# Introduction to Plasma Physics I

## CERN Course on Ion Sources

Senec, Slovakia May 2012

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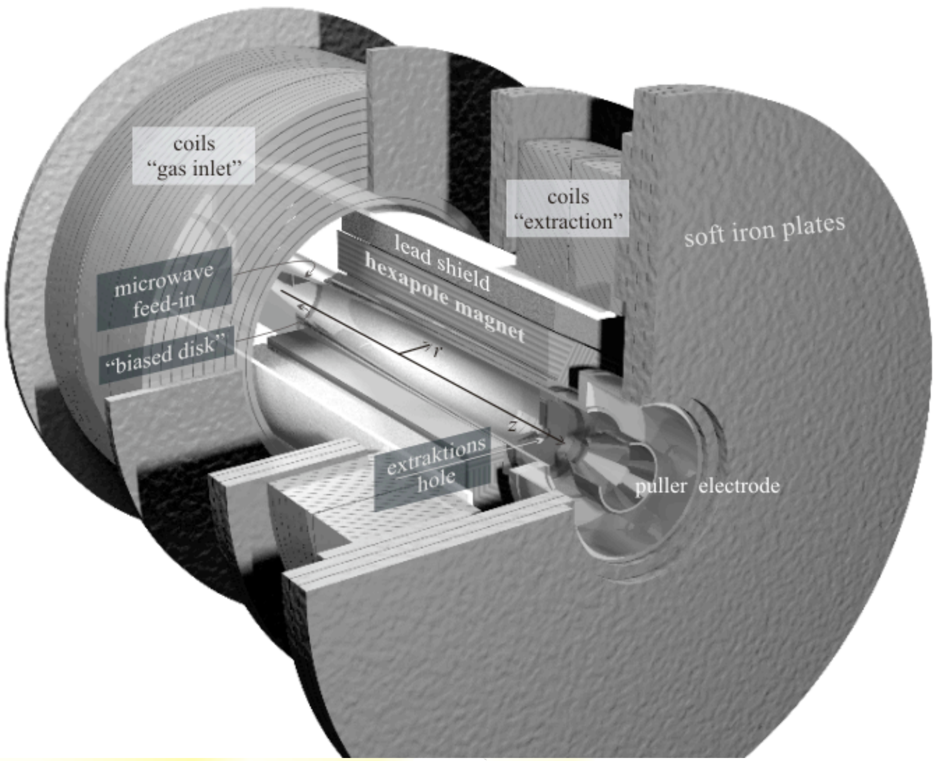
# Plasma


PLASMA IS IONIZED
MATTER
sometimes called
THE 4TH STATE OF MATTER

# ION SOURCES need COLD PLASMA




10 GHz ECRIS 2

# Main Constituents in Technical Cold Plasma

- positively (and negatively) charged ions
- electrons
- neutrals

Because of its charged components plasma is sensitive to electromagnetic fields

# Key Plasma Properties

- electric conductivity and a well defined local space potential
- quasineutrality within bulk
- screening electric fields by sheath formation (e.g. at walls and electrodes)
- collective phenomena (e.g. plasma waves, drift)

# Quantities Characterizing a Plasma

- temperatures of the constituents $T$
- number densities of the constituents $n$
- ionization degree $\eta$
- Debye length $\lambda_D$
- plasma frequency $\omega_{pl}$
- (plasma parameter $g$)

# Temperatures in Plasma

Even in so-called “cold“ plasma electrons have temperatures of the order of $10^4$ K and more, while ions and neutrals remain cold (~$10^3$ K). Thus “cold plasma” is better characterized as “low enthalpy plasma”, because it transfers little heat to its environment.

It has become international usage to use the symbol *T* not for the thermodynamic temperature measured in Kelvin, but instead for the characteristic energy $k_B T$ ($k_B$ Boltzmann constant) measured in eV. The “speaking” is: “The temperatures are measured in eV“

1 eV corresponds to $1.160 \cdot 10^4$ K

$10^4$ K correspond to 0.862 eV

# Number Density of Particles Type i

definition
$$n_i = \frac{\text{number of particles}}{\text{volume}},$$

units
$$[n_i] = 1\,\mathrm{cm}^{-3}, \text{ or } [n_i] = 1\,\mathrm{m}^{-3}.$$

$$10^{-6}\,\mathrm{cm}^{-3} = 1\,\mathrm{m}^{-3} \text{ or } 10^{6}\,\mathrm{m}^{-3} = 1\,\mathrm{cm}^{-3}$$

# Quasineutrality

For plasma with singly charged ions only

$$n_{\mathrm{i}} \approx n_{\mathrm{e}}$$

For plasma with multiply charged ions

(z charge number of the ions)

$$n_{\mathrm{e}} \approx \sum_{\mathrm{z}} \mathrm{z} \cdot n_{\mathrm{z}}$$

# Ionisation degree $\eta$ or $\eta'$

two different definitions are used

$$\eta = \sum_z n_z \Big/ \left( n_a + \sum_z n_z \right)$$

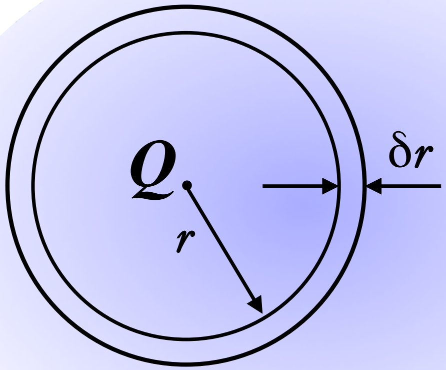
$n_a$ neutral particle density

$$(\eta' = \sum_z n_z \big/ n_a \qquad for\, n_a \to 0\; we\; get\; \eta' \to \infty)$$

$\eta << 1$ "weakly ionised plasma"

$\eta \approx 1$ "strongly" or "fully ionised plasma"

# Screening, Debye-Length

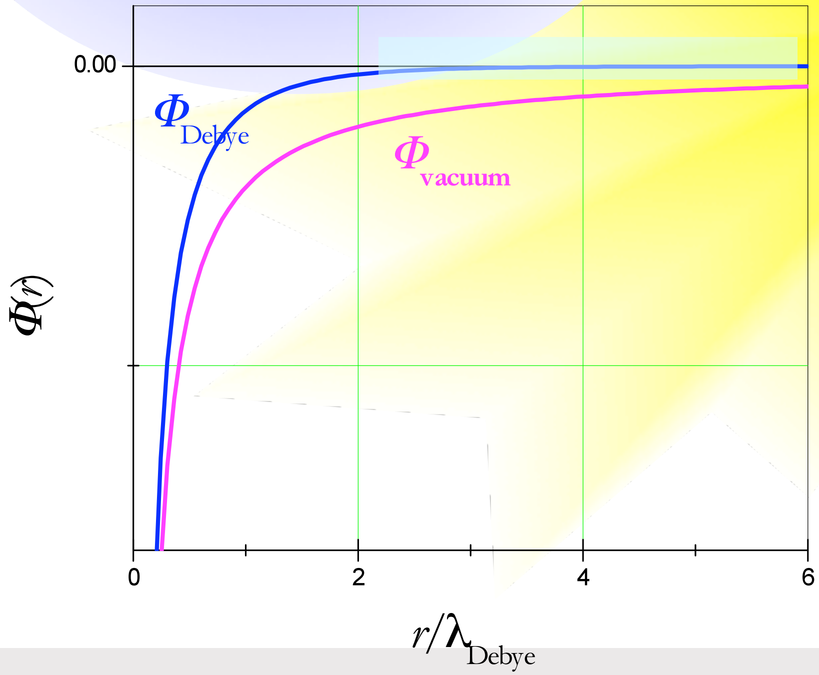



$$\Phi_{vacuum}(r) = Q/\ 4\pi\varepsilon_0\, r$$

$$\delta n = 4\pi n r^2 \delta r \Rightarrow \textit{screening}$$

$$\Phi_{Debye}(r) = (Q/\ 4\pi\varepsilon_0\, r) * \exp(-r/\lambda_{Debye})$$

*„Debye-Hückel-potential“*

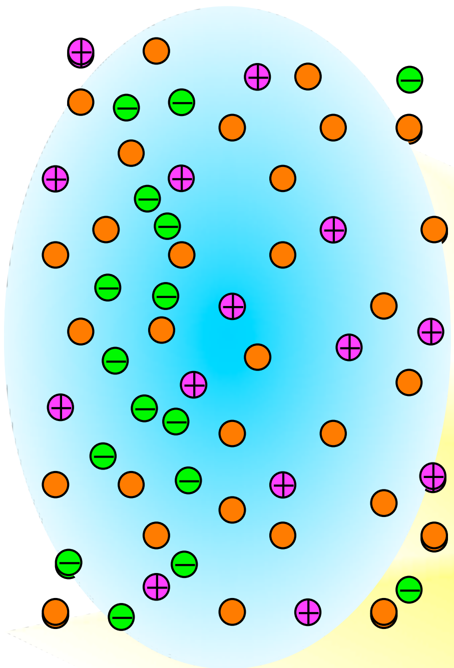



$$\lambda_{Debye} = \left( \frac{\varepsilon_0 k_B T_e}{e^2 n} \right)^{\frac{1}{2}}$$

*“Debye length”*

**P. Debye, E. Hückel 1923**

# Quasineutrality I



restoring electric field $E$ due to charge separation

$$E = \frac{en\Delta x}{\varepsilon_0}$$

Example: fluorescent tube

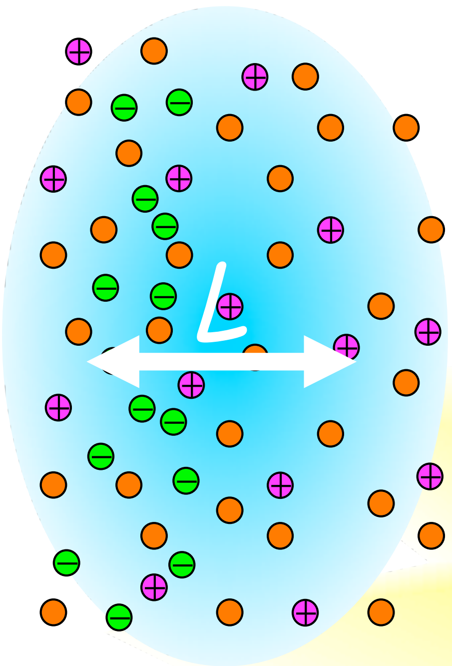
$n_e$ = $10^{16}$m$^{-3}$; $\Delta x$ = 1mm

$E_{max}$ = 180 kV/m; $U = (E\Delta x)$ = 180V

potential energy of an ion traversing a space charge sheath

$$W_{\text{pot}} = \int_0^{\Delta x} eE\mathrm{d}x = \frac{e^2 n_{\text{e}} (\Delta x)^2}{2\varepsilon_0}; \quad \tfrac{1}{2} k_{\text{B}} T = W_{\text{pot}} \Rightarrow \Delta x = \left( \frac{\varepsilon_0 k_B T}{e^2 n_{\text{e}}} \right)^{\frac{1}{2}} \equiv \lambda_{\text{D}}$$

# Quasineutrality II




What amount of deviation from neutrality can exist over a length $L$?

The increase in potential energy must not surmount $k_B T/2$

$$\tfrac{1}{2} k_{\mathrm{B}} T \approx \tfrac{1}{2} \frac{e^2 \Delta n L^2}{\varepsilon_0}$$

Substituting $k_B T$ by $\lambda_{\mathrm{D}}$ yields

$$\Delta n / n \approx \left(\lambda_{\mathrm{D}} / L\right)^2$$

Ionized gas is plasma only, if its extension is much larger than the Debye-length

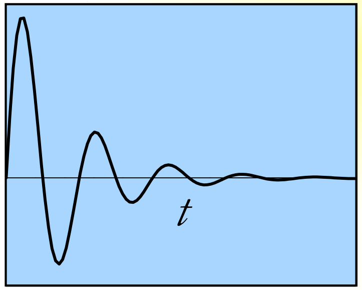
# Plasma Oscillations

Equation of motion for electrons moving under the action of the electric field generated by charge separation

$$F = eE = e^2nx/\varepsilon_0 = m_e \mathrm{d}^2x/\mathrm{d}t^2$$

This is the equation of a harmonic oscillator with the so-called electron plasma frequency as natural frequency

$$\omega_{pe} = \sqrt{e^2n/\varepsilon_0 m_e}$$




$$\omega_{pe}/\mathrm{s}^{-1} = 2\pi \cdot 8.98 \cdot \sqrt{n_e/\mathrm{m}^{-3}}$$

Typical value: for $n_e = 10^{17}\mathrm{m}^{-3}$ we have $\omega_{pe} = 2\pi * 2.8$ GHz

# Ion Plasma Frequency

If we replace electron charge and mass by the ion charge and mass we obain:

$$\omega_{\mathrm{pi}_z} = \sqrt{(ze)^2 n_z / \varepsilon_0 m_z}$$

This is the characteristic frequency of ion space charges (e.g. ion sheaths at walls).

