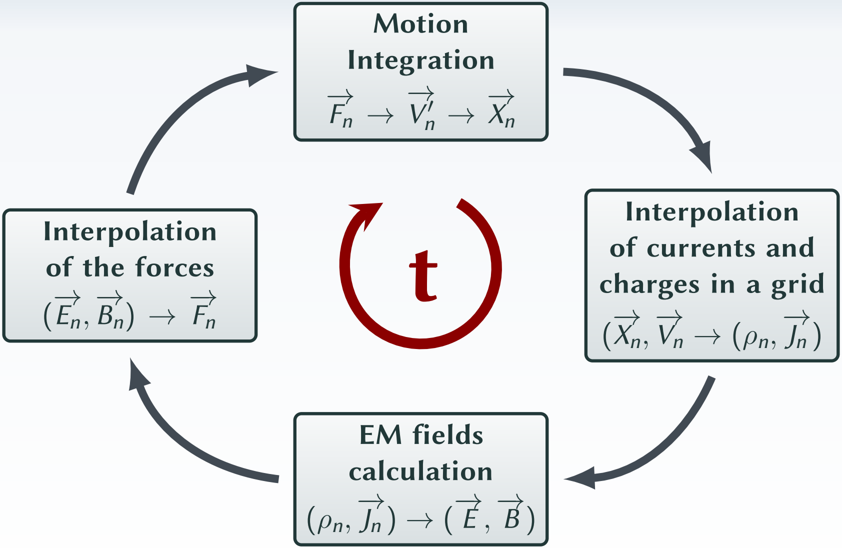
The beam is represented by N **macro-particles** that can be considered as a statistical sample of the beam with the same dynamics as the real particles. The macro-particles are transported through the accelerator step by step and at each time step ***dt***:

- the external forces acting on each macro-particle are calculated
- the space charge and the resulting forces are calculated
- the equation of motion is solved for each macro-particle

Principal methods to compute a beam space charge

- Particle-Particle Interaction (PPI) method
- Particle In Cells (PIC) method

Typical PIC code algorithm

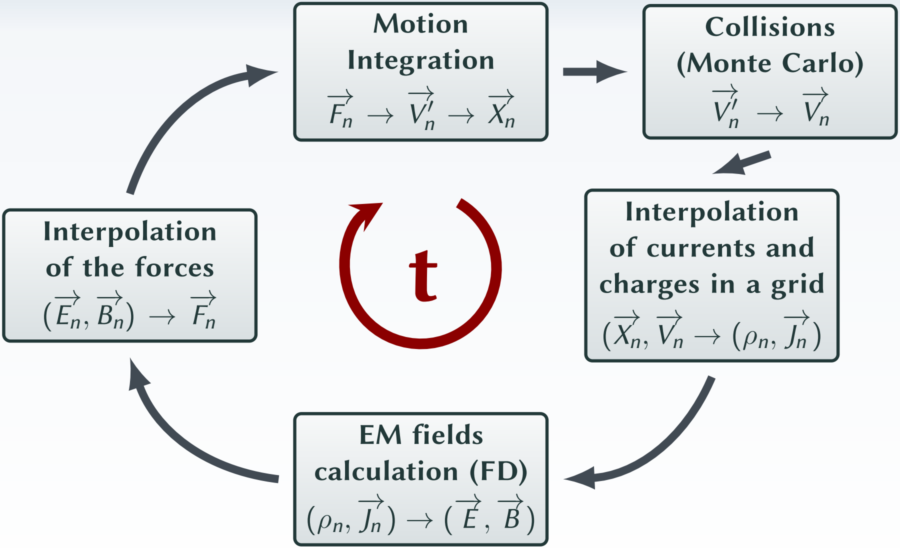


Transport with space charge compensation

- Tracking particle codes (Tracks, Parmilla, Trace3D, TraceWin ...) are used with a **constant space charge compensation degree** along the beam line (or slightly dependant of z).
- For more realistic beam transport simulations of high intensity ion beams at low energy, it is necessary to take into account **the space charge compensation of the beam on the residual gas**.
- For that, it is necessary to use a **self-consistent** code that simulate the beam interactions with the gas (ionization, neutralization, scattering) and the beam line elements (secondary emission). The dynamics of main beam is calculated **as well as the dynamics of the secondary particles**. Example of such codes: WARP [Grote et al., 2005]or SOLMAXP (developed by R. Duperrier at CEA-Saclay).

Grote, D. P., Friedman, A., Vay, J.-L., and Haber, I. (2005).
The warp code: Modeling high intensity ion beams.
AIP Conference Proceedings, 749(1):55–58.

SOLMAXP: basic algorithm



SOLMAXP, a PIC code for SCC simulations

SOLMAXP inputs

- Ion source output distributions (ex: H^+, H_2^+, H_3^+).
- Beam line external fields maps (solenoids, source extraction, RFQ cone injection trap...).
- Pressure and gas species in the beam line.

SOLMAXP outputs

- Particle distributions in the beam line (gas, electron, ions).
- Space charge potential map $\Rightarrow$ compute the space charge electric field map.

Outline

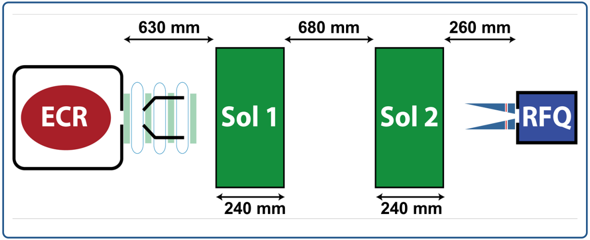
The IFMIF injector

Main parameters

- D^+ beam.
- Energy : 100 keV.
- Intensity : 140 mA.
- Final emittance : $\leqslant 0.25\ \pi$ mm.mrad

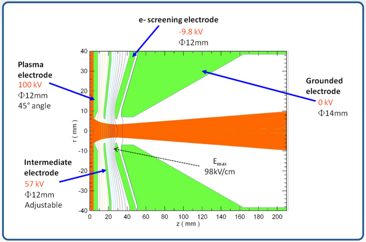
IFMIF injector

- SILHI-like source.
- 4 electrodes extraction system.
- LEBT with 2 solenoids.
- Kr injection in the LEBT for space charge compensation.



