1 Motivation- physics and mathematics

Over the past two decades, the dialogue between physicists and mathematicians has grown steadily. At the same time, plans for new colliders are actively being discussed. To uncover potential new physics at the LHC, particularly in its high-luminosity phase, it is essential that we first achieve a deep and precise understanding of established physics, namely the Standard Model. Only by reaching very high precision in our agreement with Standard Model predictions can we hope to identify new physics, through subtle deviations between experimental data and theoretical calculations. Quantum field theory provides the fundamental framework for describing the particle interactions that underpin the Standard Model.

Main tool of investigation: Scattering amplitudes

The fundamental building blocks connecting theory to experiment are scattering amplitudes, which are central to our understanding of particle physics. They are studied within well-established frameworks such as the Standard Model, effective field theories, supersymmetric theories, and even string theory. Not only these they have proven useful for developing new calculational techniques that reveal deep relations between seemingly different theories.

At colliders, we do not predict individual events but rather the probabilities of possible outcomes. These probabilities are determined by the squared absolute value of the scattering amplitudes—i.e., the quantum transition amplitudes between initial and final states. Over the past decades, scattering amplitudes have been found to exhibit remarkable properties and unexpected simplifications. As a result, this area has grown into one of the most active and influential fields in modern theoretical high-energy physics.

To summarize, the motivation for computing higher-order scattering amplitudes is twofold:

- To meet the demand for precise theoretical predictions required by present and future experiments.
- To gain deeper insight into the theory itself—in particular, to uncover the underlying mathematical structures and geometric features of these complex objects as they arise in real-world theories such as QCD.

The goal of these lectures is to present the above motivations in the simplest way possible. To this end, they are organized into the following parts:

- **From amplitudes to Feynman integrals**: finite results, dimensional regularization, and selected multi-loop computation methods.
- **Modern tools for Feynman integral computations**: integration-by-parts identities, the method of differential equations, and introduction to useful notions such as canonical form, transcendental weights and occurence of square roots.
- **How to get these results? Notion of cuts. What emerges from these calculations?**

- **Numerical evaluations**: at the end of the day, physics is about numbers! So, how do we obtain them in practice?

References:

- *Lectures on differential equations for Feynman integrals*: Henn'15
- *Feynman integrals* book: Stefan Weinzierl '22
- *Sector Decomposition*: Gudrun Heinrich '08
- *Iterated integrals in quantum field theory* lecture notes: Francis Brown '09
- *Scattering Amplitudes and Logarithmic Differential Forms* thesis: Pascal Wasser '22.

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2 From Amplitudes to Feynman integrals

Let us consider the computation of scattering amplitudes A_n with n external particles, in order to make theoretical predictions for observables. In general, it is not possible to calculate such amplitudes exactly. However, if all couplings describing the strengths of interactions among the particles are small, we may compute them within perturbation theory.

For simplicity, let us assume there is only one coupling constant, denoted by g. We can then expand the scattering amplitude in powers of g:

$$A_n = A_n^{(0)} + A_n^{(1)} + A_n^{(2)} + A_n^{(3)} + \cdots ,$$

where $A_n^{(l)}$ denotes the contribution at loop order l, and contains $(n-2+2l)$ powers of the coupling g.

The computation of scattering amplitudes proceeds via Feynman diagrams:

- Draw all Feynman diagrams for the given number of external particles and loop order l.
- Translate each graph into a mathematical expression using the Feynman rules.
- The total amplitude A_n is then obtained as the sum over all diagrams.

Let us now consider, one of these integrals, which we may call a Feynman graph G. The incoming external momenta are denoted by p_i, which can be taken as fixed vectors subject to overall momentum conservation. If all momenta are taken to be outgoing, then

$$\sum_i p_i = 0. \tag{2.1}$$

This condition must also be satisfied at each vertex of the graph.

For a tree-level graph, the external momenta uniquely determine all internal momenta. In contrast, loop graphs contain one or more closed circuits. For an l-loop graph, in addition to the external momenta, l independent loop momenta must be specified to uniquely determine all internal momenta.

Applying the Feynman rules, if the loop momenta also appear in the numerator, the corresponding integrals are called *tensor integrals*. When the numerator is independent of the loop momenta, the integrals are called *scalar integrals*. It is always possible to reduce tensor integrals to scalar integrals, and hence we focus on scalar integrals from here on.

Using the Feynman rules, a Feynman diagram is translated into a mathematical formula as follows. For each internal line j, we assign a propagator

$$\frac{i}{q_j^2 - m_j^2}, \tag{2.2}$$

where q_j is the momentum flowing through propagator j (which may depend on a linear combination of external momenta p_i and loop momenta k_r), and m_j is the mass of the particle associated with line j. Furthermore, the Feynman rules require us to integrate over all independent loop momenta:

$$\int \frac{d^4 k_r}{(2\pi)^4}. \tag{2.3}$$

Divergences

Loop integrals are generally ill-defined in four-dimensional Minkowski space, as they may contain ultraviolet (UV) or infrared (IR) divergences. UV divergences arise from the region of large loop momentum k, while IR divergences are associated with "soft" or "collinear" momentum regions.