❖ electromagnetic fields varying with frequencies well above $\omega_{pi}$ have almost no effect on the ion motion

❖ Typical values of $\omega_{pi}$ are in the MHz range. Thus ionic RF currents are very small when working with frequencies in the 10 MHz range or above. In this case ions follow the (average) DC fields only

# Plasma Kinetics

if $\lambda_n = 1/n_e^{1/3} >> \lambda_{\text{de Brogli}} = h/m_e v_{th};$

$$\left(\tfrac{1}{2} m_e v_{th}^2 = k_B T_e\right)$$

plasma can be treated by classical statistics, otherwise it is *degenerate.*

if $k_B T >> e^2/(4\pi\varepsilon_0\lambda_n) \Leftrightarrow \lambda_D >> \lambda_n \Leftrightarrow g \equiv 1/n_e\lambda_D^3 << 1$

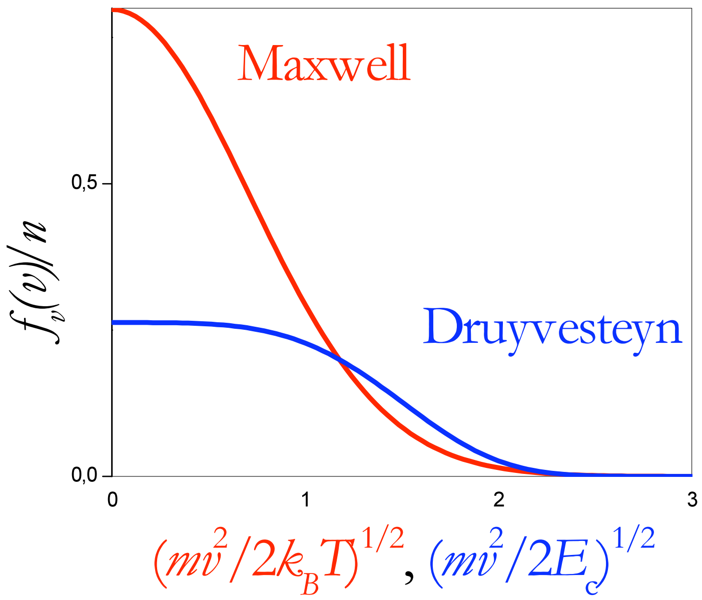
plasma can be treated as an ideal gas. Otherwise it is *strongly coupled* or *non-ideal.*

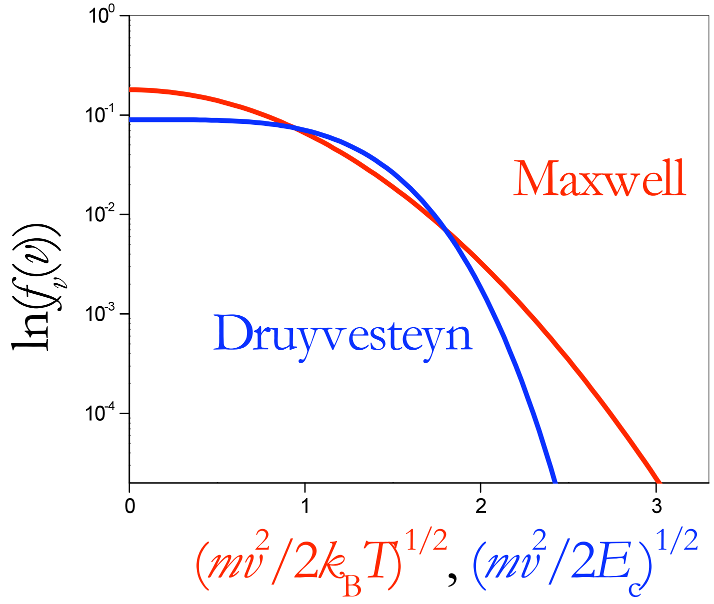
$$g \propto n_e^{1/2}/(k_B T_e)^{3/2}$$

nonideal plasmas are cold and dense

for ion source plasma we have usually $g$ « 1

# Electron Temperature and Distribution Functions




$$f_v(v) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \qquad \left\langle \frac{1}{2}mv^2 \right\rangle = \frac{3}{2}k_B T$$

$$f_v(v) = \frac{n}{\pi \cdot \Gamma(3/4)}\left(\frac{m}{2E_c}\right)^{3/2} \cdot \exp\left(-\left[\frac{mv^2}{2E_c}\right]^2\right) \qquad \left\langle \frac{1}{2}mv^2 \right\rangle = E_c \cdot \Gamma\left(\frac{5}{4}\right) / \Gamma\left(\frac{3}{4}\right)$$

# Electron Temperature and Distribution Functions

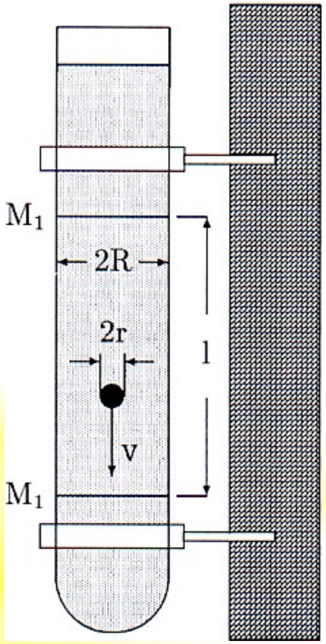
Jeans' Theorem: If the velocity distribution function can be expressed solely as a function of constants of motion, it is itself a constant of motion.

This theorem applies, for example, if there are no collisions or no transport.

In these cases the velocity distribution function can be written as a function of the **total** energy.

# Definition of the mobility *b*

constant velocity of a body moving in a viscous medium under the action of a constant external force (gravitation)




$$\vec{v} \equiv b\vec{F} \quad if \;\; \vec{F} = q\vec{E} \Rightarrow \vec{v} = bq\vec{E}$$

In weakly ionized plasma friction is due to electron (and ion) collisions with neutrals. For electrons we have:

$$b_e \equiv < 1/m_e\nu_{en} > = < 1/m_e\sigma_{en}v_e > \propto 1/\sqrt{m_e}$$

$$< 1/m_e\sigma_{en}v_e > \equiv \iiint_{\vec{v}_e}(1/m_e\sigma_{en}v_e)f(\vec{v}_e)d^3v_e \approx \frac{1}{\iiint_{\vec{v}_e}(m_e\sigma_{en}v_e)f(\vec{v}_e)d^3v_e}$$

# Diffusion

Diffusion is a consequence of the Brownian motion of electrons, ions and neutrals.

$$\overrightarrow{\Gamma_{\text{diff}}} = -D\nabla n \qquad \text{I. Fick's law}$$

$$D_{\text{e,i}} = b_{\text{e,i}} k_{\text{B}} T_{\text{e,i}} \quad \text{Einstein relation}$$

$$D_{\text{e,i}} \approx <\nu> \cdot <\lambda>^2 \propto 1/\sqrt{m_{\text{r e,i}}}; \qquad <\lambda> = <1/n\sigma>$$

$$m_{\text{r}} \text{ "reduced mass"}$$

(Collision frequency times square of mean free path between subsequent collisions)

# Diffusion



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$$\overrightarrow{\Gamma_{\mathrm{diff}}} = -D\nabla n \qquad \text{I. Fick's law}$$

$$D_{\mathrm{e,i}} = b_{\mathrm{e,i}} k_{\mathrm{B}} T_{\mathrm{e,i}} \propto 1/\sqrt{m_{\mathrm{r\,e,i}}}$$

$$\frac{D_{\mathrm{e}}}{D_{\mathrm{i}}} = \sqrt{\frac{m_{\mathrm{r\,i}}}{m_{\mathrm{e}}}}$$

# Sheath Formation in Front of a Wall

walls constitute sinks for charged particles, which are there either discharged (ions) or absorbed (ions and electrons).

As a consequence charged particles are transported from plasma to enclosing walls - generally at different speeds.

# Sheath Formation

## weakly ionized plasma

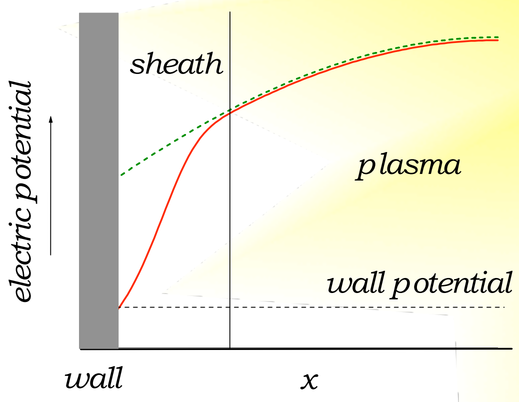
In weakly ionized, nonmagnetized plasma electron transport is fast, ion transport is slow.

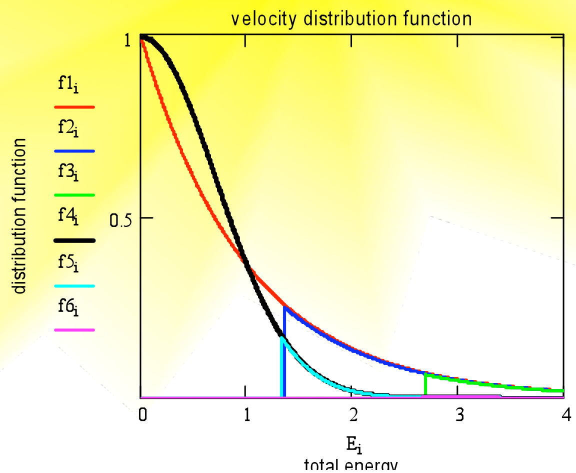
In the presence of a wall a small amount of positive space charge remains in plasma as a consequence. The plasma potential becomes **positive** with respect to enclosing walls, resp. the walls are **negatively charged**.

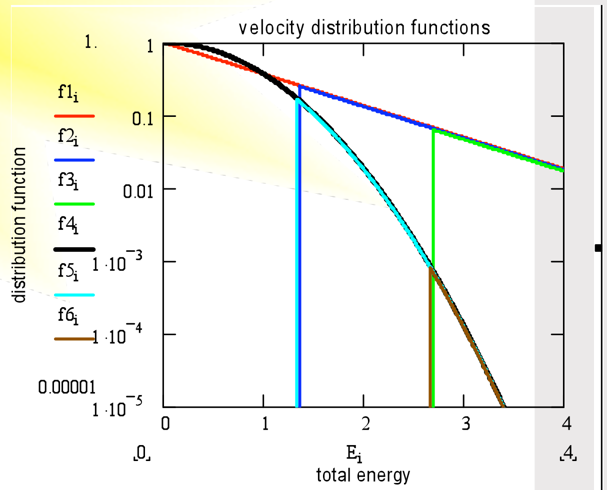
# Plasma-Sheath-Transition

## application of Jeans' Theorem

In the collisionfree case an isotropic velocity distribution function can be written as a function of the **total** energy, but is defined only for **positive kinetic energy**.

