Notes on Amplitudes for Gravitational Waves

Radu Roiban

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Abstract

We review some applications of QFT scattering amplitudes to gravitational wave physics

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1 Introduction

I was tasked to review some applications of QFT scattering amplitudes, including the double copy, in the context of gravitational wave emissions.

You already saw in Riccardo's lectures about the close connection between QFT and classical scattering observables, so we can start from there and talk about constructing the relevant amplitudes, building also on Yu-tin's lectures. One motivation for this is to compute scattering observables; another is to compute bound observables via analytic continuation. We are going to discuss methods to compute gravitational amplitudes, we will see some examples, and then we will discuss some aspects of the gravitational waveforms that can be inferred without calculations that are very involved.

This said, perturbation theory takes us only so far. To get closer to the strong-field regime, which is an inevitable regime for bound systems, some kind or resummation is necessary. Some resummation methods are available in QFT (cf e.g. Ira's lectures); for the time being however, the method of choice for the classical two-body problem is the EOB. That is several lectures in themselves, and perhaps Michele Valisneri mentioned some of it last week. We won't go into it, but the important bit for us is that it takes two inputs: a conservative Hamiltonian and radiation reaction forces (which can be evaluated from the waveform).

Thus, for the purpose of feeding scattering data into EOB, a Hamiltonian is necessary. Given scattering observables, we can construct a Hamiltonian that reproduces them. Depending on how fast we cover the rest, we will also talk a little about constructing Hamiltonians, building on Ira's lectures. At the lowest power of Newton's constant this relation is very easy to state, and the justification can be inferred from standard NR quantum mechanics lectures:

$$A = \text{Fourier transform of the potential} \tag{1}$$

We will derive this in due course, but

$$\mathcal{M}_4^{\rm GR}(\phi,\phi',\phi',\phi) = i\kappa^2\,\frac{1}{q^2}\,\frac{1}{4E_1E_2}\Big((p_1{\cdot}p_2)^2 + \frac{1}{D-2}m_1^2m_2^2 + \text{non-classical}\Big)\,, \tag{2}$$

In the COM $q^2 \to -\boldsymbol{q}^2$; and the Fourier transform gives

$$V(\boldsymbol{r}) \sim -\frac{Gm_1m_2}{|\boldsymbol{r}|}\,\frac{m_1m_2}{E_1E_2}\,\left(2(u_1\cdot u_2)^2-1\right)\ . \tag{3}$$

Expanding in velocity, gives Newton's potential and further velocity corrections, the first one of which is part of the Einstein-Infeld-Hoffman 1PM potential (up to a canonical transformation).

2 Classical limit and structure of classical amplitudes

There are several ways to describe the classical limit of scattering amplitudes, and you saw some of them in earlier lectures, such as restoring and counting factors of $\hbar$. Another way to think about it is to demand that the variance of any observable is much smaller than the observable, $\Delta O/O \ll 1$. Yet another is Bohr's Correspondence Principle, which equates the classical limit with the limit of large changes – *i.e.* large mass, momenta and angular momenta:

$$\frac{1}{m_i} \ll R_i \quad \frac{1}{m_i v_i} \ll R_i\ , \quad \frac{1}{m_i} \ll b\ , \quad \frac{1}{m_i v_i} \ll b\ , \tag{4}$$

where m_i is the mass of the scattering objects, v_i their velocities, R_i is their characteristic size (*e.g.* for black holes the *only* characteristic length is their Schwarzschild's radii $R_i = Gm_i$) and b is typically the impact parameter. The 3rd and 4th follow from the 1st and 2nd and the assumption that we want to use perturbation theory so we are in a weak field regime. There is in fact a more precise form of this –

$$R_i \ll b\,v^2 < b\ , \tag{5}$$

bv^2 is a measure of the minimum approach distance. In particular, this hierarchy of scales corresponds to the so-called *post-Minkowskian* approximation.

We will consider the expectation value of the metric; this introduces a further scale the distance (and time) between the event and the observer, $r = |\vec{x}| \sim x^0$. We assume that they are larger than any other typical length of the system:

$$\frac{1}{m_i} \ll R_i \ll b \ll r\ , \tag{6}$$

and we will perform an asymptotic expansion at large values of r. This allows us to use regular scattering theory, with LSZ reduction, etc.