Total length: 2.05 m

Ion source extraction system

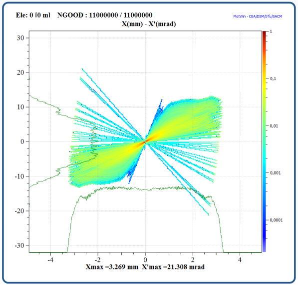


4 electrodes extraction system

- total extracted current : 175 mA (1560 A/m^2).
- Divergence minimization.
- Max. electric field: 98 kV.cm^{-1}.

Total Beam

- D$^+$: 141 mA – ε = 0.06 π mm.mrad
- D^{2+} : 26 mA
- D^{3+} : 9 mA



Simulation conditions

Simulation Conditions

- D^+, D_2^+, D_3^+ are transported.
- Residual pressure of D_2 gas (10^{-5} hPa) coming from the source & gas injection (Kr or D_2).
- Homogeneous pressure in the beam line.
- Ionisation of gas by incoming beams.

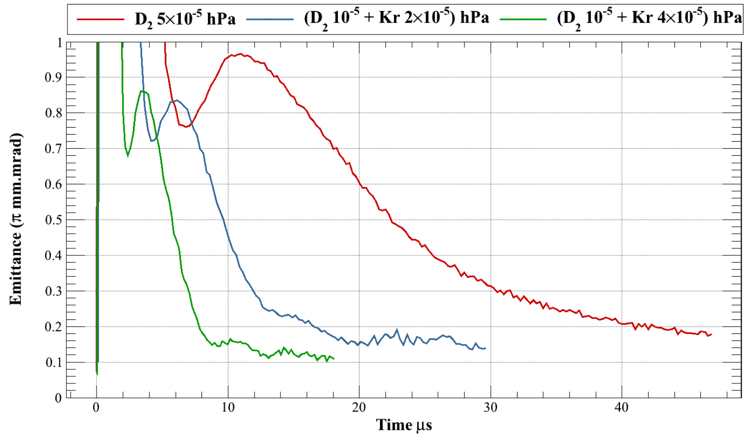
  $\Rightarrow$ $D^+ + D_2 \rightarrow D^+ + e^- + D_2^+$

  $\Rightarrow$ $D_2^+ + D_2 \rightarrow D_2^+ + e^- + D_2^+$

  $\Rightarrow$ $D_3^+ + D_2 \rightarrow D_3^+ + e^- + D_2^+$
- Ionisation of gas by created electrons

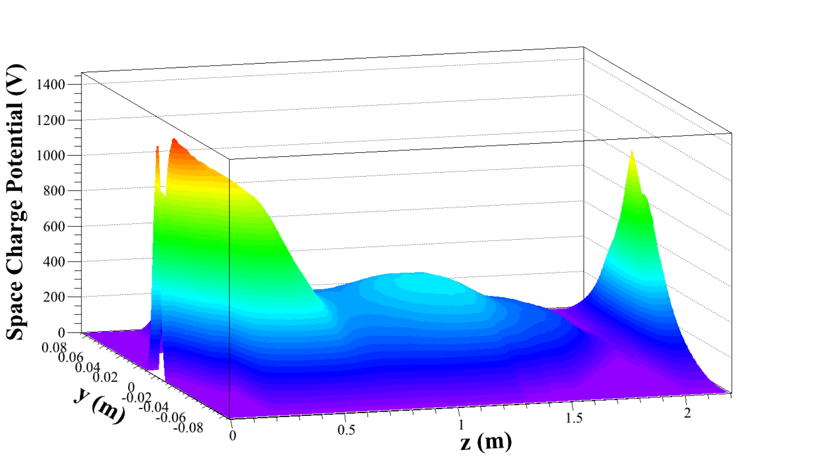
  $\Rightarrow$ $e^- + D_2 \rightarrow 2e^- + D_2^+$
- No secondary electron created by ion impact on beam pipe.
- No beams beam scattering on gas.

SCC transient



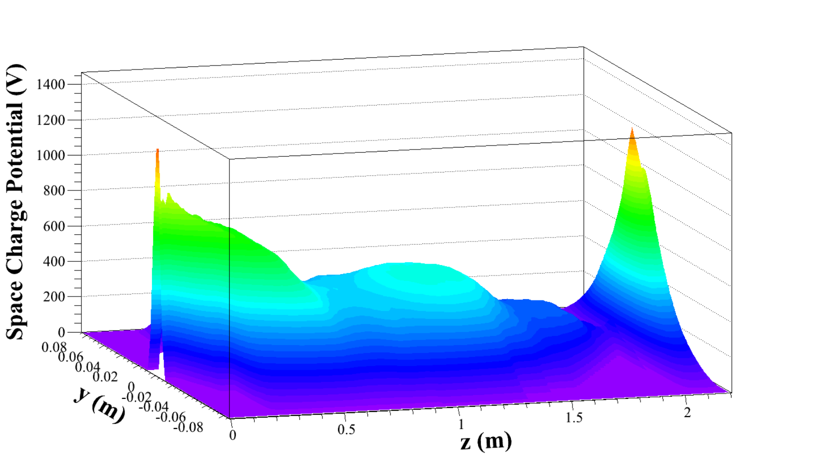
Steady state (for Kr 4×10^{-5} hPa) is reached after $\approx 15\ \mu s$