The first step in handling such divergences is to introduce a regulator that makes the integrals well defined. The standard approach is to use dimensional regularisation: we replace the 4-dimensional loop integral with a D-dimensional one, where D may be non-integer or even complex. We then perform the integration as a function of D and study its behaviour as $D \to 4$. It is customary to write

$$D = 4 - 2\epsilon. \tag{2.4}$$

The divergences then appear as poles in $1/\epsilon$. At l loops, ultraviolet divergences can generate poles up to $1/\epsilon^l$, whereas infrared divergences may lead to poles as severe as $1/\epsilon^{2l}$. The Kinoshita–Lee–Nauenberg (KLN) theorem guarantees that infrared divergences cancel against contributions from real emissions of soft particles in physical observables. Ultraviolet divergences, on the other hand, must be removed by renormalisation of the coupling constants, masses, and wavefunctions.

As an example, consider a scalar Feynman integral with n internal lines in general D dimensions:

$$I_n = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{D/2}} \prod_{j=1}^{n} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}}, \tag{2.5}$$

where $q_j = k + p_j$. The factors i and $(4\pi)^{D/2}$ frequently appear in the evaluation of Feynman integrals and are therefore conventionally absorbed into the prefactor. Finally, a general Feynman integral admits a Laurent expansion in the dimensional regularisation parameter ϵ:

$$I = \sum_{j=-2l}^{\infty} c_j \, \epsilon^j. \tag{2.6}$$

These coefficients c_j will be our main interest in the later part of these lectures.

Since obtaining the Laurent expressions of Feynman integrals is our main goal, first let us look at some traditional ways of computing these integrals. The first basic way to try is using direct integration (in suitable parametrization). One of the commonly used parametrization is the so called Feynman parametrization, which we look at below.

Feynman parametrization

Performing the integral in eq.(2.5) is difficult and not even well defined for general values of D. A standard trick is to rewrite the integral in Feynman parametrization, effectively trading the integrals over the loop momenta for integrals over newly introduced Feynman parameters. In this parametrization, the momentum integrals can be performed for general values of D. Essentially, we make use of the formula

$$\frac{1}{A^{\lambda_1} B^{\lambda_2}} = \frac{\Gamma(\lambda_1 + \lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty \frac{x^{\lambda_2 - 1} dx}{(A + Bx)^{\lambda_1 + \lambda_2}} \tag{2.7}$$

to rewrite propagators in terms of parameters, that can be integrated over. Here $\nu = \sum_{i=1}^{n} \nu_i$

Using this, the integral takes the form

$$I = \frac{\Gamma(\nu - lD/2)}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_{x_j \geq 0} d^n x \delta(1 - \sum_{i=1}^{n} x_i)(\prod_{j=1}^{n} dx_j x_j^{\nu_j - 1}) \frac{\mathcal{U}^{\nu-(l+1)D/2)}}{\mathcal{F}^{\nu - lD/2}} \tag{2.8}$$

where these $\mathcal{U}$ and $\mathcal{F}$ are called Symanzik graph polynomials that can be calculate from spanning trees and spanning 2-forest:

$$\mathcal{U}(x) = \sum_{T \in T_1} \prod_{e_{\notin} T} x_i \tag{2.9}$$

$$\mathcal{F}(x) = \sum_{(T_1, T_2) \in \mathcal{T}_2} \Big(\prod_{e_i \notin (T_1, T_2)} x_i \Big) \Big(\frac{-s(T_1, T_2)}{\mu^2} \Big). \tag{2.10}$$

In the next chapters we will move to studying more analytic ways of computing Feynman integrals, especially using the method of differential equations.

3 Modern Multi-loop Techniques

Scattering amplitudes can be represented as a perturbative expansion in terms of Feynman diagrams. As the perturbative order increases, the corresponding multi-loop diagrams become progressively more difficult to evaluate. To address this, one seeks efficient methods for computing such integrals. A central idea is to reduce the large set of integrals appearing in a given calculation to a smaller, minimal set of linearly independent integrals, known as *master integrals*, from which all others can be expressed. To achieve this reduction, several important classes of relations between Feynman integrals can be derived:

- **Integration-by-parts (IBP) identities**: These follow from the Poincaré invariance of the integrals, which is preserved in dimensional regularisation. They may be viewed as the D-dimensional extension of Gauss' theorem. Explicitly,

$$\int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_i^\mu} V^\mu \prod_{j=1}^{n} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}} = 0, \tag{3.1}$$

  where q denotes any combination of scalar products involving loop and external momenta, V is any loop or external momentum, and ν_j labels the propagator powers.