As the separation of particles is Fourier-conjugate to the change in the momentum of each particle, it follows that the momentum transfer q_i must be much smaller than the momenta of the two particles. If there is any massless radiation in the initial or final state, its momentum should be of the same order as the momentum transfer, i.e. much smaller than the massive particles. A way to implement this is by the rescaling

$$(q, k, \ell) \to (\lambda q, \lambda k, \lambda \ell)\,, \tag{7}$$

where k and ℓ are, respectively, the momenta of external and internal gravitons, and expanding at small λ. This process is the *soft expansion*. Accounting for the classical nature of Newton's potential, classical L-loop four-scalar amplitudes depend on Newton's constant and λ as

$$\mathcal{M}^{\text{cl.}}_{4+n_g,L\text{-loop}} \sim G^{L+1+\frac{1}{2}n_g}\lambda^{-2+L}\ , \tag{8}$$

where n_g is the number of external gravitons. Effective potential:

$$\mathcal{F}[V, q] = \sum_{ijk} \frac{c_{ijk}}{q^3}(Gm|q|)^i(qS/m)^j(Rq)^k \tag{9}$$

These considerations have a couple of consequences on the structure of classical amplitudes and consequently on the structure the generalized cuts determining them.

- We assumed that the matter particles are separated by some distance larger than their characteristic scale. Therefore in (Feynman) diagrams matter lines should not intersect. If they do, then they have triangle topology so Fourier-transform to b-space produces $\delta(b)$. From perspective of unitarity – the only relevant cuts separate the matter lines

- Intuitively, graviton loops are part of quantum gravity, not classical physics. One may also count the powers of momenta in the soft region, and notice that they don't satisfy the expected power counting. From the perspective of diagrams this means that every loop must have at least one matter line. From the perspective of unitarity this means that in a generalized cut, every loop must have a cut matter line.

- Terms with too high a scaling in q or ℓ are dropped.

Depending on purposes, the soft region can be further split into parts. Here it is convenient to split it into soft and off-shell, and soft and close to on-shell: (connection with Martin's and Ira's lectures)

$$\text{potential}: \quad (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}|) \sim J^{-1}\left(m|\boldsymbol{v}|^2, m|\boldsymbol{v}|\right), \tag{10}$$

$$\text{radiation}: \quad (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}||\boldsymbol{v}|) \sim J^{-1}\left(m|\boldsymbol{v}|^2, m|\boldsymbol{v}|^2\right), \tag{11}$$

The only scale is really the velocity of the matter particles, so this is what we used to separate the space-like and time-like components, but on-shell and off-shell is physically more important.

Because gravitons in the potential carry little energy, they mediate effectively instantaneous interactions, which can be modeled by a potential. In particular, such gravitons cannot be lost to (null) infinity, so the potential region is conservative. In contrast, radiation region mediates long-range interactions, and also the corresponding gravitons can be lost to infinity. This does not necessarily mean that everything in the radiation region is dissipative; however, at sufficiently high loop orders the conservative and dissipative pieces are difficult to separate. Options:

- particle production is usually captured by the imaginary part of an amplitude, so one may define the real part as *the conservative part*;

- conservative physics is time-reversal invariant so one may define the even-in-velocity part as *the conservative part*

- real particles correspond to the pole of propagators; removing poles of propagators prevents them from becoming real. So use principal-value as graviton propagators.

First time this appears all give the same answer, but they start to diverge and become confusing.

One may identify in unitarity cuts which gravitons are potential and which gravitons are real. Rule(s) of thumb: <draw figures>

- 3-point matter-matter-graviton amplitude factors force graviton off-shell/potential
- radiation + radiation = off-shell; potential+radiation = off-shell

Restriction to potential region allows dropping some more diagrams and unitarity cuts. Because potential gravitons are instantaneous, they cannot start and end on the same matter particle. If they do, then those two points must coincide. So, if we are interested in the potential region, we may drop all diagrams and cuts with gravitons starting and ending on the same matter line.

In a KMOC calculation of observables there is one further constraint: one can cut a single graviton. Everything else does not scale clasically. Consider cutting n gravitons:

$$\frac{1}{q^3}(Gq)^{L_1+1}\kappa^n\frac{1}{q^3}(Gq)^{L_3+1}\kappa^n\frac{1}{(q)^2}\frac{q^{d(n+1)}}{(q^2)^n}\overset{d=4}{=}\frac{1}{q^4}(Gq)^{L_1+L_2+(n+1)+1}q^{n-1} \tag{12}$$

So, if $n \geq 2$, this does not scale classically and can be ignored (unless one in interested in corrections to classical physics).