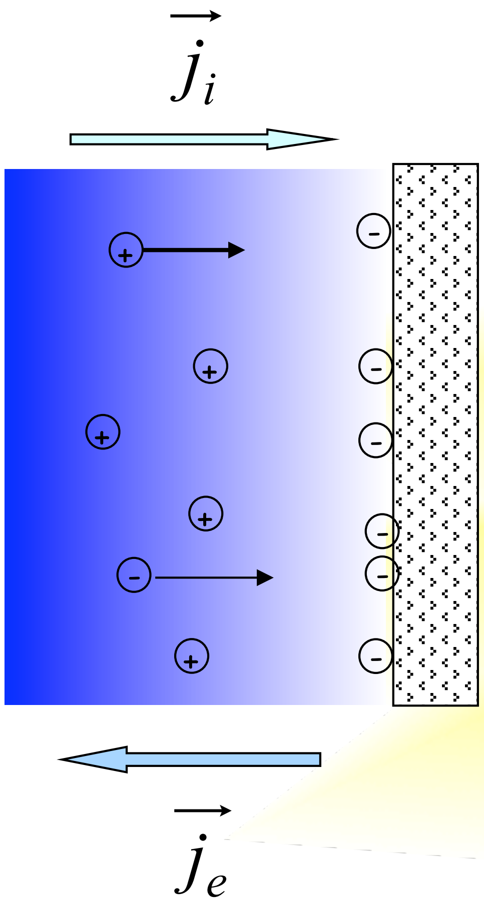


# Sheath Formation

## in weakly ionized plasma




### Charging of a dielectric wall

$$\vec{j} = \vec{j}_e + \vec{j}_i = 0$$

$$\left|\vec{j}_i\right| = \left|\vec{j}_e\right| = \left|\vec{j}_{e0}\right| \exp(-qU_{wall} / kT_e)$$

$$\text{when } U_{plasma} = 0$$

$$U_{wall} = -\frac{kT_e}{e} \ln\left(\frac{\left|\vec{j}_i\right|}{\left|\vec{j}_{e0}\right|}\right)$$

# Sheath Formation

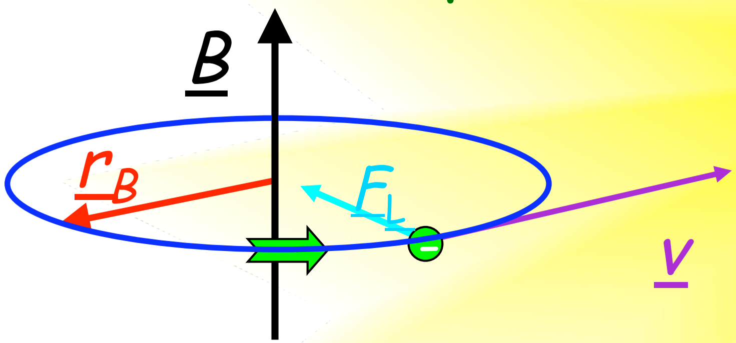
## weakly ionized plasma

At insulating walls the negative charge on a wall locally regulates itself in such a way that the ion and electron wall **current densities** become equal.

At metallic walls such a condition may hold for the **total currents** only. This is important in case of microwave discharges with external magnetic field

# Gyration of ions and electrons under the action of a static magnetic field

❖ A static magnetic field of induction $\underline{B}$ interacts with particles of mass $m$ and charge $q$ by the so-called Lorentz Force $\underline{F}_L$. A circular motion, a gyration is the consequence (non-relativistic case in plasma).




$$\overrightarrow{F_L} = q\vec{v} \times \vec{B}$$

The radius (cyclotron radius) $r_B$ of the circular trajectory is given by $r_B = mv/qB$

The corresponding cyclotron frequency $\omega_B$ does not depend on the particle velocity $v$: $\omega_B = qB/m$

# Particle Motion in Magnetized Plasma

$m\dot{\vec{v}} = q\vec{v} \times \vec{B} + \vec{F}$ vector equation of motion

$$\vec{F} \equiv \vec{F}_{\perp} + \vec{F}_{\parallel}, \quad \vec{v} \equiv \vec{v}_{\perp} + \vec{v}_{\parallel}$$

(components $\parallel$ and $\perp$ to $\vec{B}$)

$m\dot{\vec{v}}_{\perp} = q\vec{v}_{\perp} \times \vec{B} + \vec{F}_{\perp}$ $\vec{B}$ - dependent part

$m\dot{\vec{v}}_{\perp} = q\vec{v}_{\perp} \times \vec{B}$ homogeneous part (Gyration)

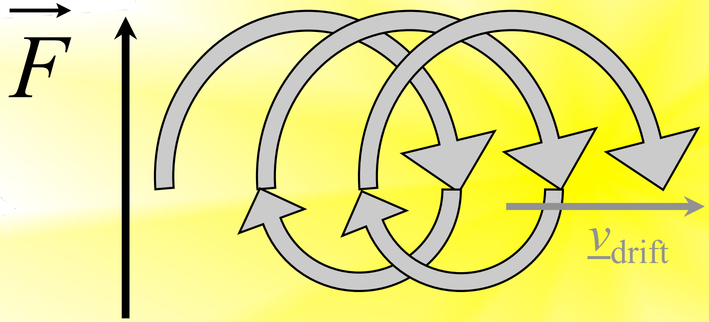
$-q\vec{v}_{\perp} \times \vec{B} = \vec{F}_{\perp}$ inhomogeneous part

(no acceleration, drift)

# Drift

Solution of the inhomogeneous part

$$qB^2\vec{v}_\perp = \vec{F}_\perp \times \vec{B} \quad \text{or} \quad \vec{v}_\perp = \frac{\vec{F}\times\vec{B}}{qB^2} \equiv \vec{v}_{drift}$$

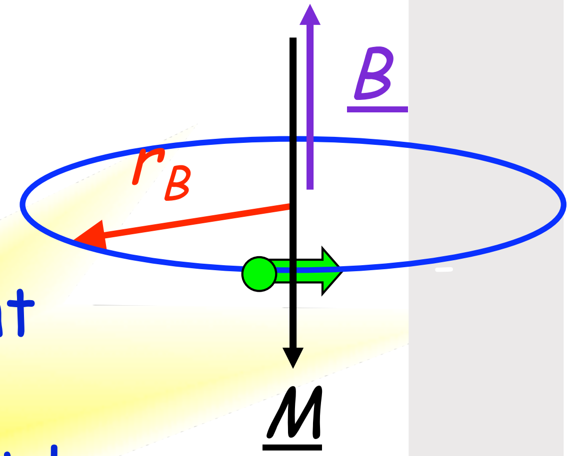



special case: $\vec{F} = q\vec{E} \Rightarrow \vec{v}_{\text{drift}} = \vec{E}\times\vec{B}/B^2$

'EcrossB drift', not dependent on charge and mass

# Magnetic Moment

gyrating particle as a pseudo particle with a magnetic moment M:




$$I = q/\tau_B = q\omega_B/2\pi$$ circular current

$$A = r_B^2\pi$$ area of current loop

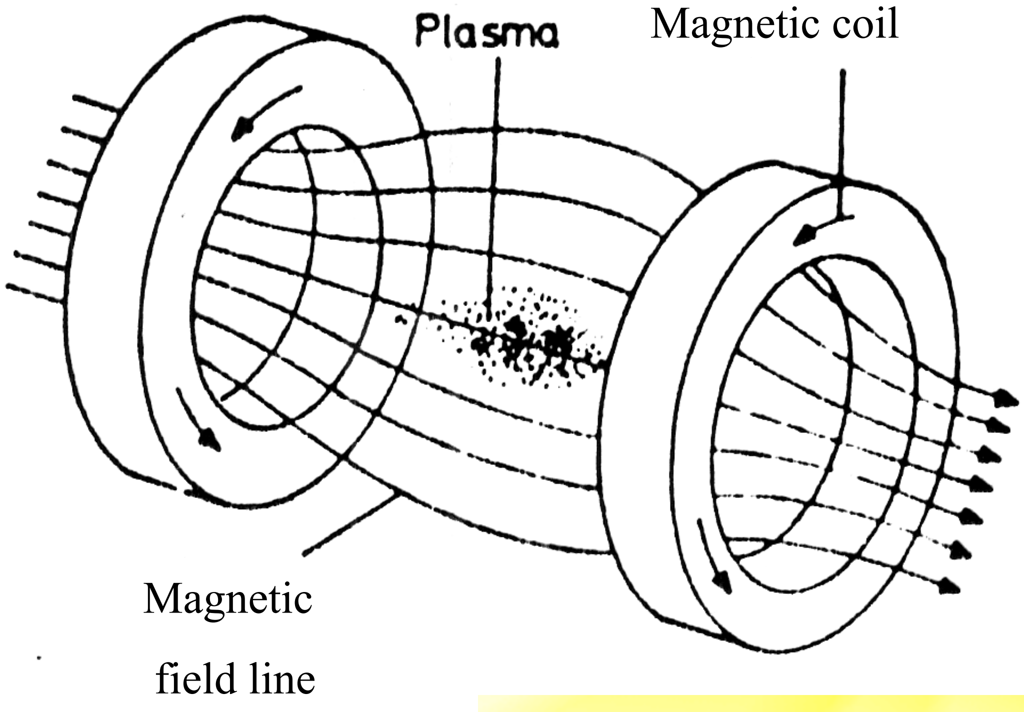
$$M = IA = \frac{q^2B}{2\pi m}\frac{\pi m^2 v_\perp^2}{q^2B^2} = \frac{\frac{1}{2}mv_\perp^2}{B} = \frac{W_\perp}{B}$$

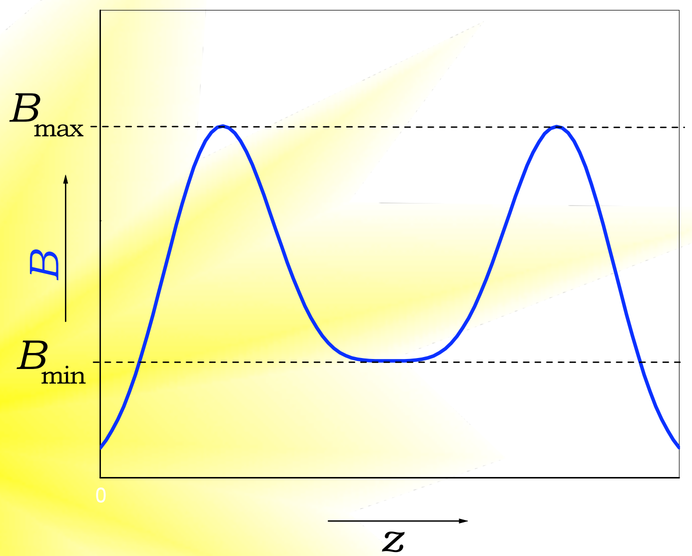
$$M = const$$

$$W = W_\perp + W_\parallel = const \quad \Rightarrow W_\parallel = W - MB$$

"adiabatic constant"