  As an example, consider the box integral ($n = 4$):

$$I_{2,1,1,1} = \frac{D-5}{s} I_{1,1,1,1} - \frac{4(D-5)(D-3)}{(D-6)\, s\, t^2} I_{0,1,0,1}, \tag{3.2}$$

$$I_{1,1,0,1} = \frac{2(D-3)}{(D-4)\, t} I_{0,1,0,1}. \tag{3.3}$$

- **Lorentz invariance identities**: Since Feynman integrals are Lorentz scalars, under an infinitesimal Lorentz transformation $p_i \to p_i + \delta p_i$ with $\delta p_i = \omega_{\mu\nu} p_i^\nu$ and $\omega_{\mu\nu}$ an antisymmetric tensor, we have

$$I(p_i + \delta p_i) = I(p_i). \tag{3.4}$$

  Expanding this relation yields

$$I(p_i + \delta p_i) = I(p_i) + \sum_n \frac{\partial I(p_i)}{\partial p_{n,\mu}} \delta p_{n,\mu} \tag{3.5}$$

$$= I(p_i) + \omega_{\mu\nu} \sum_n p_n^\nu \frac{\partial I(p_i)}{\partial p_{n,\mu}}. \tag{3.6}$$

  Using the antisymmetry of $\omega_{\mu\nu}$ gives the identity

$$\sum_n \left(p_n^\nu \frac{\partial}{\partial p_{n,\mu}} - p_n^\mu \frac{\partial}{\partial p_{n,\nu}} \right) I(p_i) = 0. \tag{3.7}$$

  Contracting with antisymmetric combinations of external momenta $p_i^\mu p_j^\nu$ yields additional relations. These are not linearly independent of the IBP identities and hence not strictly necessary for the reduction to master integrals, but they can accelerate the reduction process.

- **Symmetry relations**: Additional constraints arise from symmetries of the underlying Feynman graphs (such as permutations of external legs or propagators).
- **Dimensional shift relations (DSR)**: Dimensional recurrence relations relate integrals in different dimensions. For instance,

$$I^{2-2\epsilon}_{1,0,1,0} = -2\, I^{4-2\epsilon}_{1,0,2,0}. \tag{3.8}$$

Let us define operators $\mathbf{j}^+$

$$\mathbf{j}^+ I^{(D)}_{\nu_1,\ldots\nu_j,\ldots,\nu_n} = \nu_j I^{(D)}_{\nu_1,\ldots,\nu_{j+1},\ldots,\nu_n}, \qquad \mathbf{j}^- I^{(D)}_{\nu_1,\ldots\nu_j,\ldots,\nu_n} = I^{(D)}_{\nu_1,\ldots,\nu_{j-1},\ldots,\nu_n} \tag{3.9}$$

One can raise the dimension by 2 with the recurrence identity:

$$I^{(D-2)}_{\nu_1,\ldots,\nu_n} = \mathcal{U}(\mathbf{1}^+,\ldots,\mathbf{n}^+) I^{(D)}_{\nu_1,\ldots,\nu_n} \tag{3.10}$$

Lowering by 2 requires a bit more work, but can also be worked out,

$$I^{(D+2)}_{\nu_1,\ldots,\nu_n} \sim \mathcal{B}(\mathbf{1}^-,\ldots,\mathbf{n}^-) \tag{3.11}$$

where $\mathcal{B}$ is a polynomial originating from Baikov representation. We may choose D according to our needs, e.g. to make integrals less divergent, or for easier integration.

The method of Differential Equations

Among modern techniques for evaluating multi-loop Feynman integrals, the method of differential equations stands out as one of the most powerful. To gain some intuition, let us begin with a toy example: the massless one-loop box integral, defined as

$$I_{a_1,a_2,a_3,a_4} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{[-k^2]^{a_1}[-(k+p_1)^2]^{a_2}[-(k+p_1+p_2)^2]^{a_3}[-(k-p_4)^2]^{a_4}}. \tag{3.12}$$

The independent kinematic invariants for this integral are only two:

$$s = (p_1+p_2)^2, \qquad t = (p_2+p_3)^2.$$

Let $\vec{f}$ denote the vector of master integrals. One can derive a system of differential equations for $\vec{f}$ with respect to these invariants. The system takes the form

$$\partial_s \vec{f}(s,t,\epsilon) = A_s(s,t,\epsilon)\, \vec{f}(s,t,\epsilon), \tag{3.13}$$

$$\partial_t \vec{f}(s,t,\epsilon) = A_t(s,t,\epsilon)\, \vec{f}(s,t,\epsilon), \tag{3.14}$$

with explicit coefficient matrices

$$A_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{-\epsilon}{s} & 0 \\ \frac{-2(1-2\epsilon)}{st(s+t)} & \frac{2(1-2\epsilon)}{s^2(s+t)} & -\frac{s+t+\epsilon t}{s(s+t)} \end{pmatrix} ds, \quad A_t = \begin{pmatrix} \frac{-\epsilon}{t} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-2(1-2\epsilon)}{t^2(s+t)} & \frac{-2(1-2\epsilon)}{st(s+t)} & -\frac{s+t+\epsilon t}{t(s+t)} \end{pmatrix} dt. \tag{3.15}$$