Space charge potential evolution



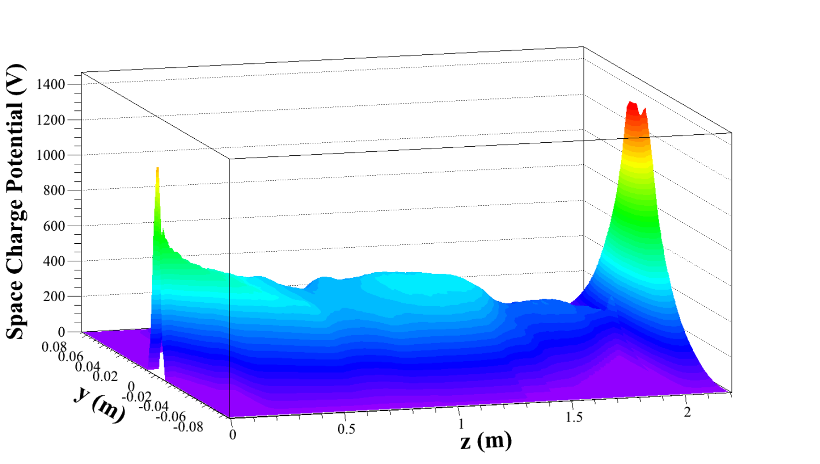
$\boxed{t = 2\ \mu s}$

Space charge potential evolution



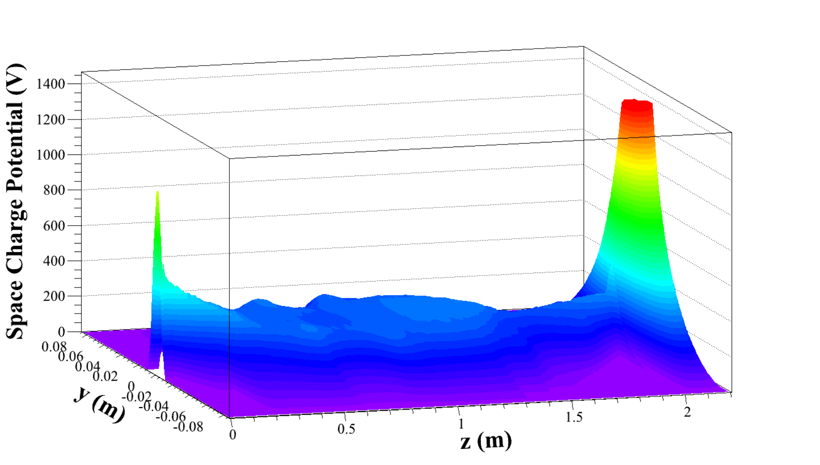
$t = 4\ \mu s$

Space charge potential evolution



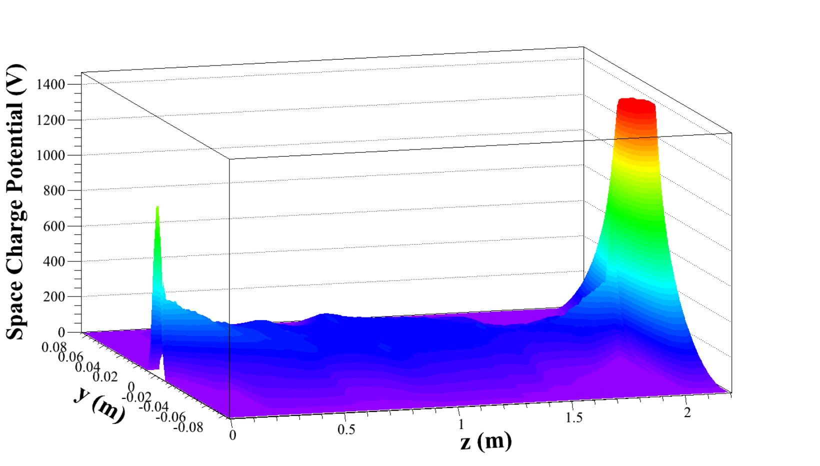
$\boxed{t = 6\ \mu s}$

Space charge potential evolution



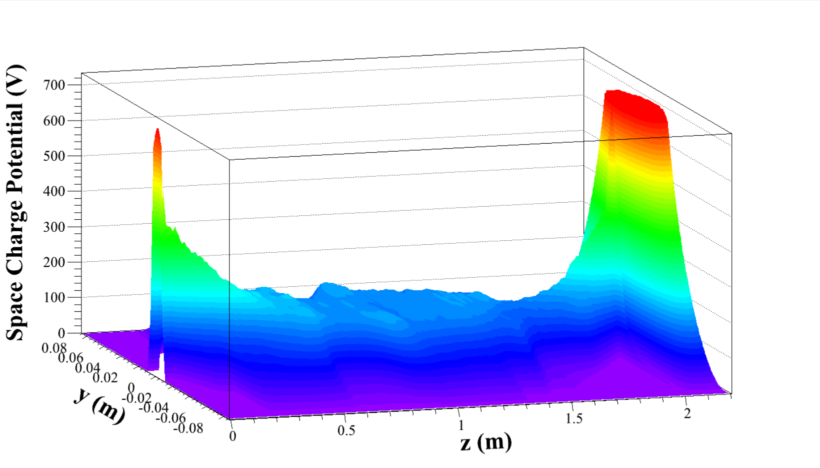
t = 8 μs

Space charge potential evolution



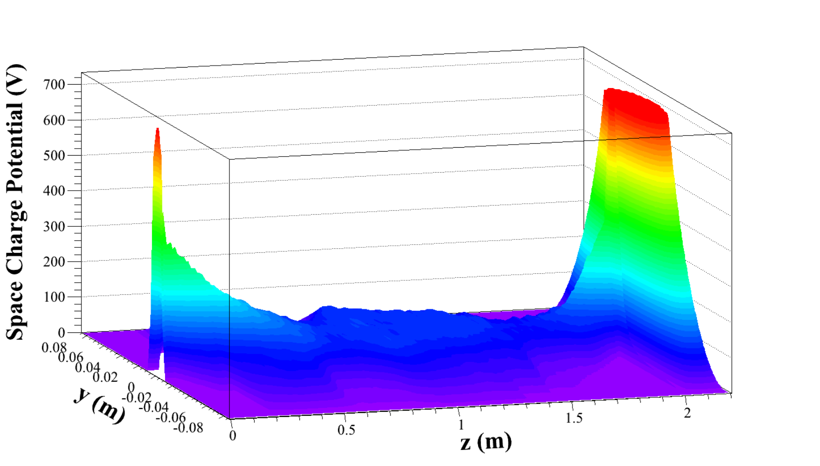
t = 10 μs

Space charge potential evolution



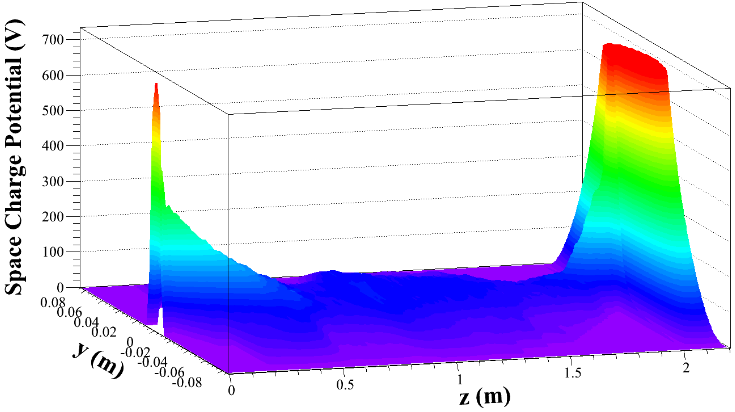
t = 10 μs – ! Cut at 700 V !