# Magnetic Mirror Trap

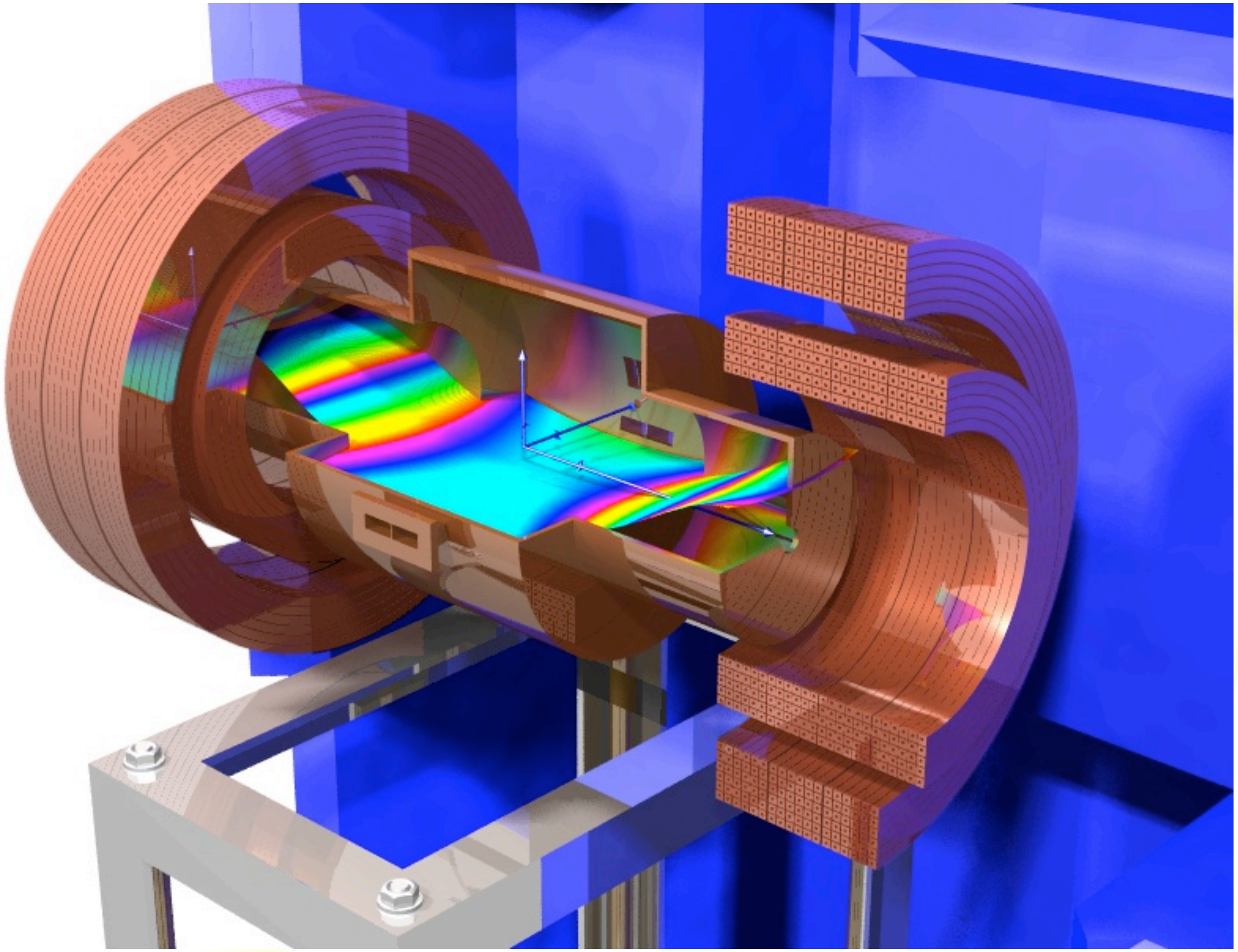



$$M = W_{\perp 0} / B_{min}; \quad W_{\| mirror} \leq 0, \text{if } W \leq M \cdot B_{max} = W_{\perp 0} \frac{B_{max}}{B_{min}}$$

$$\text{condition for containment}: \frac{W_{\perp 0}}{W} \geq \frac{B_{min}}{B_{max}} \equiv \frac{1}{R}; \text{R "mirror ratio"}$$

# A „Hammock" for charged particles



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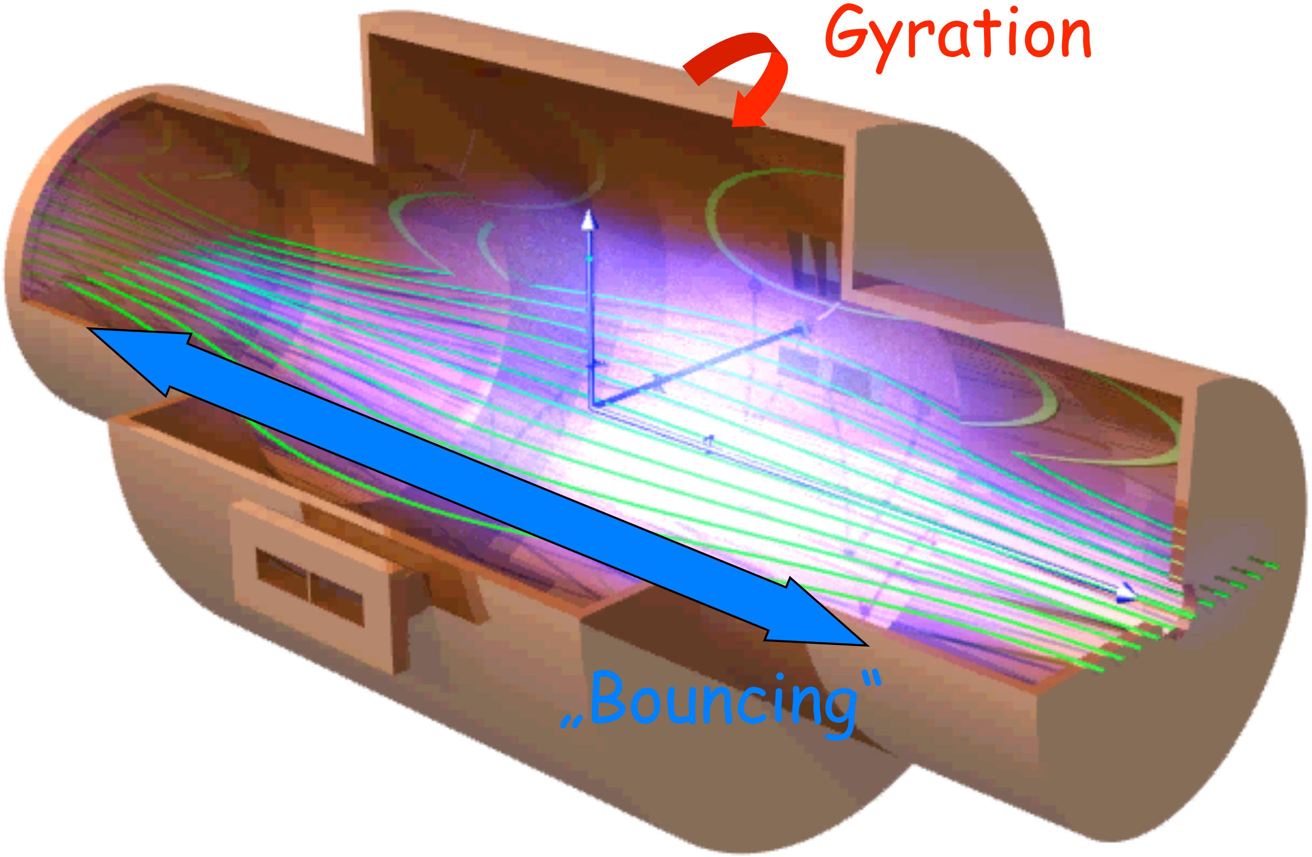
# Charged particle movement between magnetic mirrors

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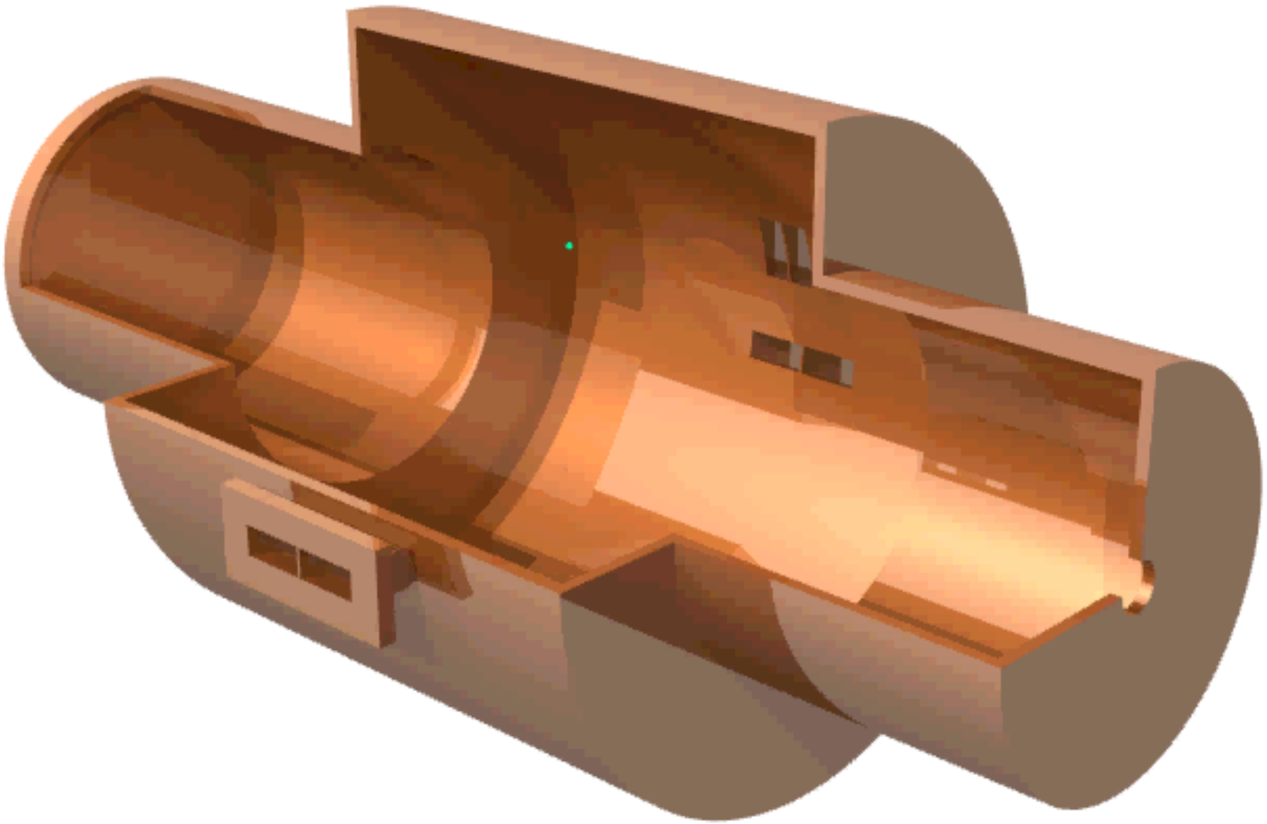
# Azimuthal drift due to radial B-field gradient



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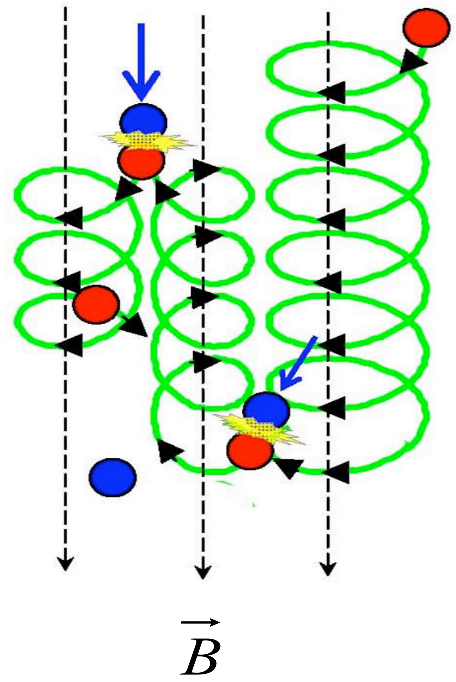
# Diffusion in an External Magnetic Field

transport **along** the field lines resembles transport in the absence of a magnetic field. Thus:

$$\frac{D_{\|\mathrm{e}}}{D_{\|\mathrm{i}}} = \sqrt{\frac{m_{\mathrm{ri}}}{m_{\mathrm{e}}}}$$

# Diffusion in an External Magnetic Field

transport **across** the field lines is a hopping from field line to field line as a consequence of collisions.




Thus the average displacement per collision is the average gyration radius $r_B$, not the mean free path

$$D_\perp \approx < \nu \cdot r_B^2 > \propto \sqrt{m} \Rightarrow \frac{D_{\perp e}}{D_{\perp i}} = \sqrt{\frac{m_e}{m_{ri}}}$$

# Fluid Description of Plasma - *Magneto Fluid Dynamics*

The kinematics of a point mass *m* is defined by:

position vector r(t) and its time derivatives $\dot{r}(t) = v(t)$ and $\ddot{r}(t)$

Kinematics of a fluid (extended medium):

$\Rightarrow$ velocity - and acceleration **fields**: $\dot{r}(r,t)$ and $\ddot{r}(r,t)$

"fluid-particles" keep their identity and thus their motion describes a trajectory

$$\vec{r}(0) = \{x(0), y(0), z(0)\} \equiv \vec{a} \equiv \{\xi, \eta, \varsigma\}$$

$\vec{r}$ coordinate of a fluid particle at $t = 0$, $\vec{a}$ fluid particle;

$$\text{at } t \neq 0 \quad \vec{r}(\vec{a},t) \equiv \{x(\vec{a},t), x(\vec{a},t), z(\vec{a},t)\}$$

$$\Rightarrow \vec{a}(\vec{r},t) = \{\xi(\vec{r},t), \eta(\vec{r},t), \varsigma(\vec{r},t)\}$$

$\vec{r}(\vec{a},t)$ coordinate of a fluid particle, which was at $t = 0$ at the position $\vec{a}$

*Euler coordinates*

$\vec{a}(\vec{r},t)$ identifies a fluid-particle, which at $t$ is at the position $\vec{r}$

Lagrange or convective resp. material coordinates

velocity field:

$\bar{v}(\vec{r},t)$ velocity of the fluid at the position $\vec{r}$ at $t$.

