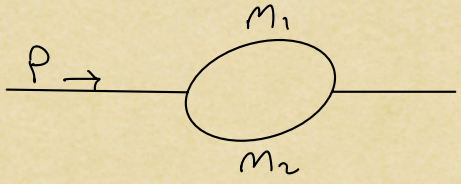
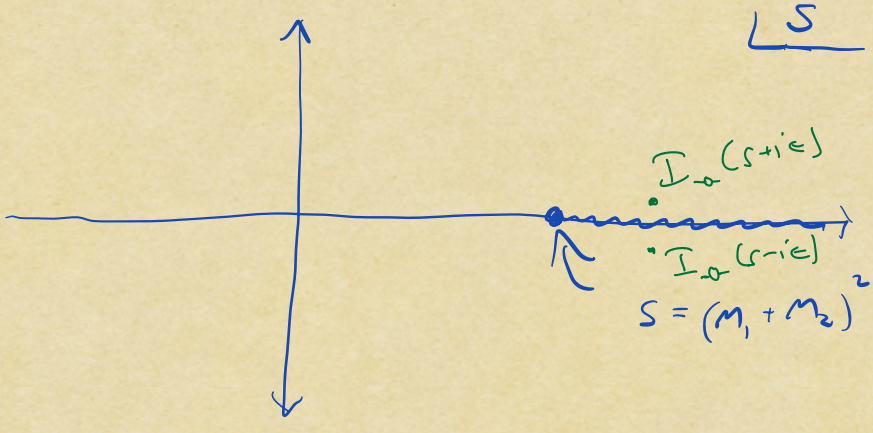
Multivalued Functions

Unitarity tells us that scattering amplitudes and related quantities (like correlators) are multi-valued functions.

⇒ For instance, if we were to evaluate the massive bubble integral:

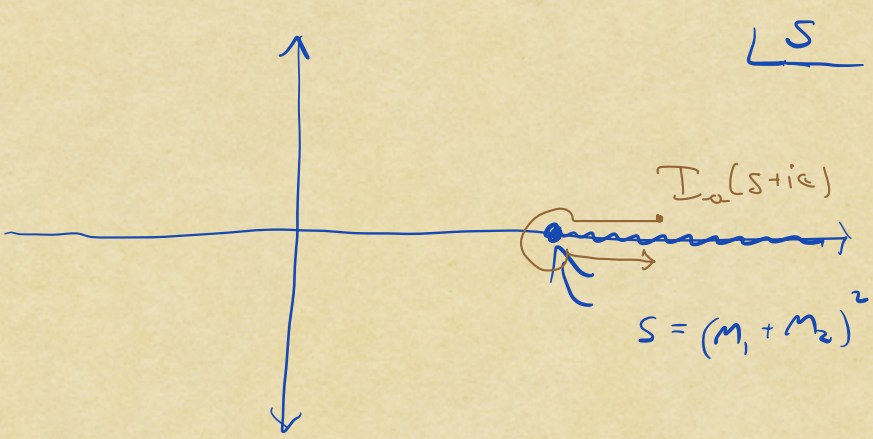


we would observe that the result has a branch cut starting at $p^2 = s = (m_1 + m_2)^2$



$$\text{Disc}_{s-(m_1+m_2)^2}(I_{-0-}) = I_{-0}(s-i\epsilon) - I_{-0}(s+i\epsilon)$$

⇒ We can compute this discontinuity by directly evaluating the integral on either side of the branch cut, or we can analytically continue $I(s+i\epsilon)$ around the branch point at $s=(m_1+m_2)^2$



⇒ The advantage of this second way of thinking is that it is well-defined for any value of s, not just on the real line

(while the location of the branch point is unambiguous, the placement of the branch cut is just convention)

⇒ in subsequent lectures, we will want to be able to make sense of sequences of discontinuities such as

$$\cdots \text{Disc}_s \text{Disc}_{s-4m^2} \text{Disc}_{m^2} \quad (\text{I})$$

for arbitrary values of $s, m^2, \ldots$

⇒ We can do this by interpreting each discontinuity to be the difference between the analytically

continued integral and the original integral

$$\text{Disc}_x(I) \to \left(\circlearrowright_{x=0} - \mathbb{1}\right) I$$

Bubble Example

Let's see what this looks like at the level of an integral. Consider the massive bubble in $D=2$, which can be written in Feynman parameter space (up to constants, which I will ignore) as

$$I_{-O-}^{D=2} = \int_0^1 d\alpha \underbrace{\frac{1}{s\alpha(1-\alpha) - m_1^2\alpha - m_2^2(1-\alpha) + i\epsilon}}_{F}$$

("the F polynomial" or second Symanzik polynomial)