Space charge potential evolution



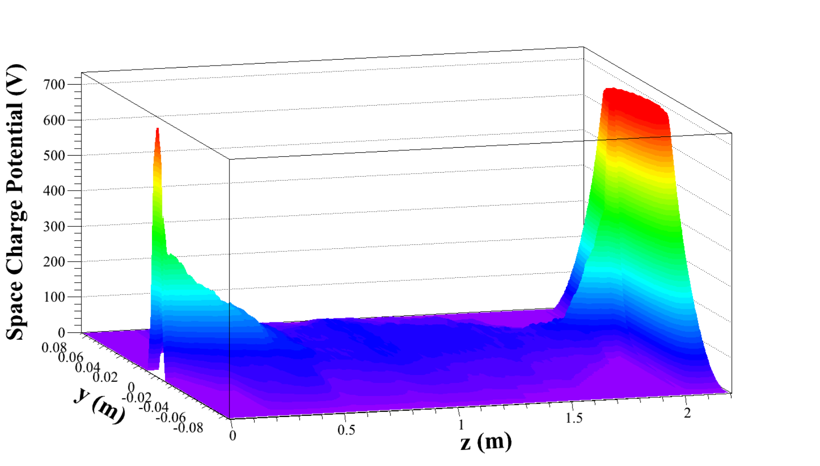
$\mathbf{t = 12\ \mu s}$

Space charge potential evolution



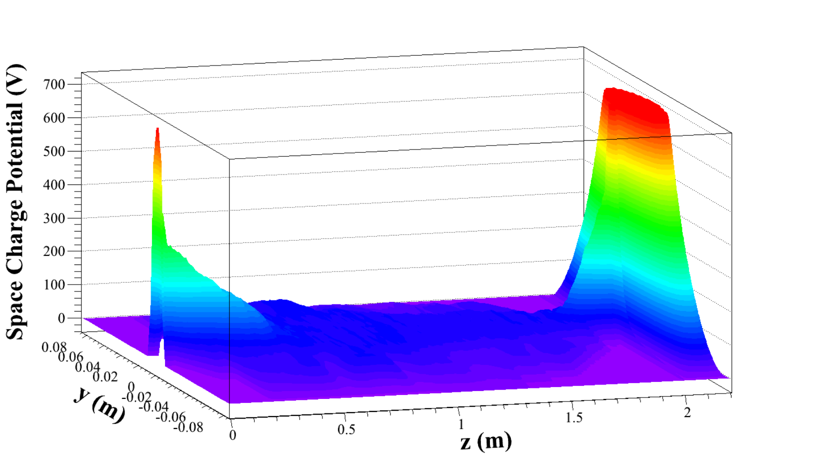
t = 14 μs

Space charge potential evolution



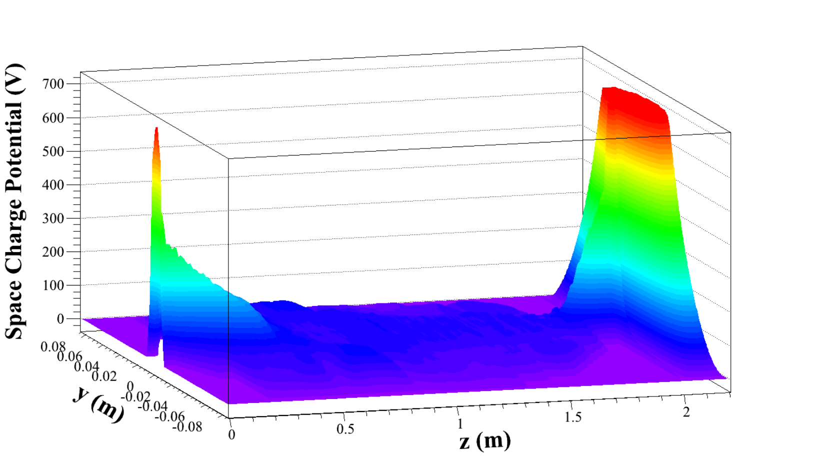
t = 16 μs

Space charge potential evolution