4 Canonical differential equations

As one can see, this representation is not particularly convenient for integrating out the components of $\vec{f}$ and obtaining explicit analytic solutions. A much more effective approach is to cast the system into a so-called *canonical form*, in which the dependence on ϵ completely factorizes. In such a basis $\vec{f_T}$, the system takes the form

$$\frac{\partial}{\partial x}\,\vec{f_T}(x,\epsilon) = \epsilon\, A(x)\,\vec{f_T}(x,\epsilon), \tag{4.1}$$

for any kinematic variable x.

In this canonical form, the system can be solved almost straightforwardly. Moreover, one can often arrange $A(x)$ such that its entries have only simple poles in the external invariants. That is,

$$\frac{\partial}{\partial x}\,\vec{f_T}(x,\epsilon) = \epsilon \sum_i \frac{a_i}{x-c_i}\,\vec{f_T}(x,\epsilon), \tag{4.2}$$

where the a_i are constant matrices and the c_i are rational numbers. The denominators $(x-c_i)$ specify the types of functions that appear in the solution and are referred to as the *letters* of the differential equation.

Integrating iteratively order-by-order in ϵ yields a Laurent expansion, whose coefficients are *iterated integrals*. In fact, generic Feynman integrals can be expressed in terms of iterated integrals over appropriately chosen differential forms on Riemann surfaces of various genera. Geometric considerations provide guidance: for example, taking the abelian differential on the genus-zero Riemann sphere leads to the class of multiple polylogarithms, while abelian differentials on a genus-one Riemann surface give rise to elliptic polylogarithms.

This lets us also discuss more about iterated integrals, about the necessary condition to express the results of the differential equations through them and so on and additionally conditions of independence of one forms. Before exploring these function spaces and their properties in detail in the next chapters, let us first discuss two important concepts: *transcendental weight* and *purity*.

Uniform transcendentality

The concept of *transcendental weight* plays a central role in the study of Feynman integrals. Roughly speaking, the weight counts the minimal number of integrations required to define a function using rational kernels with simple poles (with suitable generalizations to other one-forms).

For example, the logarithm is a transcendental function of weight one. At higher weights, polylogarithms satisfy the recursive relation

$$\frac{d}{dx}\,\mathrm{Li}_n(x) = \frac{\mathrm{Li}_{n-1}(x)}{x}, \qquad \mathrm{Li}_0(x) = \frac{x}{1-x}, \qquad \mathrm{Li}_n(0) = 0. \tag{4.3}$$

The weight of a product of two uniformly transcendental (UT) functions is defined as the sum of their individual weights. Transcendental constants such as zeta values, $\zeta_n = \mathrm{Li}_n(1)$, or $\pi = (-i)\,\log(-1)$, are special values of transcendental functions of definite weight and are therefore assigned that same weight.

As an explicit example, consider the massless one-loop box integral:

$$st\, e^{\epsilon\gamma_E}\,(-s)^{\epsilon}\, I^{4-2\epsilon}_{1,1,1,1} = \frac{4}{\epsilon^2} - 2\frac{\log(x)}{\epsilon} - \frac{4\pi^2}{3} + \epsilon\left(\frac{7\pi^2}{6}\,\log(x) + \frac{1}{3}\log^3(x) - \pi^2\log(1+x) - \log^2(x)\,\log(1+x) - 2\log(x)\,\mathrm{Li}_2(-x) + 2\,\mathrm{Li}_3(-x) - \frac{34}{3}\zeta_3\right) + \mathcal{O}(\epsilon^2), \tag{4.4}$$

where $x = \frac{t}{s}$. Where we may normalize with $c = (-s)^{\epsilon} \times e^{\epsilon\gamma_E}$ to render the integrals dimensionless and to control proliferation of Euler-gammas at each order in ϵ.

UT functions of weight n with the property that their derivative is also a UT function of weight $n-1$ are called pure.