Implementing these observations leads to enormous simplifications at the integrand level, effectively by truncating away quantum contributions.

Now that we know what kind of generalized cuts we need to consider, let's get back to constructing amplitudes – or rather the relevant parts of amplitudes. Since we are mainly interested in gravity we will focus on this, and on double-copy as a means to construct the corresponding trees. Why only trees will become clear shortly.

3 Double copy and unitarity

3.1 General comments

As we understand it today, the double-copy is a machine that takes two colored theories and produces a third one, either colorless or with a different color than the parents. The main example is getting gravity out of gauge theories.

The first indication of double copy is for free particles – there is graviton polarization tensor in the tensor product of gluon polarization tensors. For a generic $(D-2)$-2-tensor f_ij

$$f^{ij}\epsilon_i^\mu\epsilon_i^\nu \equiv H^{\mu\nu} = h^{\mu\nu} + B^{\mu\nu} + \frac{1}{D-2}\left(\eta^{\mu\nu} - \frac{p^\mu q^\nu + p^\nu q^\mu}{p\cdot q}\right)\phi \tag{13}$$

$$h^{\mu\nu} = \frac{1}{2}(H^{\mu\nu} + H^{\nu\mu}) - \frac{1}{D-2}\left(\eta^{\mu\nu} - \frac{p^\mu q^\nu + p^\nu q^\mu}{p\cdot q}\right)H^\rho{}_\rho \tag{14}$$

The two somewhat related variants of double copies:

- KLT: "convolution" of color-ordered amplitude with some kernel.

$$\mathcal{A} = \sum_\sigma T_{\sigma_i}A(\sigma_i) \longmapsto \mathcal{M} = \sum K_{\sigma_i\sigma_j}A(\sigma_i)A(\sigma_j) \tag{15}$$

- BCJ: replace color factor by kinematic factors IF color factors and kinematic factors obey the same algebraic relations demanded by gauge invariance:

$$\mathcal{A} = \sum_{\text{cubic graphs}} \frac{c_i n_i}{d_i} \longmapsto \mathcal{M} = \sum_{\text{cubic graphs}} \frac{n_i \tilde{n}_i}{d_i} \tag{16}$$

Kinematic factors obey the same algebraic relations as color factors is referred to as *c/k duality.*

Latter variant manifestly obeys factorization properties expected of amplitudes because it manifestly has all the propagators. Factorization of the former is not manifest by any means. In the products of color-ordered amplitudes one encounters combinations of propagators that don't belong to the same graph, including squared propagators. The kernel – usually referred to as the KLT kernel – removes these unphysical singularities, leaving only the physical ones.

The former variant is manifestly gauge invariant; the latter is not and requires that numerator factors have the same algebraic properties as the color factors.

Both variants, in their simplest forms, work for gluon and graviton amplitudes, in arbitrary dimensions. Scalars can be obtained by dimensional reduction.

There is an interesting question when to look for such a structure. A gauge-invariant answer is when the BCJ amplitudes relations are satisfied:

$$\sum_{i=2}^{m-1} p_1 \cdot (p_2 + \ldots + p_i)\, A_m(2, \ldots, i, 1, i+1, \ldots, m) = 0\,. \tag{17}$$

I am not gong to prove this in general – we will see an example. However, one may understand intuitively the appearance of novel relations between amplitudes: color-kinematics duality requires relations between numerators, and color-ordered amplitudes are are built of ratios n_i/d_i. So, by solving the kinematic Jacobi relations and eliminating the remaining numerators between color-ordered amplitudes one finds these amplitudes relations.