$\bar{v}(\vec{a},t)$ velocity of a particle $a$ at the time $t$

*Intensive quantities:*

$$mass\ density\ \rho$$

$$pressure\ p$$ are not additive

$$Temperature\ T$$

extensive quantities are additive; obtained as volume-integrals of intensive quantities.

$$m = \iiint_V \rho\, dV$$

consider $\Phi(\vec{a}(\vec{r},t),t) = \Phi(\vec{r}(\vec{a},t),t)$ an intensive quantity

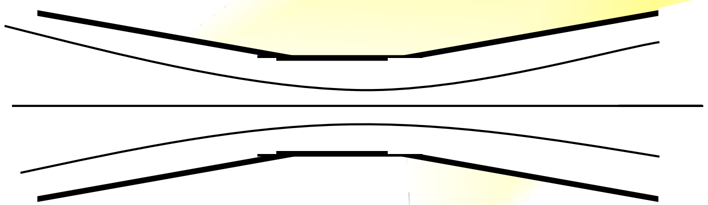
in *Euler coordinates* $d\Phi = \frac{d\Phi}{dt}dt + \frac{d\Phi}{dx}dx + \frac{d\Phi}{dy}dy + \frac{d\Phi}{dz}dz$

The *convective time derivative* $D\Phi / Dt$ is not a total derivative because $a = const.$

$$\frac{D\Phi}{Dt} = \left(\frac{\partial\Phi}{\partial t}\right)_{\vec{r}} + \frac{\partial x}{\partial t}\cdot\frac{\partial\Phi}{\partial x} + \frac{\partial y}{\partial t}\cdot\frac{\partial\Phi}{\partial y} + \frac{\partial z}{\partial t}\cdot\frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial t} + (\vec{v}\cdot\nabla)\Phi.$$

Example convective time derivative of the vector $\vec{v}$ :

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$$



Local acceleration and convective acceleration

Equation of motion:

Newton's equation for a fluid particle

$$\frac{Dm\vec{v}}{Dt} - \sum \vec{F} = \frac{D}{Dt}\iiint_{\hat{V}} \rho\vec{v}\,dV - \sum \vec{F} = 0$$

$\hat{V}$ "*material domain*", *the volume of a fluid particle*

Differentiation and integration cannot be simply interchanged because the volume may change along a trajectory.

Differentiation and integration cannot be simply interchanged because the volume may change along a trajectory.

Instead we use Reynold's transport theorem:

The temporal change of an extensive quantity in a material domain is given by the temporal change of this quantity in the fixed volume just coinciding with the material domain and the flow out of the fixed volume.

Thus

$$\frac{D}{Dt}\iiint_{\hat{V}} \rho\, dV - \sum \vec{F} = \iiint_{V}\left(\frac{\partial \rho\vec{v}}{\partial t} + (\nabla\cdot\vec{v})\rho\vec{v}\right)dV - \sum\vec{F} = 0$$

# Specification of forces

1. Surface forces – example pressure force

$$\vec{F}_{\mathrm{p}} = -\iint_A p\,\overrightarrow{dA} = -\iiint_V (\nabla p)\,dV \qquad \text{(Gauss' theorem)}$$

2. Volume forces

$$\vec{F}_V = \iiint_V \vec{f}\,dV \qquad \text{here } \vec{f} \text{ is a force density}$$

Thus

$$\iiint_V \left(\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} + \nabla p - \vec{f}\right)dV = 0$$

# Momentum equation

$$\iiint_V \left( \rho \frac{\partial \vec{v}}{\partial t} + \rho \left( \vec{v} \cdot \nabla \right) \vec{v} + \nabla p - \vec{f} \right) dV = 0$$

This must be valid for any volume! Thus the integrand must be zero.

$$\Rightarrow \rho \frac{\partial \vec{v}}{\partial t} + \rho \left( \vec{v} \cdot \nabla \right) \vec{v} + \nabla p - \vec{f} = 0 \text{ Euler's equation}$$

Classification of external forces:

Lorentz-force acting on a single electron or ion

$$q_k(\vec{E} + \vec{v}_k \times \vec{B}) \equiv \frac{1}{n_k}\vec{f}_k \quad \text{here } \overrightarrow{f_k} \text{ is the Lorentz-force-density}$$

thus

$$\vec{f}_L = \sum \vec{f}_k = \left(\sum_k n_k q_k\right)\vec{E} + \left(\sum_k n_k q_k \vec{v}_k\right) \times \vec{B} = \rho_{el}\vec{E} + \vec{j} \times \vec{B}.$$

gravitation:

$$\vec{f} = \rho\vec{g}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} + \nabla p - \rho_{\mathrm{el}}\vec{E} - \vec{j} \times \vec{B} - \rho\vec{g} + f_{\mathrm{Fr.}} = 0$$

Navier-Stokes-Equation with Lorentz-force

The Single-fluid model uses only this equation

multi-fluid model: separate fluid equations for any plasma component:

$$\rho_{\mathrm{k}} \frac{\partial \vec{v}_{\mathrm{k}}}{\partial t} + \rho_{\mathrm{k}}(\vec{v}_{\mathrm{k}} \cdot \nabla)\vec{v}_{\mathrm{k}} + \nabla p_{\mathrm{k}} - \rho_{\mathrm{k\,el}}\vec{E} - \vec{j}_{\mathrm{k}} \times \vec{B} + \sum_{\mathrm{l}} f_{\mathrm{k,l}} = 0$$

$$f_{\mathrm{k,l}} = n_{\mathrm{k}} \frac{m_{\mathrm{k}} m_{\mathrm{l}}}{m_{\mathrm{k}} + m_{\mathrm{l}}} \nu_{\mathrm{k,l}} (\vec{v}_{\mathrm{l}} - \vec{v}_{\mathrm{k}})$$

Frictional force between particles of kind k and l.

## additional equations:

$$\frac{\partial \rho_{\mathrm{el}}}{\partial t} + (\vec{v}\cdot\nabla)\rho_{\mathrm{el}} + \rho_{\mathrm{el}}\nabla\cdot\vec{v} = 0 \text{ continuity equation}$$

definition of the conductivity (called Ohm's law)

$$\vec{j} = \sigma\left(\vec{E} + \vec{v}_{\mathrm{drift}} \times \vec{B}\right) \text{resp. } \vec{j} = \overleftrightarrow{\sigma}\left(\vec{E} + \vec{v}_{\mathrm{drift}} \times \vec{B}\right)$$

$$\vec{j} = \vec{j}_{\mathrm{i}} + \vec{j}_{\mathrm{e}} = en_{\mathrm{i}}\vec{v}_{\mathrm{i,drift}} - en_{\mathrm{e,drift}}\vec{v}_{\mathrm{e}}$$

transport equations for particle transport

# Introduction to Plasma Physics II

## CERN Course on Ion Sources

Senec, Slovakia May 2012

Klaus Wiesemann
Ruhr-Universität Bochum,
Germany

# AC Conductivity

$$\vec{v} = bq\vec{E}$$ definition of the mobility

per analogy for nonresonant oscillatory motion

$$\mathrm{i} = \sqrt{-1}$$

$$m\dot{\vec{v}} = q\vec{E}(t) = q\vec{E_0} \cdot \exp(-\mathrm{i}\omega t), \Rightarrow$$

$$\vec{v} = \frac{\mathrm{i}}{\omega m} q\vec{E} \equiv bq\vec{E} \Rightarrow b = \frac{\mathrm{i}}{\omega m}$$ ac mobility

# AC Conductivity

Density of electric current induced by an electric field $\underline{E}$ in a plasma with different kinds of charged particles (index k)

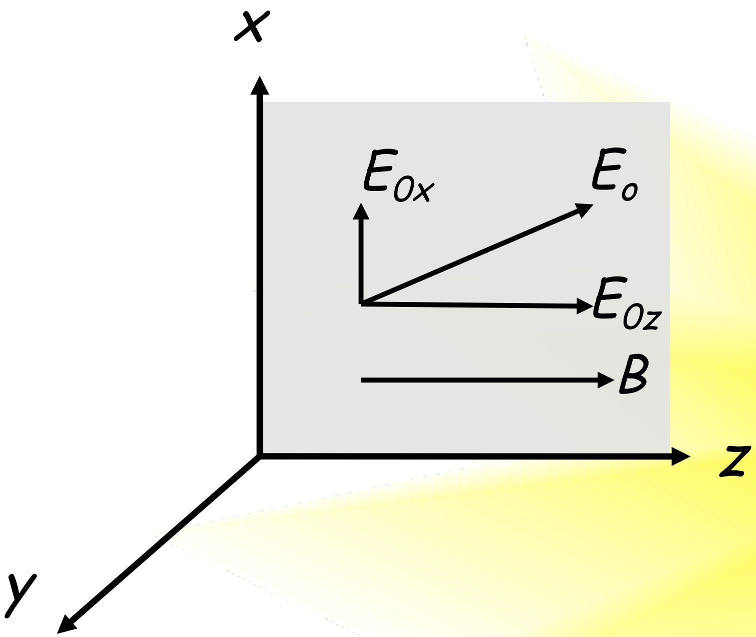
$$\vec{j} = \sum_{\mathrm{k}} q_{\mathrm{k}} n_{\mathrm{k}} \vec{v}_{\mathrm{k}} = \left(\sum_{\mathrm{k}} q_{\mathrm{k}} n_{\mathrm{k}} b_{\mathrm{k}} q_{\mathrm{k}}\right) \vec{E} \equiv \sigma \vec{E}$$

$$\text{thus} \quad \sigma = \sum_{\mathrm{k}} q_{\mathrm{k}} n_{\mathrm{k}} b_{\mathrm{k}} = \mathrm{i} \sum_{\mathrm{k}} q_{\mathrm{k}}^2 n_{\mathrm{k}} / \omega m_{\mathrm{k}}$$

AC Conductivity of a nonmagnetized plasma

(not dependent on charge sign !!)

# AC Conductivity of Magnetized Plasma




$$m\dot{\vec{v}} - q\vec{v}\times\vec{B} - q\vec{E} = 0$$

Ansatz

$$\vec{B} = (0,0,B)$$

$$\vec{E} = (E_x, 0, E_z)$$

$$\vec{v} = \left\{a_x(t)\vec{E}_{0x} + a_y(t)\vec{E}_{0x}\times\underline{B} + a_z(t)\vec{E}_{0z}\right\}\exp(-i\omega t)$$

# AC Conductivity of Magnetized Plasma

$$\frac{\partial a_z}{\partial t} - \mathrm{i}\omega a_z - \frac{q}{m} = 0$$

$$\frac{\partial a_x}{\partial t} - \mathrm{i}\omega a_x - \frac{q}{m} + \frac{qB^2 a_y}{m} = 0$$

$$\frac{\partial a_y}{\partial t} - \mathrm{i}\omega a_y - \frac{q a_x}{m} = 0 \quad \Big| \, \frac{\partial}{\partial t} \cdot$$

$$\frac{\partial^2 a_y}{\partial t^2} - 2\mathrm{i}\omega \frac{\partial a_y}{\partial t} + \left(\omega_{\mathrm{B}}^2 - \omega^2\right) a_y = \frac{q^2}{m^2}$$

$$a_z = \frac{\mathrm{i}q}{\omega m} \Rightarrow v_z = \frac{\mathrm{i}qE_z}{\omega m}$$

inhomogeneous solution;
homogeneous solution
yields a constant velocity

homogenous solution
describes gyration

For calculating electric currents we need to consider the drift velocities obtained from the inhomogeneous parts of the equations

# AC Conductivity of Magnetized Plasma

$$a_y = \frac{q^2}{m^2}\frac{1}{\omega_{\mathrm{B}}^2-\omega^2} \quad \text{and} \quad a_x = \frac{q}{m}\frac{\mathrm{i}\omega}{\omega_{\mathrm{B}}^2-\omega^2}$$