For the box family, the three master integrals are

$$\mathcal{I}_{1,1,1,1},\ \mathcal{I}_{1,0,1,0},\ \mathcal{I}_{0,1,0,1}. \tag{4.5}$$

For the top-sector integral, we include the normalization factor $s\,t$, as motivated earlier. For the bubble integrals, however, we cannot this particular integral and multiply with a prefactor, like we did in the case of box, to make it UT. From Feynman parametrization one sees that the bubble becomes pure when normalized with $(1-2\epsilon)$. Alternatively, one may use the fact that the bubble integral admits a UT form in $D=2$ dimensionns. In fact $s\,\mathcal{I}_{1,0,2,0}$ is a pure master integral, where the $\mathcal{I}_{1,0,2,0}$ comes from the fact that this integral is related to a bubble in 2 dimension, wheras the factor of s will be explained shortly.

So we might choose as a basis, the following integrals:

$$\vec{g} = (\epsilon\, c\, t\, I_{0,1,0,2}, \epsilon\, c\, s\, I_{1,0,2,0}, \epsilon^2\, c\, st\, I_{1,1,1,1}). \tag{4.6}$$

Another convenient choice of pure master integrals is then

$$\big(c\,t\,\mathcal{I}_{0,1,1,1},\ c\,s\,\mathcal{I}_{1,1,1,0},\ c\,st\,\mathcal{I}_{1,1,1,1}\big). \tag{4.7}$$

In this form, the integrals depend only on x, and the system of differential equations becomes

$$\frac{\partial}{\partial x}\vec{g}(x,\epsilon) = \epsilon\left(\frac{a}{x} + \frac{b}{1+x}\right)\vec{g}(x,\epsilon), \tag{4.8}$$

with

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}. \tag{4.9}$$

From here it is obvious that one can solve it order-by-order in ϵ through HPLs. One sets

$$\vec{g} = \sum_{k\geq 0} \epsilon^k \vec{g}^{(k)}(x), \tag{4.10}$$

and plugging it into the DE it becomes clear that at each order in ϵ, the RHS of that equation is known and can be integrated. Ofcourse its not always this straight-forward and the entries are not as simple as will see.

In fact from canonical form, i.e. this ϵ-factorised differential equations, in multi-varaible case can be written as Chen's iterated integrals.

$$g(\vec{x},\epsilon) = \mathbb{P}exp\left[\epsilon \int_\gamma d\tilde{A}\right]\vec{g}_0(\epsilon), \tag{4.11}$$

where $\mathbb{P}$ stands for path ordering along the integration contour γ, and $\vec{g}_0(\epsilon)$ is a value in the base-point. This formula is to be understood in an expansion in ϵ, where the k-th term in the expansion is a k-fold iterated integral (along γ).

First tools for inferring analytic properties of Feynman integrals

Square roots: Square roots typically arise in integrals with multi-scale Feynman integrals. As a simple example, consider the 2×2 system consisting of a massive bubble and a tadpole integral. The bubble depends on the invariants s and the internal mass m. Without loss of generality, we set $s = x$ and $m^2 = 1$. The differential equation takes the form in (D=2)

$$\partial_x \vec{f}(x,\epsilon) = \begin{pmatrix} 0 & 0 \\ \dfrac{\epsilon}{4-x} & \dfrac{2+\epsilon x}{(4-x)x} \end{pmatrix} \vec{f}(x,\epsilon). \tag{4.12}$$

To integrate the constant term, we introduce the transformation matrix

$$T = \mathrm{diag}\left(1,\ \frac{\sqrt{-4+x}}{\sqrt{x}}\right), \tag{4.13}$$

which is manifestly non-rational. As a note, this is the solution of the homogeneous equation. Under a transformation $\vec{f'} = T\vec{f}$, the corresponding differential equation is obtained using $TAT^{-1} - TD(T^{-1})$. Now the system of differential equations becomes

$$\partial_x \vec{f'}(x,\epsilon) = \epsilon \begin{pmatrix} 0 & 0 \\ -\dfrac{1}{\sqrt{x(x-4)}} & \dfrac{1}{4-x} \end{pmatrix} \vec{f'}(x,\epsilon). \tag{4.14}$$

Integrating the constant term in more complicated cases will be discussed later.

Maximal Cuts

Maximal cuts play a central role in Feynman integral computations and often guide the choice of master integrals. Colloquially, 'cutting' a propagator means forcing the particle propagating through it to be on-shell. This is straightforward mathematically when the propagator appears to the first power: we replace it with a Dirac δ-function, which enforces the on-shell condition. A *maximal cut* corresponds to simultaneously cutting all propagators of an integral. Mathematically, this is equivalent to taking an n-fold residue of the integrand in the complex plane, where n is the number of propagators.

Why do we care? Maximal cuts provide solutions to the homogeneous differential equations satisfied by the Feynman integral. Any such solution remains valid upon multiplication by a non-zero constant. By computing maximal cuts, one can construct integral representations for all homogeneous solutions in any dimension. The computation is simplified using the Baikov parametrization. All independent homogeneous solutions can be obtained by evaluating the maximal cut along distinct contours that avoid branch cuts of the integrand.

