In addition to identifying where singularities can appear by solving the Landau equations, it is often also possible to predict the nature of the expected singularity (for instance, whether a logarithm, square root, or simple pole will arise)

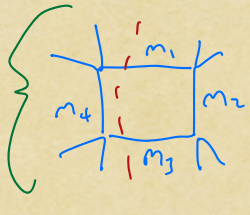
⇒ we won't have time to discuss how to make these types of predictions in these lectures, but we will make use of the following result, proven by Landau for Feynman integrals with generic external and internal masses:

$$I_G(\varphi \to 0) \sim \begin{cases} \varphi^\gamma \log \varphi & \text{if } \gamma \in \mathbb{Z},\ \gamma \geq 0 \\ \varphi^\gamma & \text{otherwise} \end{cases}$$

$$\gamma = \tfrac{1}{2}(LD - E - 1)$$

where D is the spacetime dimension, and L and E are the number of loops and edges in the Landau diagram one is left with after contracting all edges that are not on-shell in the corresponding solution to the Landau equations

For example, the threshold singularity at $s = (m_1 + m_2)^2$ behaves like:



$\sqrt{\varphi}$ when $(D-3)$ odd

$\log \varphi$ when $(D-3) \geq 0$ even

Now its time to see how all these constraints come together to let us bootstrap integrals directly. To do this, we will follow these steps:

(1) identify the types of special functions that can appear, and the possible prefactors

(in the context of Feynman integrals in or near integer dimensions, we expect iterated integrals to appear — so we are really asking what type of non-dlog integration kernels can appear)

(2) Solve the Landau equations, and construct integration kernels that are consistent with these singularities, including the type of singularity if known

(3) Derive constraints on the sequences of discontinuities that can appear in the final result

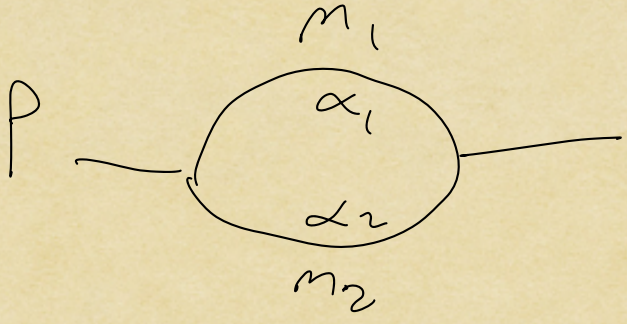
(& positivity, hierarchical constraints, Steinmann relations)

At this point we can construct all iterated integrals that are consistent with these predictions

(4) Fix any remaining freedom by expanding the integral around singular kinematic points

The massive bubble integral (again)

Let's now see how this bootstrap procedure can be used to uniquely determine the functional form of the massive bubble integral in $D=2$



① we first confirm that there will be no obstructions to evaluating any integrals in terms of MPLs

⇒ we do this by computing all possible sequences of residues of the integrand, i.e. leading singularities

⇒ in this example we already saw that the residues of the one-fold integral representation give us

$$\sim \frac{1}{\sqrt{s-r_+}\sqrt{s-r_-}}$$

where we recall $r_\pm = (m_1 \pm m_2)^2$

⇒ this tells us two things:

(i) all integrations can be done either algebraically or (poly)logarithmically

(ii) the prefactor of the iterated integrals that appear in the final answer multiply this product of roots

(2) we now solve the Landau equations, finding four solutions in $D=2$

singularity	which propagators on shell?	α-positive?	nature of singularity
$m_1^2 = 0$	1	yes	log
$m_2^2 = 0$	2	yes	log
$S = r_+$	1, 2	yes	$\sqrt{\ }$
$S = r_-$	1, 2	no	$\sqrt{\ }$

$\Rightarrow$ we also know the nature of these singularities from the formula derived by Landau

$$I(\phi \to 0) = \begin{cases} \phi^\gamma \log \phi & \text{for } \gamma \in \mathbb{Z},\ \gamma \geq 0 \\ \phi^\gamma & \text{otherwise} \end{cases}$$

$$\gamma = \frac{1}{2}(DL - E - 1)$$

here $D = 2$, $L = 1$

⇒ So, we now want to construct possible symbol letters that encode these singularities (and no more!)