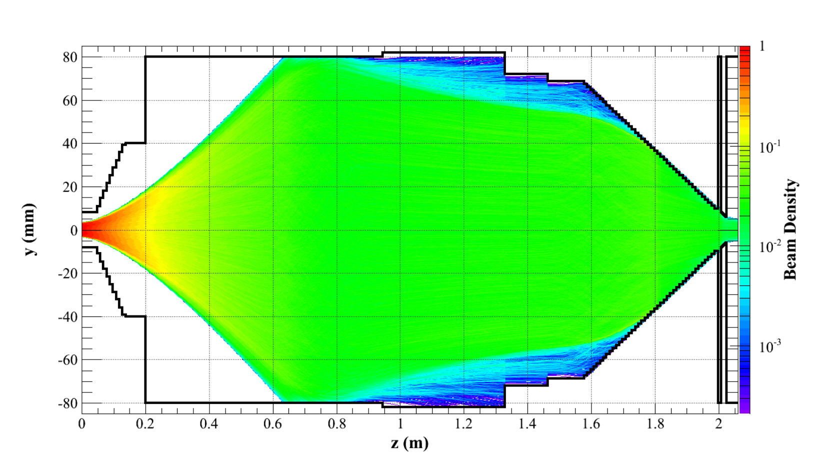
t = 18 μs

Space charge potential evolution



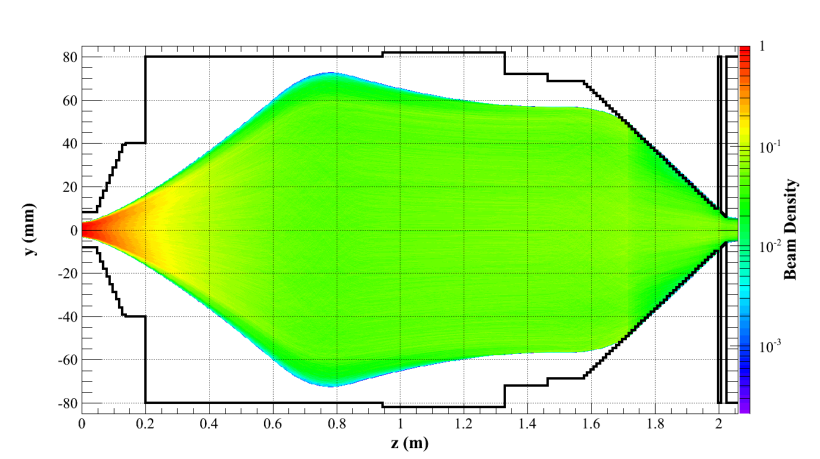
$t = 20\ \mu s$

Beam evolution



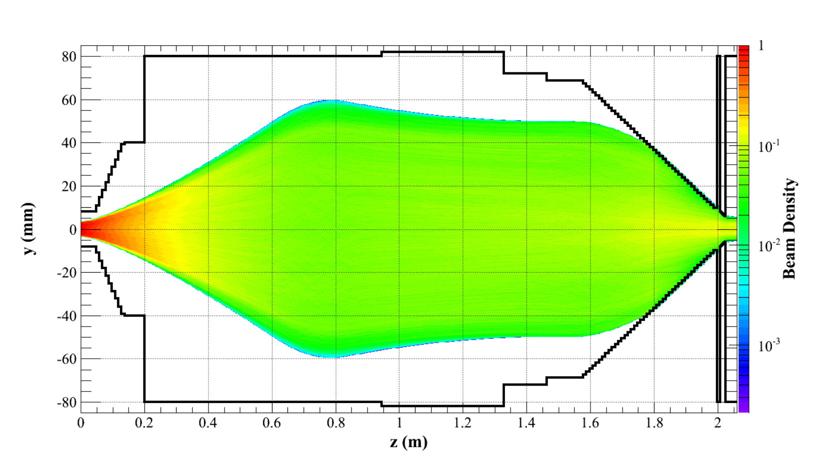
$t = 2\ \mu s$

Beam evolution



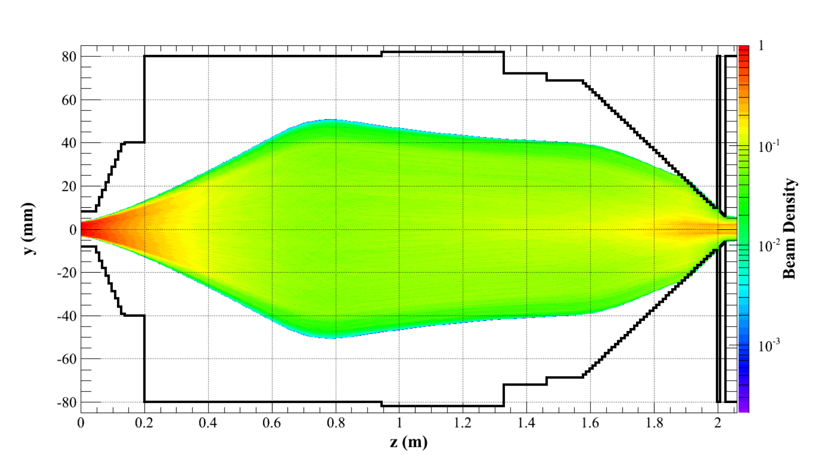
$\mathbf{t = 4\ \mu s}$

Beam evolution