3.2 Color/kinematics duality and double copy

Rather than talk a lot about generalities, let's look at an example: the 4-gluon amplitude. You saw earlier in Yu-tin's lectures the color-ordered amplitudes in spinor-helicity form. Here, in d dimensions, in terms of polarization vectors:

$$\mathcal{A} = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\,, \tag{18}$$

where the color and kinematic factors are

$$c_s = -2 f^{a_1 a_2 b} f^{b a_3 a_4} \tag{19}$$

$$\begin{aligned} n_s = \frac{i}{2}\Big\{ &\Big[(\varepsilon_1\cdot\varepsilon_2)p_1^\mu + 2(\varepsilon_1\cdot p_2)\varepsilon_2^\mu - (1\leftrightarrow 2)\Big]\Big[(\varepsilon_3\cdot\varepsilon_4)p_{3\mu} + 2(\varepsilon_3\cdot p_4)\varepsilon_{4\mu} - (3\leftrightarrow 4)\Big] \\ &+ s\Big[(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4) - (\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\Big]\Big\}\,, \end{aligned} \tag{20}$$

with $n_{t,u}$ and $c_{t,u}$ obtained via circular permutations $1 \to 2 \to 3 \to 1$. $n_{s,t,u}$ are not gauge invariant

$$n_s \overset{\epsilon_4 \to k_4}{\longrightarrow} i\frac{s}{2}\Big[(\varepsilon_1\cdot\varepsilon_2)\big((\varepsilon_3\cdot p_2) - (\varepsilon_3\cdot p_1)\big) + \text{cyclic}(1,2,3)\Big] \equiv s\,\alpha(\varepsilon, p)\,, \tag{21}$$

but the full amplitude is:

$$\mathcal{A} \to (c_s + c_t + c_u)\alpha(\varepsilon, p) = 0 \tag{22}$$

b/c of Jacobi identity obeyed by the structure constants of Lie algebras. This is an example of the algebraic relations I was mentioning and clearly it is required by gauge invariance.

The kinematic numerators turn out to also obey this relation:

$$n_s + n_t + n_u = 0 \ , \tag{23}$$

This is called *color-kinematics duality*. At 4 points it is or it isn't. At higher points there is wiggle room, and we can talk afterwards about this.

Let us then construct:

$$\mathcal{M}_4 = i\Big[\frac{n_s\tilde{n}_s}{s} + \frac{n_t\tilde{n}_t}{t} + \frac{n_u\tilde{n}_u}{u}\Big] \tag{24}$$

It has nice properties; in particular, it is invariant under the gauge transformation of one of the two polarization vectors

$$\epsilon_i^\mu \tilde{\epsilon}_i^\nu \longrightarrow p_i^\mu \tilde{\epsilon}^\nu \tag{25}$$

as a consequence of properties of $\tilde{n}$,

$$\mathcal{M} \to 2(\tilde{n}_s + \tilde{n}_t + \tilde{n}_u)\alpha(\varepsilon, p) = 0 \ . \tag{26}$$

What does this mean?

Each product of pol vectors can be decomposed in irreps of the Lorentz group

$$\epsilon^\mu \otimes \tilde{\epsilon}^\nu = \epsilon_h^{\mu\nu} + \epsilon_B^{\mu\nu} + \frac{1}{D-2}\left(\eta^{\mu\nu} - \frac{p^\mu q^\nu + p^\nu q^\mu}{p\cdot q}\right)\phi\phi \tag{27}$$

so $\mathcal{M}_4$ decomposes in pieces. The transformation decomposes as

$$\epsilon_h^{\mu\nu} \to p_i^{((\mu}\tilde{\epsilon}^{\nu))} \qquad \epsilon_B^{\mu\nu} \to p_i^{[\mu}\tilde{\epsilon}^{\nu]} \qquad \phi \to 0$$

The first is linearized diffeomorphisms, so $\epsilon_h^{\mu\nu}$ describes a graviton.

Some lessons:

- gauge invariance + color kinematics $\longrightarrow$ diffeo invariance
- global symmetry + color kinematics $\longrightarrow$ local symmetry
- relations between kinematic numerators $\longrightarrow$ relations between partial amplitudes

$$\begin{aligned} A(1,2,3,4) &= \frac{n_s}{s} - \frac{n_t}{t} = n_s\Big(\frac{1}{s} + \frac{1}{t}\Big) + \frac{n_u}{t} \ , \\ A(1,3,2,4) &= \frac{n_t}{t} - \frac{n_u}{u} = -\frac{n_s}{t} - n_u\Big(\frac{1}{u} + \frac{1}{t}\Big) \ . \end{aligned} \tag{28}$$