**cyclotron resonance**

$$a_z = \frac{\mathrm{i}q}{\omega m} \Rightarrow v_z = \frac{\mathrm{i}qE_z}{\omega m}$$

$$\overrightarrow{v_{\mathrm{drift}}} = \left(\frac{q}{m}\frac{\mathrm{i}\omega E_{\mathrm{x}}\exp(-i\omega t)}{\omega_B^2-\omega^2}, \frac{-q^2B}{m^2}\frac{E_{\mathrm{x}}\exp(-i\omega t)}{\omega_B^2-\omega^2}, \frac{\mathrm{i}qE_{\mathrm{z}}\exp(-i\omega t)}{\omega m}\right)$$

# AC Conductivity of Magnetized Plasma

For a complete solution we now set

$$\vec{E} = \vec{E}_0 \exp(-\mathrm{i}\omega t) = \vec{E}_x + \vec{E}_y + \vec{E}_z$$

and

$$\vec{v} = \frac{\mathrm{i}\omega}{\omega_\mathrm{B}^2 - \omega^2}\frac{q}{m}\left(\vec{E}_x + \vec{E}_y\right) + \frac{q^2/m^2}{\omega_\mathrm{B}^2 - \omega^2}\left(\vec{E}_y \times \vec{B} + \vec{E}_x \times \vec{B}\right) + \frac{\mathrm{i}q}{\omega m}\vec{E}_z$$

# AC Conductivity of Magnetized Plasma

component equations of the drift velocity

$$v_{drif\,t\,x} = \frac{\mathrm{i}\omega}{\omega_{\mathrm{B}}^2-\omega^2}\frac{q}{m}E_x + \frac{q^2B/m^2}{\omega_{\mathrm{B}}^2-\omega^2}E_y + 0$$

$$\equiv b_{xx}qE_x + b_{xy}qE_y + b_{xz}qE_z$$

$$v_{drif\,t\,y} = -\frac{q^2B/m^2}{\omega_{\mathrm{B}}^2-\omega^2}E_x + \frac{\mathrm{i}\omega}{\omega_{\mathrm{B}}^2-\omega^2}\frac{q}{m}E_y + 0$$

$$\equiv b_{yx}qE_x + b_{yy}qE_y + b_{yz}qE_z$$

$$v_{drif\,t\,z} = 0 + 0 + \frac{\mathrm{i}q}{\omega m}E_z$$

$$\equiv b_{zx}qE_x + b_{zy}qE_y + b_{zz}qE_z$$

# AC Conductivity of Magnetized Plasma

mobility tensor $\overleftrightarrow{b}$

$$\overleftrightarrow{b} = \frac{\mathrm{i}}{\omega m}\begin{Vmatrix} \dfrac{\omega^2}{\omega^2-\omega_B^2} & \dfrac{\mp \mathrm{i}\omega\omega_B}{\omega^2-\omega_B^2} & 0 \\ \dfrac{\pm\omega\omega_B}{\omega^2-\omega_B^2} & \dfrac{\omega^2}{\omega^2-\omega_B^2} & 0 \\ 0 & 0 & 1 \end{Vmatrix} \equiv \frac{\mathrm{i}}{\omega m}\overleftrightarrow{K}$$

conductivity tensor

$$\overleftrightarrow{\sigma} = \sum_k n_k \underline{\underline{b}}_k q_k^2 \quad \Rightarrow \quad \sigma_{ij} = \sum_k n_k \left(b_{ij}\right)_k q_k^2$$

$$\overleftrightarrow{\sigma}(\omega) = \frac{\mathrm{i}}{\omega}\sum_k \frac{n_k q_k^2}{m_k}\overleftrightarrow{K}_k$$

# Some general wave concepts

## (Electromagnetic Waves in Vacuum)

### Maxwell Equations in Vacuum: derivation of the wave equations

$$\left(-\mu_0 \frac{\partial}{\partial t}\cdot\right)\Bigg|\nabla\times\vec{H}=\varepsilon_0\frac{\partial\vec{E}}{\partial t}\quad\left(\vec{j}=0,\text{ no electric current}\right)$$

$$\varepsilon_0\mu_0=1/c^2$$

$$(\nabla\times)\left|\nabla\times\vec{E}=-\mu_0\frac{\partial\vec{H}}{\partial t}\quad\left(\nabla\cdot\vec{E}=0,\text{ no space charge, }\nabla\cdot\vec{B}=0,\text{ no magnetic monopoles}\right)\right.$$

$$-\mu_0\nabla\times\frac{\partial\vec{H}}{\partial t}=-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\left(\nabla\cdot\vec{E}\right)+\nabla^2\vec{E}$$

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}=0$$

wave equation

$$\vec{E}=\vec{E_0}\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)+\cancel{\vec{E}_0^{*}\exp\left(-i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)}$$

plane waves

$$-k^2+\omega^2/c^2=0\ or\ \omega=\pm kc$$

dispersion relation

# Some general wave concepts

## (Electromagnetic Waves in Vacuum)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$ wave equation

$$\vec{E} = \vec{E_0}\exp\left(\mathrm{i}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right) + \underline{E}_0^*\exp\left(-\mathrm{i}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right)$$ plane waves

$$-k^2 + \omega^2 c^2 = 0 \; or \; \omega = \pm kc$$ dispersion relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{k}\cdot\vec{r} - \omega t\right) = \vec{k}\cdot\dot{\vec{r}} - \omega = 0$$

$$v_{\mathrm{ph}} = \frac{\omega}{k}.$$ phase velocity

$$\mu = c/v_{\mathrm{ph}} = c \cdot k/\omega$$ refractive index

# Transversal Plasma Waves $\vec{k} \perp \vec{E}$

$$\vec{j} = \sigma \vec{E} \neq 0; \quad \nabla \cdot \vec{E} \neq 0 \qquad \vec{k}\left(\vec{k} \cdot \vec{E}\right) \equiv \overleftrightarrow{kk} \cdot \vec{E} = 0$$

$$\nabla \times \vec{H} = \vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \Bigg| \cdot \mu_0$$

plane wave

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\frac{1}{\varepsilon_0 c^2} \frac{\partial \vec{j}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}.$$

$$\vec{E} = \vec{E}_0 \exp\left(i\left[\vec{k} \cdot \vec{r} - \omega t\right]\right)$$

$$\left(\cancel{\overleftrightarrow{kk}} - k^2\right)\vec{E} = \frac{\omega^2}{c^2}\left(1 - \frac{\sum_k \omega_{pk}^2}{\omega^2}\right)\vec{E} \equiv \varepsilon \frac{\omega^2}{c^2} \vec{E}$$

$$\varepsilon = 1 + \frac{i\sigma}{\varepsilon_0 \omega} = \left(1 - \sum_k \omega_{pk}^2 \Big/ \omega^2\right)$$

(non-magnetized plasma)

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad \text{or} \quad \frac{\omega^2}{\omega_p^2} = 1 + \frac{k^2 c^2}{\omega_p^2}.$$

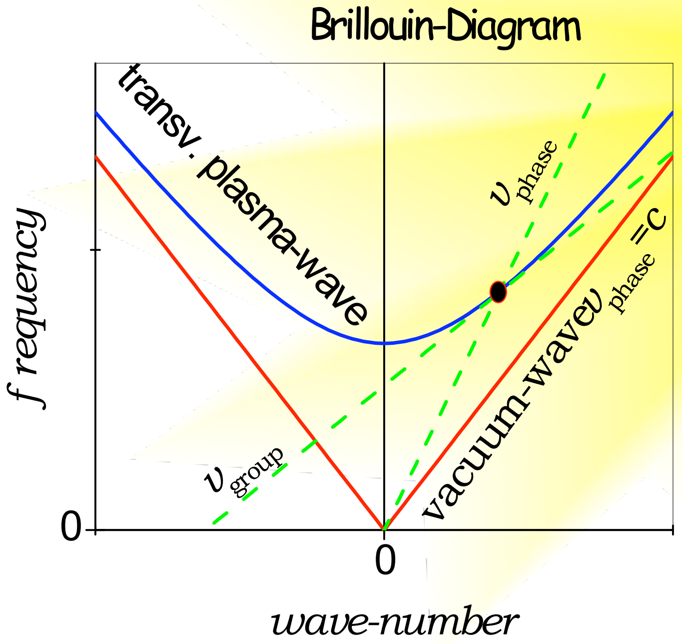
dispersion relation in collisionfree plasma (Eccles relation)

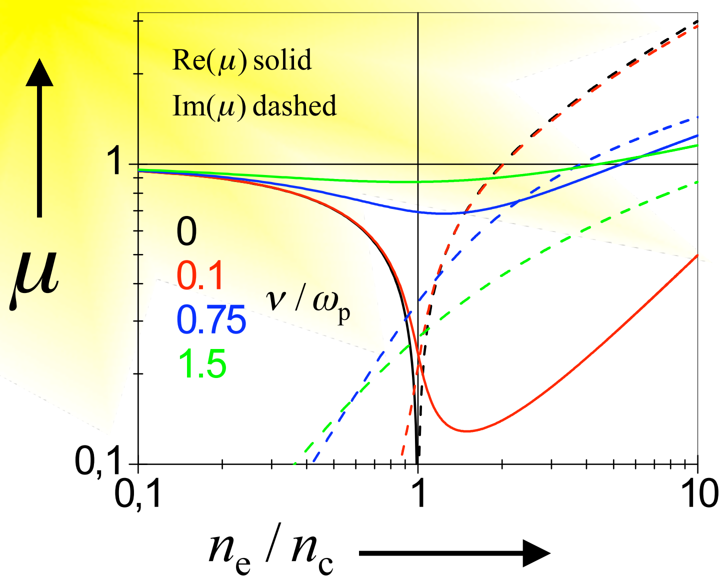
# Dispersion Relation for Transversal EM Waves in Nonmagnetized Plasma

$$\frac{\omega^2}{\omega_p^2} = 1 + \frac{k^2 c^2}{\omega_p^2}.$$

$$\mu^2 = \frac{k^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

$n_c = \omega^2 \varepsilon_0 m_e / e^2$ critical density



# Waves in Magnetized Plasma

dielectric tensor

$$\overleftrightarrow{\varepsilon} = \overleftrightarrow{1} + \frac{\mathrm{i}\overleftrightarrow{\sigma}}{\varepsilon_0 \omega} = \overleftrightarrow{1} - \sum_{\mathrm{k}} \frac{\omega_{\mathrm{pk}}^2}{\omega^2} \overleftrightarrow{K}_{\mathrm{k}}$$

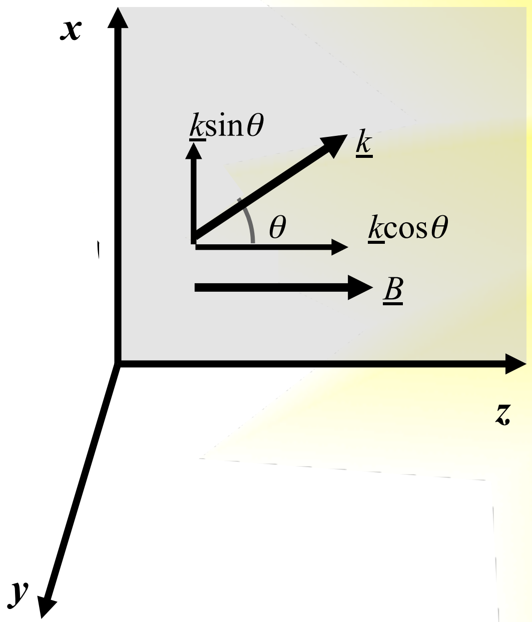
$$\left\{ k^2 \overleftrightarrow{1} - \overleftrightarrow{kk} - \frac{\omega^2}{c^2} \overleftrightarrow{\varepsilon} \right\} \cdot \vec{E} = 0 \qquad \overleftrightarrow{\mu\mu} \equiv \overleftrightarrow{kk}\, c^2 / \omega^2$$

$$\left| \mu^2 \overleftrightarrow{1} - \overleftrightarrow{\mu\mu} - \overleftrightarrow{\varepsilon} \right| = 0$$

dispersion relation

# Waves in Magnetized Plasma

$$\overleftrightarrow{\varepsilon} \equiv \left\| \begin{matrix} S & \mathrm{i}D & 0 \\ -\mathrm{i}D & S & 0 \\ 0 & 0 & P \end{matrix} \right\|$$