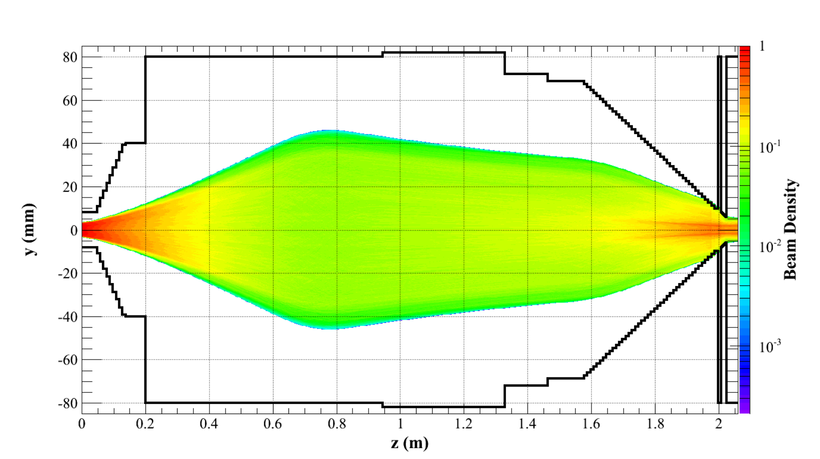
t = 6 μs

Beam evolution



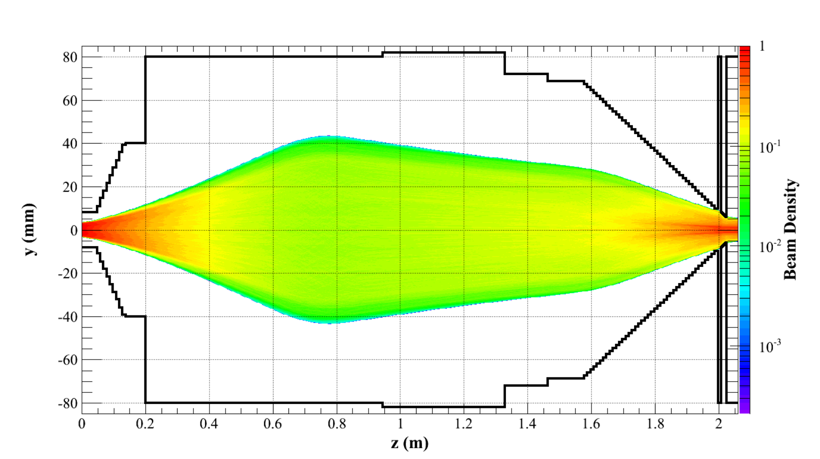
t = 8 μs

Beam evolution



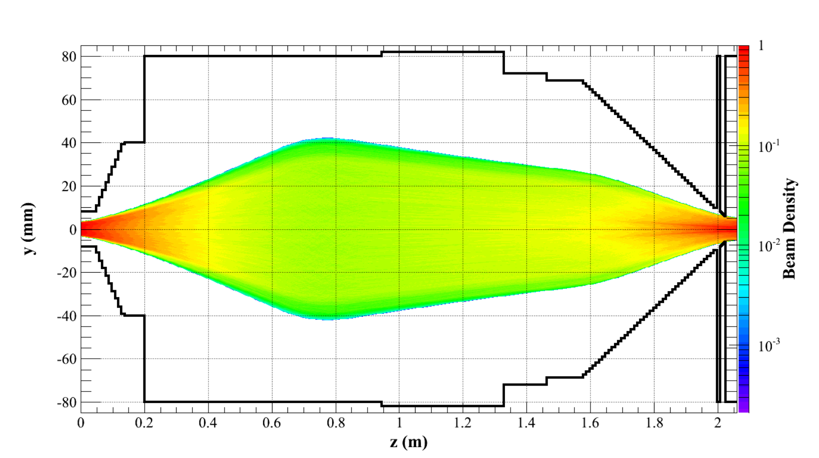
t = 10 μs

Beam evolution



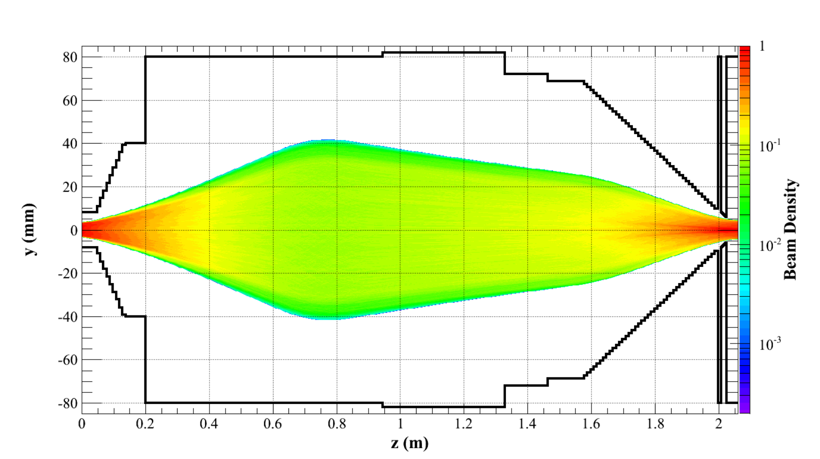
$t = 12\ \mu s$

Beam evolution



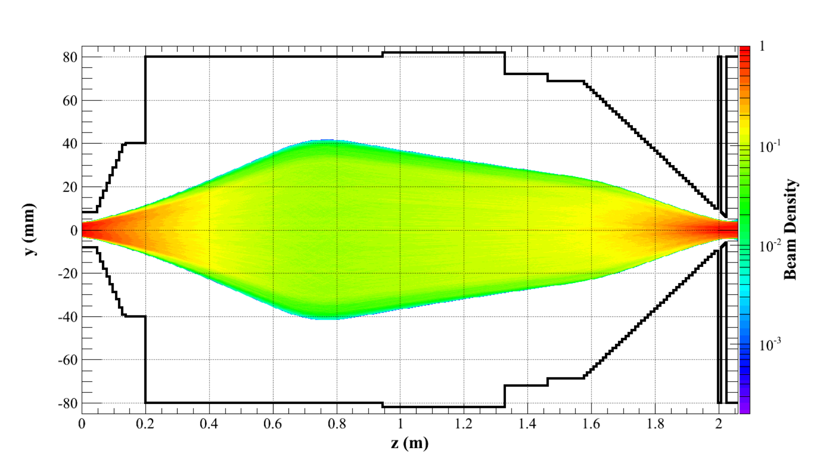
t = 14 μs