(we choose this example because it can be written as a one-fold parameter integral)

⇒ the Landau equations for a pinch singularity to arise in this integral is

$$F = 0$$

$$\frac{\partial}{\partial\alpha} F = 0$$

The roots of F occur at

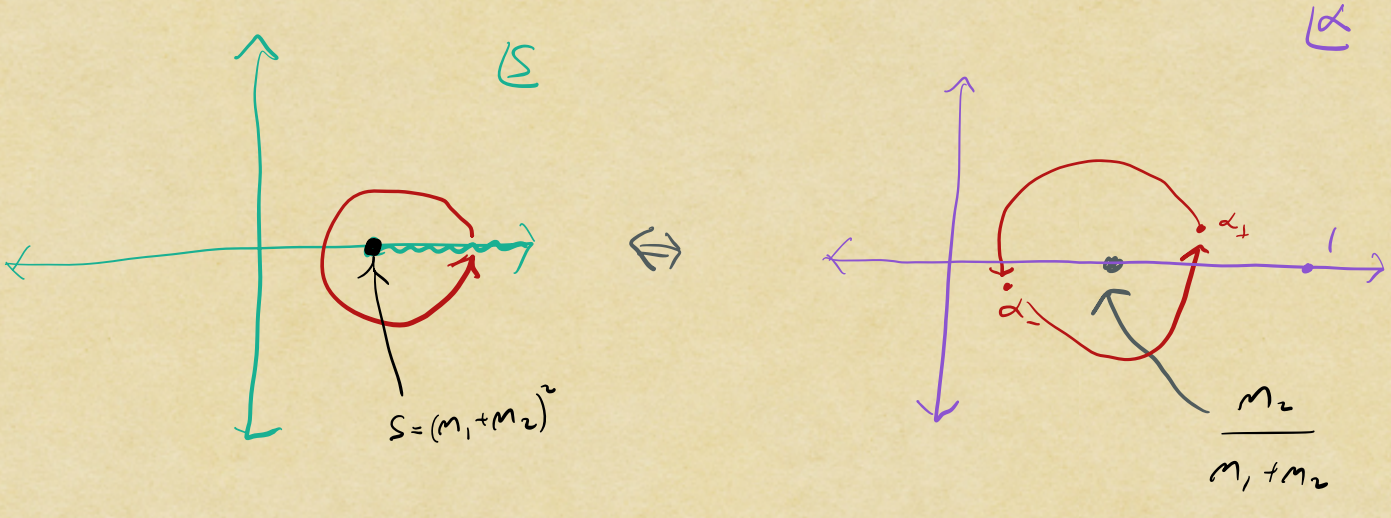
$$\alpha_\pm = \frac{s + m_2^2 - m_1^2 \pm \sqrt{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 + is\varepsilon}}{2s}$$

Once we also impose the derivative condition, we get

$$s = (m_1 + m_2)^2 - i\varepsilon\,, \qquad \alpha_\pm = \frac{m_2}{m_2 + m_1} + i\varepsilon\,\mathrm{sgn}(m_2 - m_1)$$
$$s = (m_1 - m_2)^2 + i\varepsilon\,, \qquad \alpha_\pm = \frac{m_2}{m_2 - m_1} - i\varepsilon\,\mathrm{sgn}(m_2 - m_1)$$

where the lack of symmetry in m_1 and m_2 is due to our choice of $\alpha = \alpha_1 = 1 - \alpha_2$

⇒ Now we can track what happens as we vary s around the singular point $s = (m_1 + m_2)^2$

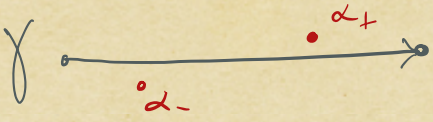


⇒ the α_+ and α_- roots only rotate half way around the point $m_2/(m_1+m_2)$ due to the square root, and end up swapping positions

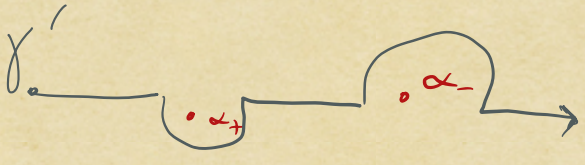
⇒ due to the $i\epsilon$ condition, they end up crossing the original integration contour

⇒ this means the analytically-continued integration contour differs from the original one

original contour



analytically continued contour



their difference is just a pair of residue contours

$$\text{Disc}_{s-(m_1+m_2)^2}\left(I^{2D}_{-a}\right) = \int_{\gamma'} \frac{d\alpha}{F} - \int_{\gamma} \frac{d\alpha}{F} = \int_{C.R} \frac{d\alpha}{F}$$