$$\begin{vmatrix} \mu^2\cos^2\theta - S & -\mathrm{i}D & -\mu^2\cos\theta\sin\theta \\ +\mathrm{i}D & \mu^2 - S & 0 \\ -\mu^2\cos\theta\sin\theta & 0 & \mu^2\sin^2\theta \end{vmatrix} = 0$$

dispersion relation

# Waves in Magnetized Plasma

$$\overleftrightarrow{\varepsilon} \equiv \begin{Vmatrix} S & \mathrm{i}D & 0 \\ -\mathrm{i}D & S & 0 \\ 0 & 0 & P \end{Vmatrix} \qquad \begin{vmatrix} \mu^2\cos^2\theta - S & -\mathrm{i}D & -\mu^2\cos\theta\sin\theta \\ +\mathrm{i}D & \mu^2 - S & 0 \\ -\mu^2\cos\theta\sin\theta & 0 & \mu^2\sin^2\theta \end{vmatrix} = 0$$

dispersion relation

$$S \equiv 1 - \sum_{\mathrm{k}} \frac{\omega_{\mathrm{pk}}^2}{\omega^2} \frac{1 - \mathrm{i}\nu_{\mathrm{mk}}/\omega}{\left(1 - \mathrm{i}\nu_{\mathrm{mk}}/\omega\right)^2 - \omega_{\mathrm{Bk}}^2/\omega^2}$$

$$D \equiv -\sum_{\mathrm{k}} \frac{\omega_{\mathrm{pk}}^2}{\omega^2} \frac{\omega_{\mathrm{Bk}}}{\omega} \frac{1}{\left(1 - \mathrm{i}\nu_{\mathrm{mk}}/\omega\right)^2 - \omega_{\mathrm{Bk}}^2/\omega^2}$$

$$P \equiv 1 - \sum_{\mathrm{k}} \frac{\omega_{\mathrm{pk}}^2}{\omega^2} \frac{1}{1 - \mathrm{i}\nu_{\mathrm{mk}}/\omega}$$

# Waves in Magnetized Plasma

$$\overleftrightarrow{\varepsilon} \equiv \begin{Vmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{Vmatrix} \qquad \begin{vmatrix} \mu^2\cos^2\theta - S & -iD & -\mu^2\cos\theta\sin\theta \\ +iD & \mu^2 - S & 0 \\ -\mu^2\cos\theta\sin\theta & 0 & \mu^2\sin^2\theta \end{vmatrix} = 0$$

dispersion relation

$$A_1\mu^4 - A_2\mu^2 + A_3 = 0$$

$$A_1 = S\sin^2\theta + P\cos^2\theta$$

$$A_2 = RL\sin^2\theta + SP(1+\cos^2\theta)$$

$$A_3 = PRL$$

$R = S + D$

$L = S - D$

# Waves in Magnetized Plasma

$$A_1\mu^4 - A_2\mu^2 + A_3 = 0$$

$$\mu^2_{\mathrm{s,f}} = \frac{A_2}{2A_1}\left(1 \pm \sqrt{1-4A_1A_3/A_2^2}\right)$$

slow and fast wave

plasma as wave guide:

$\mu^2 < 0$ *stop band*

$\mu^2 = 0$ *cutoff frequency*

$\mu^2 \to \infty$ *resonance*

$\mu^2 > 0$ *propagation*

# Cutoff Frequencies

$$A_1\mu^4 - A_2\mu^2 + A_3 = 0$$

cutoffs $\mu = 0 \quad only\ for \quad A_3 = PRL = 0$

cutoff frequencies independent of the angle of propagation

$$P = 1 - \sum_k \frac{\omega_{pk}^2}{\omega^2} \equiv 1 - \frac{\omega_p^2}{\omega^2}$$

thus $P = 0 \Rightarrow \omega = \omega_p$

the plasma frequency is a cutoff frequency

# Cutoff Frequencies

$$A_1\mu^4 - A_2\mu^2 + A_3 = 0$$

cutoffs $\mu = 0 \quad only\ f\ or \quad A_3 = PRL = 0$

for $R = 0$, $L = 0$ we obtain two cutoff frequencies

$$\omega_{1,2}\left(\omega_{1,2} \mp \omega_{\mathrm{Be}}\right) = \omega_{\mathrm{pe}}^2 \quad \text{for} \quad \omega >> \omega_{\mathrm{Bi}}$$

# Resonances

$$A_1\mu^4 - A_2\mu^2 + A_3 = 0 \;\Big|\; \div \mu^4$$

resonances
$$A_1 - A_2\frac{1}{\mu^2} + A_3\frac{1}{\mu^4} = 0$$

now *solution* $\frac{1}{\mu^2} = 0$ *only, if* $A_1 = 0$

$$A_1 = S\cdot\sin^2\theta + P\cdot\cos^2\theta = 0 \Rightarrow \tan^2\theta = -P/S$$

resonance frequencies depend on the direction of wave propagastion – *resonance cones*

# Principal Directions

solving $$A_1\mu^4 - A_2\mu^2 + A_3 = 0$$ for θ yields

$$\tan^2\theta = \frac{-P\left(\mu^2-R\right)\left(\mu^2-L\right)}{\left(S\mu^2-RL\right)\left(\mu^2-P\right)}$$

*Appleton-Lassen Equation*

Principal directions of propagation

$$along\ \vec{B}, \theta = 0 \quad \Rightarrow \tan\theta = 0$$

$$across\ \vec{\mathrm{B}}, \theta = 90^\circ \Rightarrow \tan\theta \to \infty$$

# Cyclotron Waves

$$\tan^2\theta = \frac{-P\left(\mu^2 - R\right)\left(\mu^2 - L\right)}{\left(S\mu^2 - RL\right)\left(\mu^2 - P\right)}$$

$$along\ \vec{B}, \theta = 0 \quad \Rightarrow \tan\theta = 0$$

$\Rightarrow \mu_{\mathrm{r}} = \sqrt{R}$ right circularly polarized waves

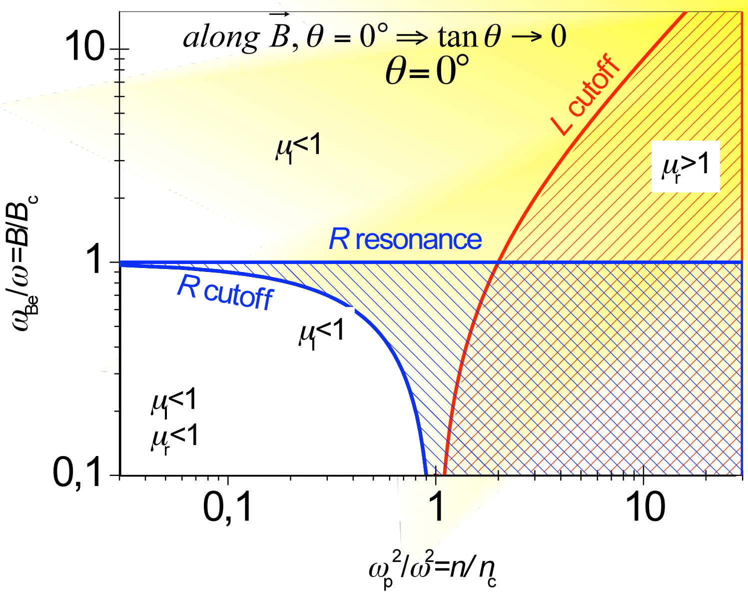
$\Rightarrow \mu_{\mathrm{l}} = \sqrt{L}$ left circularly polarized waves

# Cyclotron Waves

RUB

$$\mu_r \to \infty \quad for \quad \omega = |\omega_{Be}|; \ \mu_r = 0 \quad for \quad \frac{|\omega_{Be}|}{\omega} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\mu_l \to \infty \quad for \quad \omega = \omega_{Bi}; \ \mu_l = 0 \quad for \quad \frac{|\omega_{Be}|}{\omega} = \frac{\omega_p^2}{\omega^2} - 1$$




$$B_c \equiv \frac{\omega m_e}{q}; \quad n_c \equiv \frac{\omega^2 \varepsilon_0 m_e}{e^2}$$

# Ordinary and Extraordinary Waves

$$across\ \vec{B},\ \theta = 90° \Rightarrow \tan\theta \to \infty$$

$$\tan^2\theta = \frac{-P\left(\mu^2 - R\right)\left(\mu^2 - L\right)}{\left(S\mu^2 - RL\right)\left(\mu^2 - P\right)}$$

$$\mu_o = \sqrt{P}$$

ordinary waves, corresponds to case of waves in non-magnetized plasma

$$\mu_{ex} = \sqrt{RL/S}$$

extraordinary waves

# Ordinary and Extraordinary Waves

$$\mu_o = \sqrt{P}$$

ordinary waves, corresponds to case of waves in non-magnetized plasma

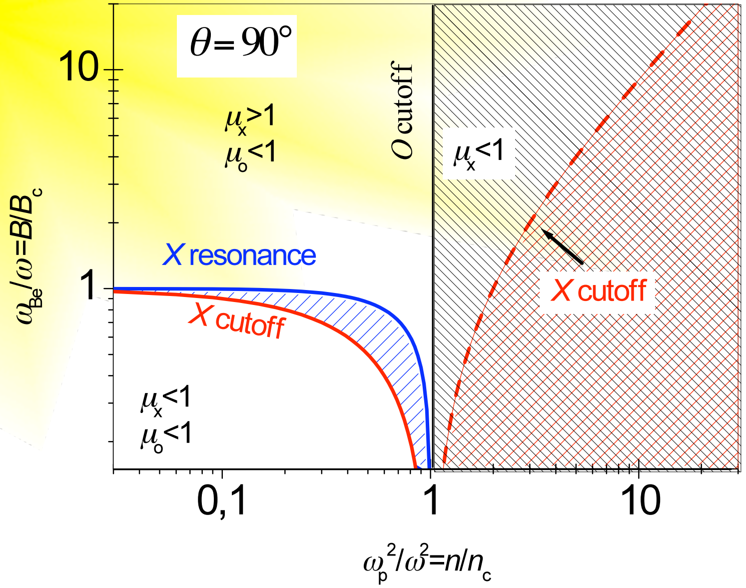
$$\mu_{ex} = \sqrt{RL/S}$$

extraordinary waves

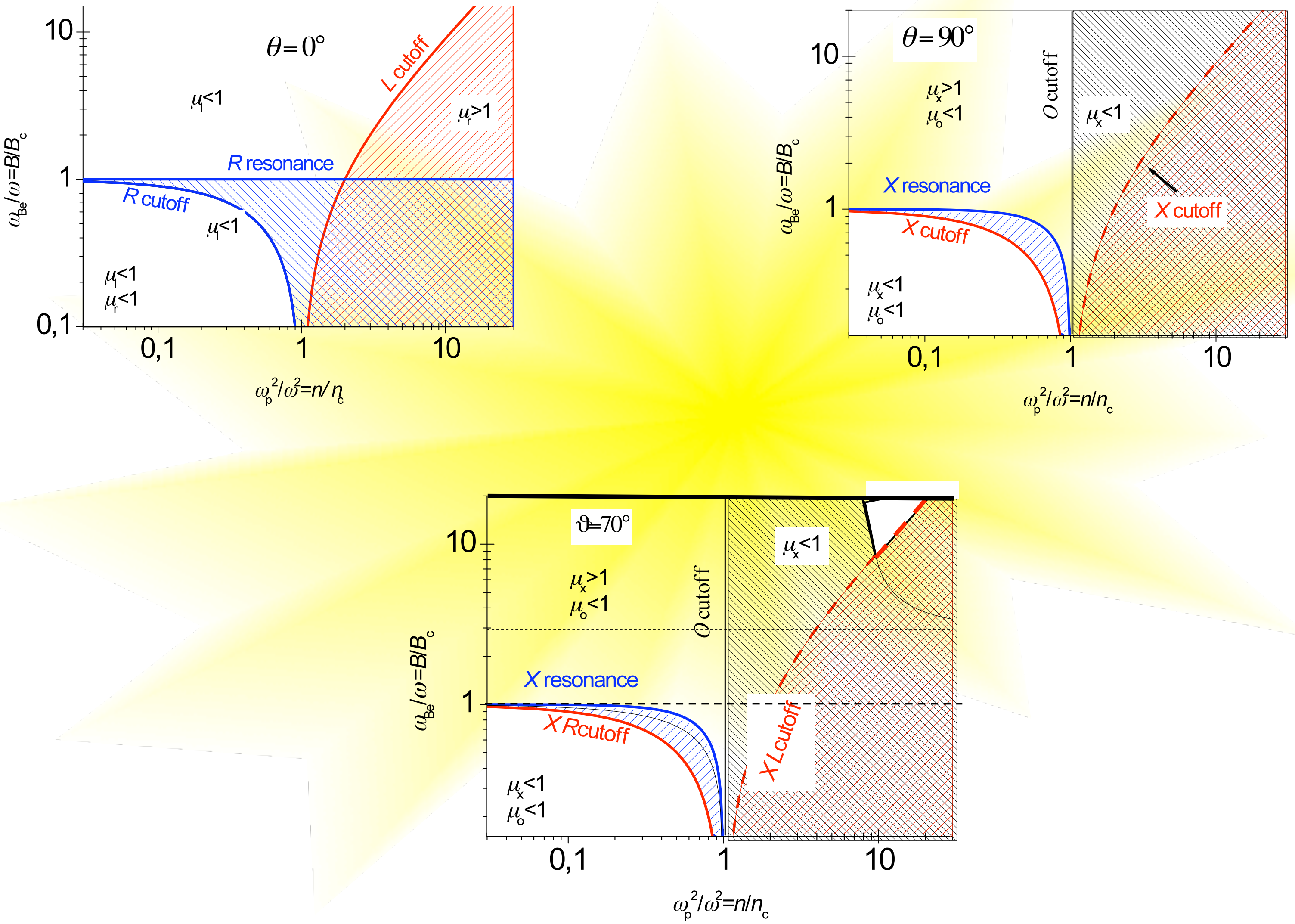
$$\mu_o = 0 \; for P = 0 \Rightarrow \omega = \omega_p \quad no\ resonance$$