Beam evolution



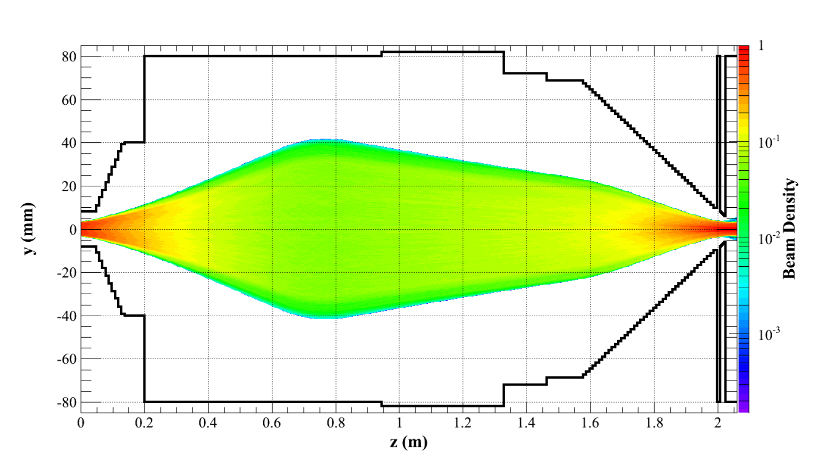
t = 16 μs

Beam evolution



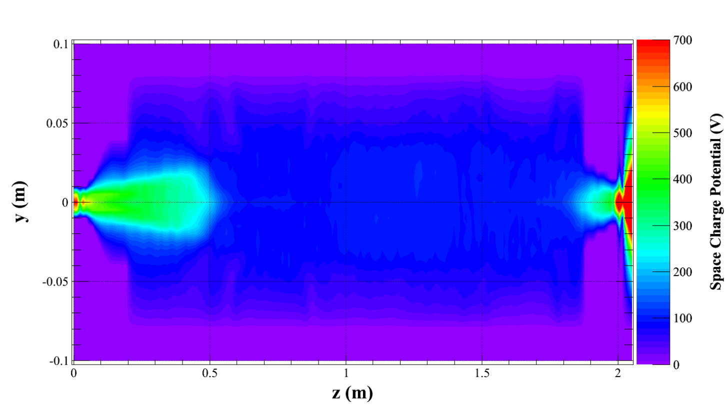
t = 18 μs

Beam evolution



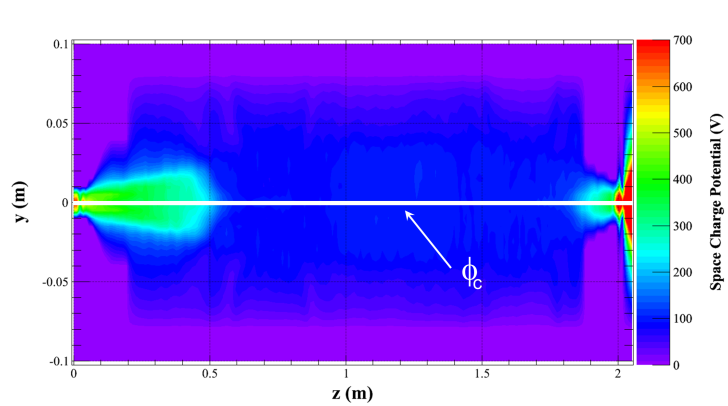
t = 20 μs

SCC degree



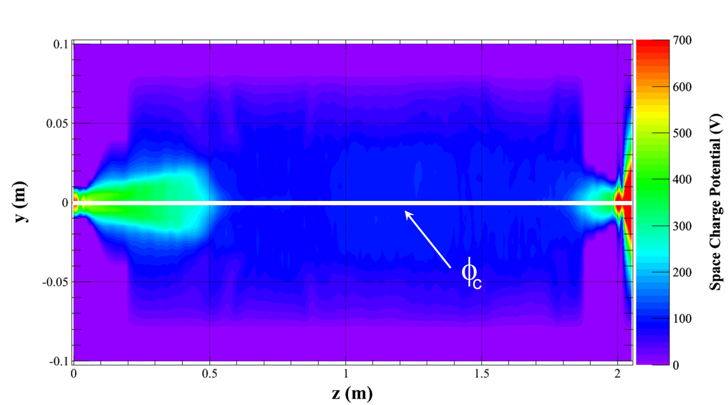
Two dimensions cut in the (z0y) plane of a space charge potential map