System is degenerate however, which is reasonable 'cause n-s are gauge dependent while A are gauge invariant. Eliminate n_s and n_u while accounting for the relation between Mandelstam invariants:

$$u\,A(1,3,2,4) = s\,A(1,2,3,4) \ . \tag{29}$$

General version:

$$\sum_{i=2}^{m-1} p_1\cdot(p_2 + \ldots + p_i)\,A_m(2,\ldots,i,1,i+1,\ldots,m) \ . \tag{30}$$

We are now ready to illustrate the relation to the KLT relations.

$$\mathcal{M}(1,2,3,4) = i\Big[\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\Big] = -i\frac{st}{u}\Big[A(1,2,3,4)\Big]^2 = -isA(1,2,3,4)A(1,2,4,3) \tag{31}$$

In the second equal sign the relation $u\,A(1,3,2,4) = s\,A(1,2,3,4)$ was used.
The 4th expression has a double pole in the s channel that doesn't belong; the s factor turns it into a single pole.
The 3rd expression has double poles in the s and t channels that don't belong; the st factor turns them into single poles.

3.3 Scalars from vectors and gravitons: Kaluza-Klein reduction amplitudes

We now discuss a simple method for obtaining tree-level matter from pure Yang-Mills scattering amplitudes. We will refer to this as Kaluza-Klein reduction, even though Kaluza-Klein is traditionally considered in the context of gravity.

Let us start with the YM Lagragian in $D+n$ dimensions, and decompose the gauge field as $A_N^a = \{A_\mu, \phi_{Af}\}$, where $\mu = 0,\ldots,D-1$, $f = D,\ldots,D+n-1$, and f is a flavor index. We get

$$-\frac{1}{4}(F_{MN})^2 = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \frac{1}{2}\sum_f \Big[(D_\mu\phi_f^a)^2 - m_f^2(\phi_f^a)^2 + f^x{}_{ab}f^x{}_{cd}\phi_f^a\phi_{f'}^b\phi_f^c\phi_{f'}^d\Big] \tag{32}$$

One interpretation of this operation is that we have toroidally compactified flat space $\mathbb{R}^{1,D+n-1} \to \mathbb{R}^{1,D-1}\times T^n$, where T^n is the n-torus.

In Yang-Mills theory with adjoint representation scalars, the standard way to obtain the scattering amplitude for two distinct scalars would be to use the Feynman diagram. Alternatively, since we already have the four gluon amplitude worked out we can immediately obtain the four-scalar amplitude, by starting in $D+4$ dimensions and reducing to D dimensions

$$\begin{aligned} &p_i^\mu \to (p_i^\mu, \pm m_1, 0, 0, 0), \quad \epsilon_i^\mu \to (0_D, 0, 0, 1, 0), \quad i = 1,4 \\ &p_i^\mu \to (p_i^\mu, 0, \pm m_2, 0, 0), \quad \epsilon_i^\mu \to (0_D, 0, 0, 0, 1), \quad i = 2,3 \end{aligned} \tag{33}$$

This is done by restricting the momenta to live in D dimensions, and having the polarizations of the two scalars in two distinct extra dimensions,

$$\begin{aligned} &\epsilon_1\cdot\epsilon_4 = \epsilon_2\cdot\epsilon_3 = -1\,, \\ &p_i\cdot\epsilon_j = 0 = \epsilon_i\cdot\epsilon_j \quad (i\neq j, \text{otherwise}) \\ &(p_1+p_4)^2 = 0\,, \quad (p_2+p_3)^2 = 0\,, \\ &(p_i+p_j)^2 = m_1^2 + m_2^2 \quad (i\neq j, \text{otherwise}) \end{aligned} \tag{34}$$

The vast majority of terms in the four-gluon amplitude vanish (see (20), (21)), and the kinematic numerators simplify to ($s\to s-m_1^2-m_2^2, u\to u-m_1^2-m_2^2$, $t\to t-m_1^2+2m_1^2-m_1^2 = t$)

$$n_s = \frac{s-m_1^2-m_2^2}{2}\,, \qquad n_t = \frac{u-s}{2}\,, \qquad n_u = -\frac{u-m_1^2-m_2^2}{2}\,, \tag{35}$$

$$A = ic_s + ic_u + c_t\frac{i}{t}(p_2-p_3)^\mu(p_1-p_4)_\mu \tag{36}$$

From Feynman diagrams, the scattering amplitude for two distinct scalars due to gluon exchange is,

$$A_4^{\rm YM}(\phi,\phi',\phi',\phi) = ig^2 f^{a_2a_3b} f^{ba_1a_4}(p_1-p_4)^\mu \,\frac{1}{t}\,(p_2-p_3)^\mu$$
$$= ig^2 f^{a_4a_1b} f^{a_2a_3b}\Big(\frac{4p_1\cdot p_2}{t}+1\Big)\,, \tag{37}$$