⇒ This is nothing more than the residue of the two roots of F

$$\sim 2\pi i \left(\text{res}_{\alpha=\alpha_+} \left(\frac{1}{F} \right) - \text{res}_{\alpha=\alpha_-} \left(\frac{1}{F} \right) \right)$$

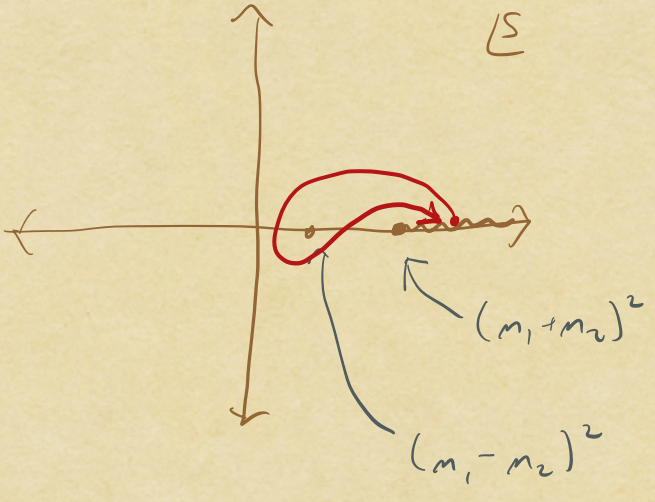
$$= \frac{4\pi i}{\sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}$$

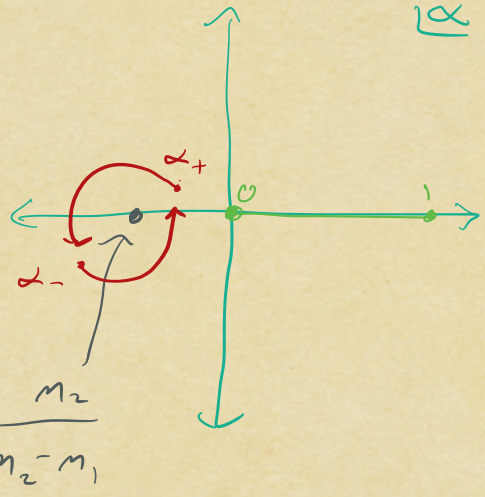
What about the pseudo threshold, where $s = (m_1-m_2)^2$?

$$s = (m_1+m_2)^2 - i\varepsilon\,, \qquad \alpha_\pm = \frac{m_2}{m_2+m_1} + i\varepsilon\,\mathrm{sgn}(m_2-m_1)$$
$$s = (m_1-m_2)^2 + i\varepsilon\,, \qquad \alpha_\pm = \frac{m_2}{m_2-m_1} - i\varepsilon\,\mathrm{sgn}(m_2-m_1)$$

We can also analytically continue s around this point

⇒ however, if $m_1 > m_2$, the roots α_+ and α_- are both negative. If $m_2 > m_1$, the roots are both larger than 1. In either cases the roots do not pinch the integration contour





As a result, the analytic continuation of s does nothing to the integration contour and the discontinuity is zero

α-positivity

This can be formalized as a general result

> Feynman integrals only have singularities that correspond to "α positive" solutions to the Landau equations when considered in the physical region and on the physical sheet

$\Rightarrow$ "α positive" – all Feynman parameters are non-negative

$\Rightarrow$ "physical region" – the regions of kinematic space where momenta/masses describe actually possible configurations of scattering particles

$\Rightarrow$ "physical sheet" – the Riemann sheet that we land on when we use the Feynman $i\epsilon$ prescription

⇒ note that this statement should be interpreted as referring to the full set of Feynman parameters, before any projective redundancy is taken care of

(in the case of the projective Feynman parameters for the bubble, we have $\frac{\alpha_1}{\alpha_2} = \pm \frac{m_2}{m_1}$, where the minus sign corresponds to the threshold)

An analogous result holds for cosmological correlators, since the integration contour over the x_i variables is also $x_i \geq 0$

The more familiar form of this result comes from Cutkosky's formula (1960)

$$\text{Disc}_{G'}(I_G) = \int \prod_{i=1}^{L} d^4\ell_i \, \frac{\left[\prod_{e \in G'} \delta^+(q_e^2 - m_e^2)\right]}{\prod_{e \notin G'} (q_e^2 - m_e^2)}$$

where G' is the graph that one gets by contracting all the propagators in G that are off-shell in the corresponding solution to the Landau equations, and

$$\delta^+(q_e^2 - m_e^2) = \delta(q_e^2 - m_e^2)\,\theta(q_e^0)$$

↑ positive energy flow

The formal proof of Cutkosky's formula was given by Pham (1967):