SCC degree



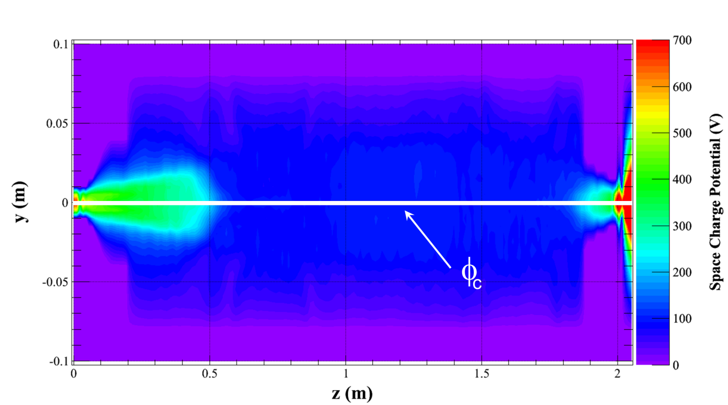
ϕ_c is the **potential on axis** of the **compensated** beam

SCC degree



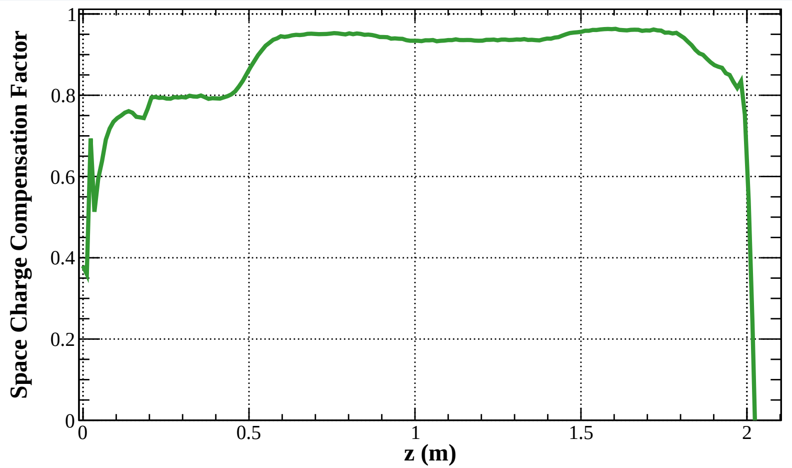
$\phi_0 = \frac{I_B}{4\pi\varepsilon_0\beta_B c}\left(1 + 2\ln\left(\frac{r_P}{r_B}\right)\right)$ is the is the **potential on axis** of the **uncompensated** beam

SCC degree



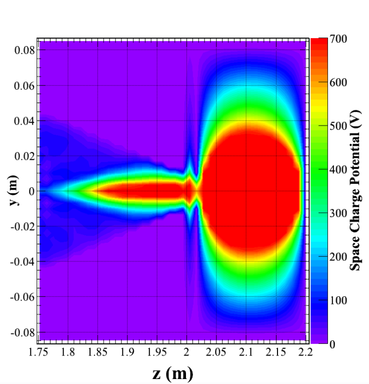
with $\eta = 1 - \frac{\phi_c}{\phi_0}$, we can compute the **space charge compensation degree** along the beam line.

SCC degree

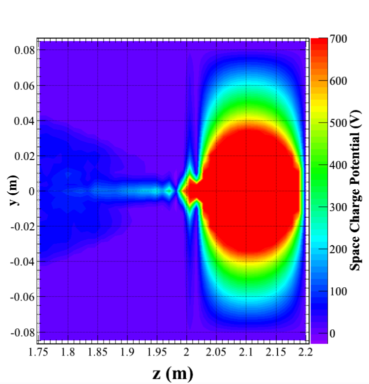


Space charge compensation degree in the IFMIF LEBT

Role of the e⁻ repeller

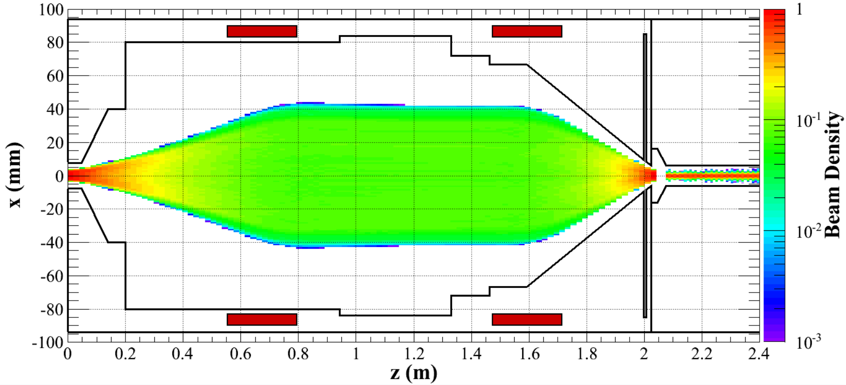


Without electron repeller



With electron repeller

Beam dynamics results



LEBT Output: ϵ_{RMS} = 0.16 π mm.mrad

IFMIF RFQ transmission : 96 %

Thank you for your attention!

For Further Reading

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M. Reiser
John Wiley & Sons, 2008.

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Charged particle beams
S. Humphries
Wiley, 1990.

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Space Charge
K. Schindl
CERN Accelerator School 2003: Intermediate Course on Accelerator, CERN-2006-002, 2006.

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A. BenIsmail, R. Duperrier, D. Uriot, N. Pichoff
Phys. Rev. ST Accel. Beams 10(7), 2007.