The 4-scalar gravitational amplitude is then

$$\mathcal{M}_4(\phi,\phi,\phi,\phi) = \Big(\frac{\kappa}{2}\Big)^2 \frac{1}{t}\Big((p_1-p_4)^\mu\,\eta_{\mu\nu}\,(p_2-p_3)^\nu\Big)^2 = \Big(\frac{\kappa}{2}\Big)^2 \frac{1}{t}\Big(4p_1\cdot p_2+t\Big)^2\,. \tag{38}$$

This is the 4-scalar amplitude in fat gravity – scalar coupled with graviton, dilaton and axion. To get pure GR we need to project out the dilaton and axion. To this end we recognize the first expression $\eta_{\mu\nu}\eta_{\rho\sigma}$ as the sum over all the states in the tensor product of two vectors. Thus, we need to replace it with the sum over the graviton physical states:

Focusing on the graviton polarization tensor – symmetric, traceless, transverse – it is

$$\varepsilon_{\mu\nu}^{\lambda\lambda'} \equiv \frac{1}{2}\big(\varepsilon_\mu^\lambda \tilde\varepsilon_\nu^{\lambda'} + \varepsilon_\nu^\lambda \tilde\varepsilon_\mu^{\lambda'}\big) - \frac{1}{D-2}\varepsilon_\mu^\omega \tilde\varepsilon_\nu^\omega\,\delta^{\lambda\lambda'}\,, \tag{39}$$

The sum over the physical polarizations of a vector is

$$\mathcal{P}_{\mu\nu} \equiv \sum_\lambda \varepsilon_\mu^\lambda \varepsilon_\nu^{\lambda *} = -\eta_{\mu\nu} + \frac{p_\mu q_\nu + p_\nu q_\mu}{p\cdot q}\,. \tag{40}$$

This follows from transversality on k and on some null reference vector q (one for each external state) and the sign is related to positivity of the inner product in the space of states. Then, the sum over polarizations becomes

$$\mathcal{P}_{\mu\nu;\rho\sigma} \equiv \sum_{\lambda\lambda'} \varepsilon_{\mu\nu}^{\lambda\lambda'}\varepsilon_{\rho\sigma}^{\lambda\lambda'} \tag{41}$$
$$= \frac{1}{2}\big(\mathcal{P}_{\mu\rho}\mathcal{P}_{\nu\sigma}+\mathcal{P}_{\mu\sigma}\mathcal{P}_{\nu\rho}\big) - \frac{1}{D-2}\mathcal{P}_{\mu\nu}\mathcal{P}_{\rho\sigma} \to \frac{1}{2}\big(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}\big) - \frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\,, \tag{42}$$

The result is

$$\mathcal{M}_4^{\rm GR}(\phi,\phi',\phi',\phi) = i\kappa^2 \frac{1}{q^2}\Big(p_1\cdot p_2\, p_1\cdot p_3 + \frac{1}{D-2}m_1^2 m_2^2 + \lambda q^4\Big)\,, \tag{43}$$

where λ is some constant.

3.4 Generalized unitarity, Double copy at loop level

The double copy construction works at loop level though some difficulties appear at high enough loop order. The difficulties relate to finding representation of loop integrands that exhibit color/kinematics duality. For our purpose however – classical loop corrections in GR coupled with matter – we also need to project out unwanted states. There are essentially two strategies for doing this.

(1) introduce ghost fields that cancel out the unwanted states

(2) project them out from generalized cuts and reconstruct integrand from the resulting cuts

We will discuss only the second version, focusing specifically on gravity. As you saw in Yu-tin's lectures, under suitable circumstances, d-dimensional generalized cuts that are products of tree amplitudes are sufficient to reconstruct amplitudes integrands. The same is true for final state

observables in the observable-based formalism that were defined in Riccardo's lectures, because they are defined in terms of S-matrix elements. For gravitational wave applications in the classical limit these conditions are met even though we are interested in theories of massive particles: tadpoles and bubbles on external lines are zero.