Baikov representation

Consider a general L-loop integral,

$$I_{a_1,\ldots,a_n} = \int \frac{d^D k_1}{i\pi^{D/2}} \cdots \frac{d^D k_L}{i\pi^{D/2}} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}}, \tag{4.15}$$

where $k_1, \ldots, k_L$ are loop momenta and $p_1, \ldots, p_E$ are independent external momenta. The number of independent scalar products involving loop momenta is $n = LE + L(L+1)/2$.

In Baikov representation, we change variables from loop momenta to the propagators: $z_1 = D_1, \ldots, z_n = D_n$. Note that for one-loop this is always possible to do, but for higher loop you also have isps, that needs to complemented. Only at one loop the n is equal to the number of denominators. The integral becomes

$$I_{a_1,\ldots,a_n} = \frac{u\, \pi^{(L-n)/2}\, U^{\frac{E+1-D}{2}}}{\Gamma(\frac{D-E-L+1}{2}) \ldots \Gamma(\frac{D-E}{2})} \int \frac{dz_1 \ldots dz_n}{P^{\frac{E+L+1-D}{2}}\, z_1 \ldots z_n}. \tag{4.16}$$

Here U and P are Gram determinants:

$$U = G(p_1, \ldots, p_E), \qquad P = G(k_1, \ldots, k_L, p_1, \ldots, p_E), \tag{4.17}$$

where for a set of four-vectors $\{v_i\}_{i=1}^n$, the Gram determinant is

$$G(v_1, \ldots, v_n) = \det(\mathcal{M}), \quad \mathcal{M}_{ij} = v_i \cdot v_j. \tag{4.18}$$

The polynomial P is called the Baikov polynomial and is expressed in terms of z_i. (irrelavant) Integration is performed in the region

$$\frac{G(k_i, \ldots, k_L, p_1, \ldots, p_E)}{G(k_{i+1}, \ldots, k_L, p_1, \ldots, p_E)} > 0, \quad i = 1, \ldots, L. \tag{4.19}$$

Example: Box integral

$$I_{\text{box}}^{(D)} = \frac{\big(-\frac{1}{4} st(s+t)\big)^{\frac{4-D}{2}}}{\pi^{3/2}\, \Gamma(\frac{D-3}{2})} \int \frac{dz_1 \ldots dz_4}{P^{\frac{5-D}{2}}\, z_1 \ldots z_4}, \tag{4.20}$$

with the Gram determinant

$$\begin{aligned} P = \frac{1}{16}\big(&s^2t^2 + 2s^2tz_2 + 2s^2tz_4 + s^2z_2^2 + s^2z_4^2 - 2s^2z_2z_4 + 2st^2z_1 + 2st^2z_3 \\ &+ 2stz_1z_2 - 4stz_1z_3 + 2stz_2z_3 + 2stz_1z_4 - 4stz_2z_4 + 2stz_3z_4 + t^2z_1^2 + t^2z_3^2 - 2t^2z_1z_3\big). \end{aligned} \tag{4.21}$$

From cuts to leading singularities

Using Baikov parametrization, let us explicitly compute the unitarity cuts of the box integral in $D = 4-2\epsilon$ dimensions where we cut all four propagators, i.e. we take the maximal cuts. By computing contour integrals around the poles of the integrand we pick up a factor of $2\pi i$ for each cut. This factor makes the correspondence between "cut" integral and the original integral.

Let us start with the box cut for general values of the dimension D:

$$(2\pi i)^4 I^{(D)}_{box,cut} = \frac{(\frac{-1}{4}st(s+t))^{\frac{4-D}{2}}}{\pi^{3/2}\Gamma(\frac{D-3}{2})} \oint_{z_1=0}\oint_{z_2=0}\oint_{z_3=0}\oint_{z_4=0} \frac{dz_1...dz_4}{P^{\frac{5-D}{2}} z_1...z_4} \tag{4.22}$$

$$= \frac{(-\frac{1}{4}st(s+t))^{\frac{4-D}{2}}}{\pi^{3/2}\Gamma(\frac{D-3}{2})(\frac{st}{4})^{5-D}}(2\pi i)^4. \tag{4.23}$$

Now we insert different values for D. For $D = 4 - 2\epsilon$, we obtain:

$$(2\pi i)^4 I^{(4-2\epsilon)}_{box,cut} = \frac{(-\frac{1}{4}st(s+t))^{\epsilon}}{\pi^{3/2}\Gamma(\frac{1}{2}-\epsilon)(\frac{st}{4})^{1-2\epsilon}}(2\pi i)^4 =_{\epsilon\to 0} \frac{64\pi^2}{st}. \tag{4.24}$$

Comparing the same result to the UT integral for box diagram, we can see the same normalization factor st and both having transcendental weight two.