$$\mu_{ex} = 0 \; for R, L = 0 \Rightarrow like \; for \theta = 0$$

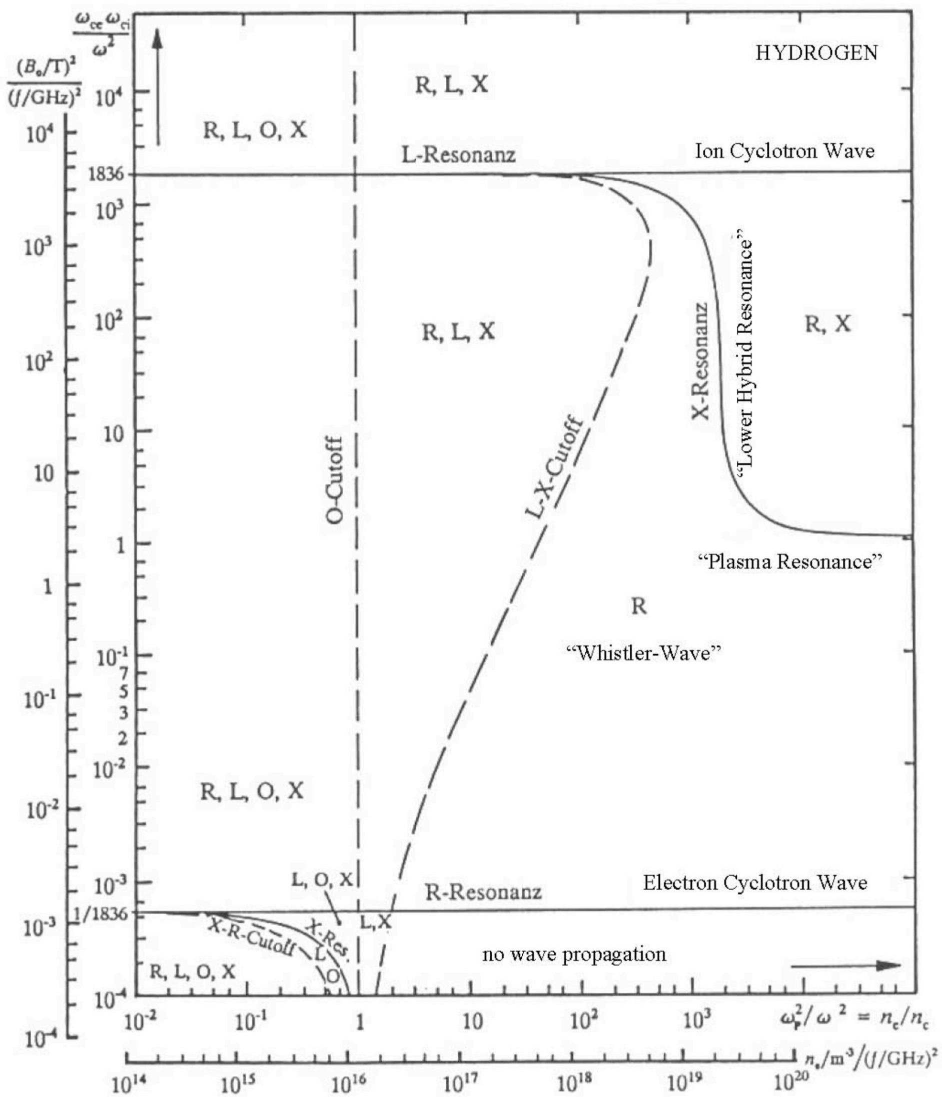
$$\mu_{ex} \to \infty \; for S = 0 \Rightarrow \frac{\omega_{Be}}{\omega} = \sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}$$



# Oblique Propagation



# CMA Diagram

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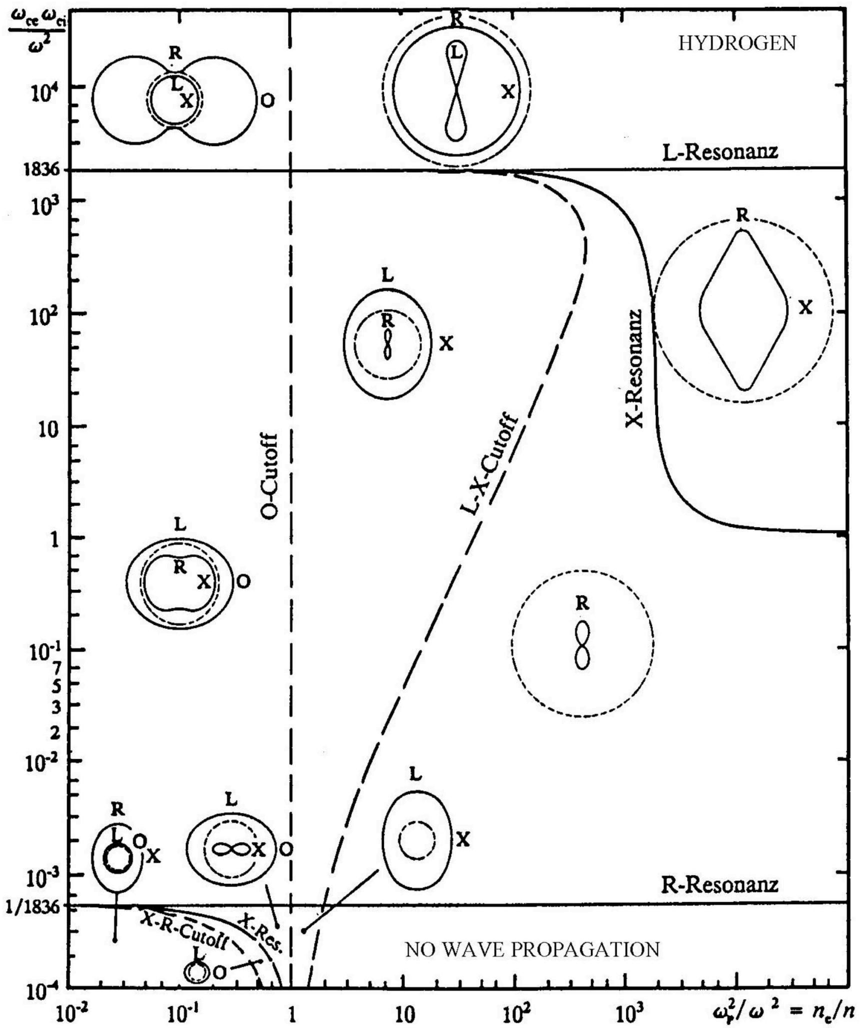
# CMA Diagram




polar plots of the phase velocity

$$\vec{v}_{phase} = \omega\vec{k}/k^2$$

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# Longitudinal Waves

Phase velocity of sound waves $c_s = \sqrt{\kappa p / \rho}$

$\kappa = (N+2)/N$ ratio of specific heats

from adiabatic gas law $p/n^\kappa = const \Rightarrow \nabla p/p = \kappa \nabla n/n$

restoring force due to pressure at compression

analogously in plasma

$c_{si} = \sqrt{\kappa_i p_i / \rho_i}$ *ion sound speed*

$c_{se} = \sqrt{\kappa_e p_e / \rho_e}$ *electron sound speed*

# Longitudinal Waves

$$\rho_e \frac{\partial \underline{v}_e}{\partial t} = -n_e e \vec{E} + \nabla p_e$$

for homogeneous plasma !!

$$\rho_i \frac{\partial \underline{v}_i}{\partial t} = n_i e \vec{E} + \nabla p_i \qquad \kappa_i = \kappa_e = 3 \left(\text{one dimension}\right);\ \underline{k} \parallel \underline{E}$$

high frequency electron waves

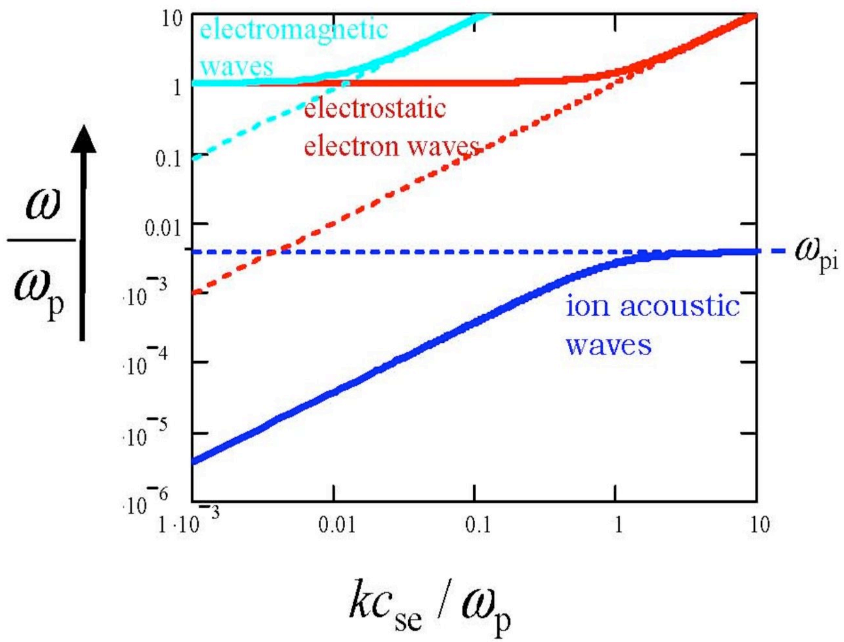
$$\omega^2 = \omega_p^2 + k^2 \left(c_{se}^2 + c_{si}^2\right) \approx \omega_{pe}^2 + k^2 \kappa_e \frac{k_B T_e}{m_e}$$

low frequency ion waves (approx. for small *k*)

$$\omega^2 \approx k^2 \left(c_{si}^2 + \frac{m_e}{m_i} c_{se}^2\right) = k^2 \left(\kappa_i \frac{k_B T_i}{m_i} + \kappa_e \frac{k_B T_e}{m_i}\right) \approx k^2 \kappa_e \frac{k_B T_e}{m_i}$$

# Longitudinal Waves

exact dispersion plot for em and es plasma waves




$n_{e,i} = 10^{12}\,\mathrm{cm}^{-3}$, argon

$k_B T_e = 4\mathrm{eV}$, $k_B T_i = 0.1\mathrm{eV}$

The dashed slanting lines correspond to em-wave propagation in vacuum, resp. to es-wave propagation in an ideal gas. At short wavelengths i. e. big *k* es-waves are strongly damped

# Introduction to Plasma Physics

# The END

CERN Course on Ion Sources

Senec, Slovakia May 2012

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