So, consider such a cut, which is a product of tree amplitudes, in a gravitational theory:

$$\mathcal{C}^{\text{gravity}} = \sum_{\substack{\text{gravity}\\ \text{states}}} \mathbb{M}^{\text{tree}}_{(1)} \mathbb{M}^{\text{tree}}_{(2)} \cdots \mathbb{M}^{\text{tree}}_{(m)} \ . \tag{44}$$

Assume that this gravitational theory is a double copy at tree level and writing each factor as

$$\mathbb{M}^{\text{tree}}_n = \sum_{i\in\Gamma} \frac{n_i \tilde{n}_i}{d_i} \longleftarrow \mathcal{A}^{\text{tree}}_n = \sum_{i\in\Gamma} \frac{c_i n_i}{d_i} \ \oplus \ \tilde{\mathcal{A}}^{\text{tree}}_n = \sum_{i\in\Gamma} \frac{c_i \tilde{n}_i}{d_i} \ . \tag{45}$$

The double copy assumes that the gravity states are products of gauge th. states –

$$\text{asymptotic gravity states} = \text{asymptotic gt states} \times \ \text{asymptotic gt states}$$

It therefore follows that

$$\mathcal{C}^{\text{gravity}} = \sum_{j_1\in\Gamma_1} \cdots \sum_{j_m\in\Gamma_m} \frac{1}{d_{j_1}\dots d_{j_m}} \sum_{\substack{\text{gauge}\\ \text{states}}} n_{j_1}\dots n_{j_m} \sum_{\substack{\text{gauge}\\ \text{states}}} \tilde{n}_{j_1}\dots \tilde{n}_{j_m} \ . \tag{46}$$

We recognize the two sums as giving the kinematic numerators of the corresponding gauge theory cuts,

$$\begin{cases} \mathcal{C}^{\text{gauge}} = \displaystyle\sum_{\substack{\text{gauge}\\ \text{states}}} \sum_{j_1\in\Gamma_1} \sum_{j_2\in\Gamma_2} \cdots \sum_{j_m\in\Gamma_m} \frac{c_{j_1} n_{j_1}}{d_{j_1}} \frac{c_{j_2} n_{j_2}}{d_{j_2}} \cdots \frac{c_{j_m} n_{j_m}}{d_{j_m}} = \sum_{\text{states}} \mathbb{A}^{\text{tree}}_{(1)} \mathbb{A}^{\text{tree}}_{(2)} \mathbb{A}^{\text{tree}}_{(3)} \cdots \mathbb{A}^{\text{tree}}_{(m_i)} \\ \widetilde{\mathcal{C}}^{\text{gauge}} = \displaystyle\sum_{\substack{\text{gauge}\\ \text{states}}} \sum_{j_1\in\Gamma_1} \sum_{j_2\in\Gamma_2} \cdots \sum_{j_m\in\Gamma_m} \frac{c_{j_1} \tilde{n}_{j_1}}{d_{j_1}} \frac{c_{j_2} \tilde{n}_{j_2}}{d_{j_2}} \cdots \frac{c_{j_m} \tilde{n}_{j_m}}{d_{j_m}} = \sum_{\text{states}} \mathbb{A}^{\text{tree}}_{(1)} \mathbb{A}^{\text{tree}}_{(2)} \mathbb{A}^{\text{tree}}_{(3)} \cdots \mathbb{A}^{\text{tree}}_{(m_i)} \end{cases} . \tag{47}$$

The next thing to discuss is the projection of unwanted states. In the case at hand, we have the dilaton and antisymmetric tensor; in a spinor helicity form this can be done by simply following the phase weight, i.e. by projecting out the states with vanishing phase weight in the sum. Similarly, in d dimensions this amounts to revisiting the sum factorization in Eq. (46) and using instead the sum over states – formally

$$\mathcal{C}^{\text{gravity}} = \sum_{j_1\in\Gamma_1} \cdots \sum_{j_m\in\Gamma_m} \frac{1}{d_{j_1}\dots d_{j_m}} \mathcal{P}\left[\sum_{\substack{\text{gauge}\\ \text{states}}} n_{j_1}\dots n_{j_m}, \sum_{\substack{\text{gauge}\\ \text{states}}} \tilde{n}_{j_1}\dots \tilde{n}_{j_m} \right] \ ; \tag{48}$$

We discussed the graviton projector in our discussion of tree-level amplitudes. Because of the presence of the reference vector, simply plugging it in makes expressions quite nasty. To start with, the pole sits on the real line, because there is no $i\epsilon$ prescription for it. They go away if one goes to an integrand basis (no integration-by-parts relations).