So far, we have seen examples of only rational functions from the maximal cut. However, in general they can be algebraic as well: connection to the geometry. For instance for massive bubble in $2 - 2\epsilon$ dimension, we have to leading order in ϵ:

$$(2\pi i)^2 I_{mb,cut} = \frac{-4\pi}{\sqrt{x(x-4)}} + \mathcal{O}(\epsilon), \tag{4.25}$$

with $x = p^2/m^2$.

We may also consider the case of the 2-loop massive sunrise diagram which is defined as

$$I_{\nu_1\nu_2\nu_3} = e^{(2\epsilon\gamma_E)}(m^2)^{\nu_{123}-D}\int \frac{d^Dk_1}{i\pi^{D/2}}\frac{d^Dk_2}{i\pi^{D/2}}\frac{1}{(-q_1^2+m^2)^{\nu_1}(-q_2^2+m^2)^{\nu_2}(-q_3^2+m^2)^{\nu_3}}, \tag{4.26}$$

with $x = -p^2/m^2$ and $q_1 = k_1$, $q_2 = k_2 - k_1$, $q_3 = -k_2 - p$, and we have set $\mu^2 = m^2$. The three master integrals are $(I_{110}, I_{111}, I_{211})$ and we can obtain the DE:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(D-3) & -3 \\ 0 & (D-3)(3D-8)/6 & (3D-8)/2 \end{pmatrix}\frac{dx}{x} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{4} & -(D-3)(3D-8)/8 & -(D-3) \end{pmatrix}\frac{dx}{x+1}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & (3-D)(3D-8)/24 & -(D-3) \end{pmatrix}\frac{dx}{x+9} \tag{4.27}$$

Similarly, for massive sunrise, we obtain (using the loop-by-loop approach)

$$(2\pi i)^3 I_{Msunrise,cut}(2-2\epsilon, x) = \frac{(2\pi i)^3}{\pi^2}\int_{C_{maxcut}} \frac{dz}{\sqrt{z(z+4)(z^2+2(1+x)z+(1-x)^2)}} + \mathcal{O}(\epsilon). \tag{4.28}$$

One can convince themselves that there is a coupled system of 2x2 differential matrix, which cannot be "removed" without transcendental transformations.

5 Towards numerical evaluation of Feynman integrals

Numerical integration: sector decomposition

Another approach is numerical evaluation, based on *sector decomposition*. This method operates directly in the Feynman parameter space and is particularly useful for extracting singularities regulated by dimensional regularisation. The basic idea is to decompose the integration region into sectors with a simple singularity structure, expand in the regularisation parameter ϵ, and finally evaluate the finite coefficients numerically.

We are interested in resolving the singularities of Feynman integrals. Lets consider a triangle diagram, with $p_1^2 = p_2^2 = 0$ and $(p_1+p_2)^2 = s$. Here $\mathcal{U} = \sum \alpha_i$ and $\mathcal{F} = -s\alpha_1\alpha_3$. We get from the general formula:

$$\int_{\alpha_i \geq 0} d^3\alpha \delta(1 - \sum_i \alpha_i) \frac{(\alpha_1+\alpha_2+\alpha_3)^{(-1-\epsilon)}}{(-s\alpha_1\alpha_3)^{1-2\epsilon}}. \tag{5.1}$$

This integral clearly has endpoint singularities. We split this integration into three sectors, to resolve these singularities:

$$\begin{aligned} &\theta(\alpha_1-\alpha_2)\theta(\alpha_1-\alpha_3) \\ &\theta(\alpha_2-\alpha_1)\theta(\alpha_2-\alpha_3) \\ &\theta(\alpha_3-\alpha_1)\theta(\alpha_3-\alpha_2) \end{aligned} \tag{5.2}$$

these are the sectors with largest α_i.

Lets look at the first sector: We first map $\alpha \to t$, $\alpha_2 \to t\alpha_2$ and $\alpha_3 \to t\alpha_3$ we get

$$\int_0^1 d\alpha_2 \int_0^1 d\alpha_3 (-s)^{-1+2\epsilon} \alpha_3^{-1+2\epsilon} (1+\alpha_2+\alpha_3)^{-1-\epsilon}. \tag{5.3}$$

where we have used the two rules

$$\int \frac{dt}{t} \delta(1 - t\sum x_i) = 1, \tag{5.4}$$

$$\int_0^1 t^{-1+a\epsilon} = \frac{1}{a\epsilon}. \tag{5.5}$$

Now we add the subtraction term:

$$\alpha_3^{-1+2\epsilon} \Big((1+\alpha_2+\alpha_3)^{(-1-\epsilon)} - (1+\alpha_2)^{-1-\epsilon} \Big) + \alpha_3^{-1+2\epsilon} (1+\alpha_2)^{-1-\epsilon}. \tag{5.6}$$

Here, the first term is free of end-point divergence, and can be evaluated numerically and the second term isolates the divergences and can be integrated numerically using the second formula above. We get for this term:

$$\frac{1}{2\epsilon} \int_0^1 d\alpha_2 (1+\alpha_2)^{-1-\epsilon}, \tag{5.7}$$

which can also be integrated numerically. This procedure also provides a constructive interpretation of Feynman integrals as periods.