Theorem (Pham): the discontinuity of a Feynman integral I_G with respect to α-positive singularities in which each loop in G has at least one nonzero α is given by Cutkosky's formula

Cutkosky's formula holds for all first discontinuities of Feynman integrals, but there is no analogous (general) formula for second (or higher) discontinuities

⇒ However, there is still more that can be said, as we will see in the next lecture

But first, let us quickly see how this picture is modified in dimensional regularization. This is easiest to illustrate with an endpoint singularity

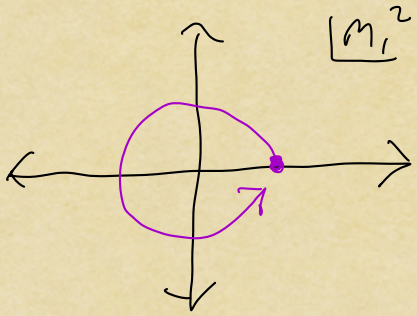
⇒ going back to the bubble example, we can check that endpoint singularities occur when one of the internal masses vanishes

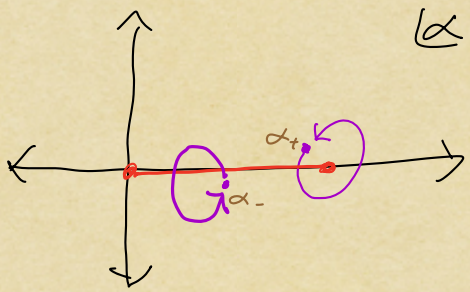
$$M = 0$$

$$\alpha\left(\alpha-1\right)\left(\frac{2}{s\alpha} M\right) = 0$$

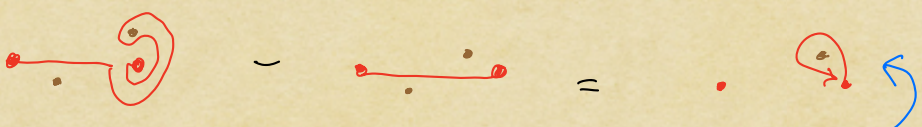
$$M = s\alpha(1-\alpha) - m_1^2\alpha - m_2^2(1-\alpha) \quad \rightarrow \quad -m_1^2 = 0$$

⇒ If we analytically continue m_1^2 around zero in $D=2$ dimensions, we find the following picture:





⇒ the difference contour becomes in this case a residue around just one of the poles

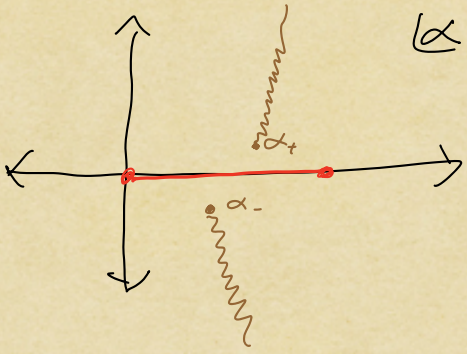


⇒ In particular, this becomes a residue contour because the two straight portions of this contour cancel

⇒ But what about in $D = 2 - 2\epsilon$ dimensions?

$$\int_0^1 \frac{d\alpha}{F} \rightarrow \int_0^1 \frac{d\alpha \; U^{2\epsilon}}{F^{1+\epsilon}}$$

⇒ Since the F polynomial is raised to a non-integer power, we now have branch cuts extending from its roots



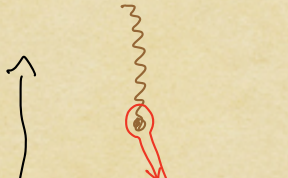
⇒ this also means the straight portions of the difference contour no longer cancel

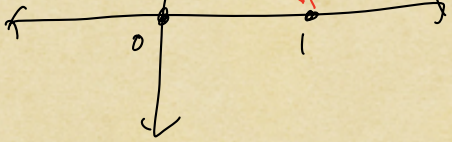
⇒ after passing through the branch cut, we pick up a factor of $e^{-2\pi i \epsilon}$

$$\delta^{-1-\epsilon} \rightarrow \left(\delta e^{2\pi i}\right)^{-1-\epsilon} = \delta^{-1-\epsilon} e^{2\pi i(-1-\epsilon)}$$

$$= \delta^{-1-\epsilon} e^{-2\pi i \epsilon}$$

⇒ as a result the full difference contour is the full line integral





⇒ This lesson holds more generally — in dim reg the difference contours will be "extended" compared to the integer-dimensional ones

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Useful references for the material in this lecture include

"Constraints on Sequential Discontinuities from the Geometry of On-shell Spaces", 2211.07633

"Singularities and Discontinuities of Feynman Amplitudes", Cutkosky (1960)