⇒ The following algorithm helps us do that

⇒ first, any logarithmic-type singularity can itself appear as a symbol letter. Thus we have

$$\log(m_1^2), \log(m_2^2)$$

as possibilities

⇒ what about algebraic symbol letters? Let's imagine we have a letter of the form

$$\log\left(A \pm \sqrt{B}\right)$$

⇒ these latter can only vanish when

$$A^2 - B = 0$$

which implies that

$$A^2 - B = c \prod_i L_i^{n_i}$$

where c is a number, the n_i are integers and the L_i are the set of logarithmic branch points

⇒ however, we also know that B can be nothing more than a product of the possible square root singularities R_j

⇒ therefore

$$A^2 = c \prod_i L_i^{a_i} + \prod_j R_j$$

where we have a finite number of polynomials L_i, R_j to

put in these products, and we can bound the n_i from above

$\Rightarrow$ we thus scan through all possibilities on a computer, for "reasonable" n_i, and ask "is there a number C that makes the right hand side a perfect square"

$\Rightarrow$ there are very few solutions in practice; in this case we find just the pair of solutions

$$\log\left(s - r_+ \pm \sqrt{s-r_+}\sqrt{s-r_-}\right)$$

(note that the SOFIA code by Correia, Giroux, and Mizera has an implementation of this algorithm now

(itself by Matijašić and Miczajka))

⇒ one can check that the product of these two arguments is proportional to $m_1^2 m_2^2$, therefore only one of these letters is new

⇒ so we are free to choose as our three possible symbol letters

$$\log(m_1^2), \log(m_2^2),$$

$$\log\left(\frac{s - r_+ - \sqrt{s-r_+}\sqrt{s-r_-}}{s - r_+ + \sqrt{s-r_+}\sqrt{s-r_-}}\right)$$

$$\| \quad \log(R)$$

At this stage, our ansatz for the symbol takes the form

$$S(I) = \frac{1}{\sqrt{s-r_+}\sqrt{s-r_-}} \quad \left(\begin{array}{c}\text{some symbol constructed} \\ \text{from these three letters}\end{array}\right)$$

but we can't say anything about this symbol yet.

$\Rightarrow$ to cut down on what can appear, we next derive some hierarchical constraints

$\Rightarrow$ since the two $m_i^2 = 0$ singularities involve putting different propagators on shell, we have that

$$\text{Disc}_{m_1^2} \cdots \text{Disc}_{m_2^2} \cdots (I) = 0$$

$$\text{Disc}_{m_2^2} \cdots \text{Disc}_{m_1^2} \cdots (I) = 0$$

$\Rightarrow$ this immediately means we cannot have any symbol terms of the form

$$\cdots m_1^2 \otimes \cdots \otimes m_2^2 \otimes \cdots$$

⇒ in fact, when $m_1 = 0$ or $m_2 = 0$ R also vanishes. This means that R also cannot appear in a symbol term with either m_1^2 or m_2^2, or with itself

$$\cdots R \otimes \cdots \otimes m_2^2 \otimes \cdots$$

$$\cdots R \otimes \cdots \otimes R \otimes \cdots$$

⇒ this leaves only three possible types of terms in our ansatz

(i) $\log(R)$ by itself

(ii) $m_1^2 \otimes \cdots \otimes m_1^2$ of any length

(iii) $m_2^2 \otimes \cdots \otimes m_2^2$ of any length

so our ansatz for the symbol is

$$c_0 + c_1 \log(R)$$

$$+ \sum_{i=1}^{\infty} a_i \log^i(m_1^2) + \sum_{i=1}^{\infty} b_i \log^i(m_2^2)$$

where the c_i, a_i, b_i are just numbers

⇒ next, we cut down on our ansatz further by comparing to Landau's formula for the asymptotic behavior as $m_i^2 \to 0$ again