Numerical Evaluation of the function space

We would like to get for specified input parameters (i.e. for specified kinematic variables) a number for a Feynman integral (more precisely for the coefficients of the Laurent expansion in the dimensional regularisation parameter). We begin by first reviewing the definition of Chen's iterated integrals.

Iterated integrals

Consider a path γ on an n-dimensional manifold M with starting point $x_i = \gamma(0)$ and end-point $x_f = \gamma(1)$, $\gamma : [0,1] \to M$. Let us also consider a set of differential 1-forms $\{\omega_i\}$ and their pullbacks to the interval $[0,1]$, which we denote by $f_j(\lambda)\, d\lambda = \gamma^* \omega_j$. The k-fold iterated integral over $\omega_1, ..., \omega_k$ along γ is defined as

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \, ... \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k) \tag{5.8}$$

$$= \int_0^\lambda d\lambda_1 f_1(\lambda_1)\, I_\gamma(\omega_2, ..., \omega_k; \lambda_1)\,, \tag{5.9}$$

where we define the 0-fold integrals as $I_\gamma(;\lambda) = 1$. The iterated integrals defined in Eq. (5.8) have many useful properties. Some well-known examples of iterated integrals are the multiple polylogarithms (MPLs), which are a special class of functions, where the 1-forms ω_i are such that $\gamma^* \omega_j = \frac{d\lambda}{\lambda - c}$, for some $c \in \mathbb{C}$. The class of MPLs is quite well understood. There exist many tools for their algebraic manipulation and numerical evaluation. However, it is not always possible to express all master integrals in terms of these functions. This is especially true in computations that involve massive internal particles, where we often encounter elliptic curves in the analytic structure of the master integrals.

For our work, the most important property is regarding the decomposition of the path γ. Let there be two paths α, β: $[0,1] \to M$, such that $\beta(0) = \alpha(1)$ and let $\gamma = \alpha\beta$ be the path obtained by concatenating α and β. Then we can decompose

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) = \sum_{i=0}^{k} I_\beta(\omega_1, ..., \omega_i; \lambda)\, I_\alpha(\omega_{i+1}, ..., \omega_n; \lambda)\,. \tag{5.10}$$

How to get numbers out of it?

- In simplest cases, i.e. rational one-forms one can express these iterated integrals as MPLs. These can be numerically evaluated.
- In generic cases, we might use the freedom in choice of the path to integrate our integrals. For instance, we might choose a path such that the one-forms simplify on that path.
- For generic one-forms and generic paths, we may employ series expansion techniques to do numerical evaluations.

Series expansion of the one-forms

At each order in ϵ, the master integrals are expressed as a linear combination of the iterated integrals.

$$MI = \sum c_i I^i_\gamma(\omega_{i_1}, ..., \omega_{i_k}; \lambda) \tag{5.11}$$

We need to be able to numerically evaluate each of these components. Lets first consider a path, say γ on which we parametrize our iterated integrals, aka our one-forms. We basically integrate along a one-dimensional path connecting two fixed points in the space of the Mandelstam variables.

$$x(\lambda) = x_i + (x_f - x_i)\lambda, \quad \lambda \in [0, 1]. \tag{5.12}$$

We may perform series expansions of our one-forms around the base point of this path:

$$\gamma^* \omega_i(\lambda) = d\lambda f_i(\lambda) = d\lambda \sum_j c_j(x_f, x_i)\lambda^j. \tag{5.13}$$

Integrating this expression to give us a form

$$\int_0^\lambda d\lambda' f_i(\lambda') = \sum c_{jk} \lambda^j (log(\lambda))^k. \tag{5.14}$$

This step can be done iteratively. This is closed under integration.

$$\int \lambda^j (log(\lambda))^k = \sum c_{lm} \lambda^l (log(\lambda))^m. \tag{5.15}$$

Integral of this form again gives us series of that form. If we choose a path thats within the radius of convergence then its trivial to do the integration. However, every series expansion has a radius of convergence and it might be that it is not sufficient to cover the whole path. In that case, we might choose another path and series expand around the base point of that path. With these two paths our full path is covered. We may integrate our iterated integral on this path and use the path-concatenation formula to patch up the full result of the integration.

After constructing the solution along a path, there are ofcourse other things that we need to understand, for example, analytic continuation around the different branch-points, the determination of the boundary values, and the estimation of the numerical precision of our solutions. The number of paths, the number of digits we require for precision, all affect the speed of these numerical evaluations.