⇒ in particular, we know that

$$I(m_1^2 \to 0) \sim \log(m_1^2)$$

for just a single log. This means $a_i = b_i = 0$ for all $i \geq 2$

⇒ by the symmetry $m_1 \leftrightarrow m_2$, we also have that $a_1 = b_1$. Thus, our ansatz becomes

$$\frac{c_0 + c_1 \log(R) + a_1 \log(m_1^2 m_2^2)}{\sqrt{s - r_+}\sqrt{s - r_-}}$$

with just three undetermined coefficients c_0, c_1, and a_1

⇒ Finally, we can use our knowledge of α-positivity to recognize that there should be no branch cut at $s = \Gamma_-$ on the physical sheet

⇒ This implies $c_0 = c_1 = 0$, which leaves us with

$$c_1 \frac{\log(R)}{\sqrt{s-\Gamma_+}\sqrt{s-\Gamma_-}}$$

⇒ But wait, what is the problem with this final answer?

- The function is symmetric with respect to both the threshold and pseudothreshold square roots

⇒ It turns out we can fix this by dividing the numerator/denominator of R by $\sqrt{s-\Gamma_+}$

$$\Rightarrow R' = \left(\frac{\sqrt{s-r_+} - \sqrt{s-r_-}}{\sqrt{s-r_+} + \sqrt{s-r_-}} \right)$$

⇒ One can check this gives you the right analytic structure

⇒ Therefore, up to a proportionality factor, we have

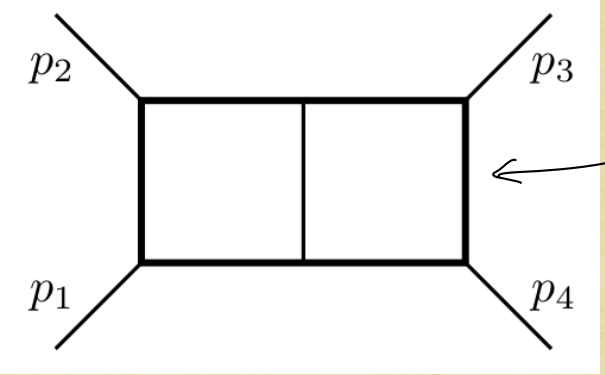
$$I_{-0}^{D=2} \propto \frac{1}{\sqrt{s-r_+}\sqrt{s-r_-}} \log(R')$$

This illustrates how these bootstrap methods work in practice. Additional simple examples are done in 2410.02424

⇒ note in particular one gets a different answer for the bubble in $D=3$, since the discontinuities at $m_i^2 = 0$ become square root type and the threshold/pseudothreshold discontinuities become logarithmic

$$I_{-0}^{D=3} = \frac{1}{\sqrt{s}} \log\left(\frac{\sqrt{m_1^2} + \sqrt{m_2^2} - \sqrt{s}}{\sqrt{m_1^2} + \sqrt{m_2^2} + \sqrt{s}} \right)$$

We also see there a more nontrivial example:



← single internal mass m in outer loop

⇒ After computing the leading singularity to confirm this integral will evaluate to polylogarithms and to determine the algebraic prefactor, we ...

We first solve the Landau equations, finding five α-positive solutions:

$$s = 4m^2, \qquad s \to \infty, \qquad t = 4m^2, \qquad t \to \infty, \qquad m^2 = 0$$

and five more non-α-positive solutions:

$$s = 0, \qquad t = 0, \qquad m^2 \to \infty, \qquad s + t = 0, \qquad st - 4m^2 s - 4m^2 t = 0$$

* Sequences of four positive symbol letters 20,736

Constraint	Remaining Parameters
Integrable weight-four symbols	6993
α-positivity for square roots	861
α-positivity for logarithms	161
Genealogical constraints	28
Only algebraic α-positive $s - 4m^2$ thesholds	6
Only algebraic $t - 4m^2$ thesholds	1

Hierarchy!

Useful references for this lecture include

"Applications from the Landau bootstrap", 2410.02424

"Implications of the Landau Equations for Iterated Integrals", 2109.09744