As we saw at tree level, for a single sewn propagator q drops out manifestly b/c of gauge invariance IF one is allowed to use the transversality of all the other polarization vectors/tensors. In a cut, where we sew more than one polarization vector. It turns out that one can choose representations of gauge theory tree amplitudes so that they obey gauge invariance on all but 2 legs without making

use of transversality of the polarization vectors. This form is compatible w/ c/k. To reach this form one chooses to solve momentum conservation (while using transversality) so that all the $\epsilon \cdot k$ that can appear upon gauge transforms are already in the basis of scalar products.

For such representations q drops out and the projector collapses to

$$\mathcal{P}_{\mu\nu;\rho\sigma} \to \frac{1}{2}\big(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\big) - \frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma} \ , \tag{49}$$

i.e. the numerator of the graviton propagator in de Donder gauge.

Example: Scalar-QED Compton amplitude:

$$A = 2i\left(\frac{p_4\cdot\epsilon_1 p_3\cdot\epsilon_2}{p_1\cdot p_4} + \frac{p_4\cdot\epsilon_2 p_3\cdot\epsilon_1}{p_1\cdot p_3} + \epsilon_1\cdot\epsilon_2\right) \tag{50}$$

vs.

$$A = 2i\left(\frac{(-p_2-p_3)\cdot\epsilon_1(-p_1-p_4)\cdot\epsilon_2}{p_1\cdot p_4} + \frac{(-p_2-p_3)\cdot\epsilon_2(-p_2-p_4)\cdot\epsilon_1}{p_1\cdot p_3} + \epsilon_1\cdot\epsilon_2\right) \tag{51}$$

4 Some explicit calculations

4.1 Aspects of the scattering waveform

To recall from Riccardo's lectures, the waveform is refined as

$$W(x,t) = \epsilon^{\mu\nu}\lim_{r\to\infty} r(g_{\mu\nu} - \eta_{\mu\nu}) \tag{52}$$

where $g_{\mu\nu}$ is the metric sourced by the scattering process.

The corresponding operator, whose expectation value gives the wave form is

$$\mathbb{h}_{\mu\nu}(x) = \sum_\eta \int \mathrm{d}\Phi(k)\, e^\eta_{\mu\nu}(k) a_\eta(k)\, e^{-ik\cdot x} + \text{h.c.} \simeq \sum_\eta \frac{-i}{4\pi|\boldsymbol{x}|}\int_0^\infty \hat{\mathrm{d}}\omega\, e^\eta_{\mu\nu}(k) a_\eta(k) e^{-i\omega u}|_{k=\omega n} + \text{h.c.}\,,$$

where $u = x^0 - |\boldsymbol{x}|$ and

$$\mathrm{d}\Phi(k) \equiv \hat{\mathrm{d}}^4k\,\hat{\delta}_+(k^2)\,, \quad \hat{\mathrm{d}}k \equiv \frac{\mathrm{d}k}{2\pi}\,, \quad \hat{\delta}(k) \equiv 2\pi\,\delta(k)\,, \quad \delta_+(k^2) \equiv \delta(k^2)\Theta(k^0). \tag{53}$$

The calculation justifying the second equal sign is as follows.

$$\int \hat{\mathrm{d}}^4k\,\hat{\delta}_+(k^2)\, a_\eta(k)\, e^{-ik\cdot x} = \int_{0^\infty} \hat{\mathrm{d}}\omega \int \hat{\mathrm{d}}^3n \hat{\delta}(1-\boldsymbol{n}^2)\omega e^{-i\omega x^0} e^{i\omega \boldsymbol{n}\cdot\boldsymbol{x}}\, a_\eta(k) \tag{54}$$

$$= \int_{0^\infty} \hat{\mathrm{d}}\omega \int \hat{\mathrm{d}}^3n \int d\lambda e^{i\lambda(1-\boldsymbol{n}^2)} e^{-i\omega x^0} e^{i\omega\boldsymbol{n}\cdot\boldsymbol{x}}\,\omega\, a_\eta(k) \tag{55}$$

We integrate via saddle-point approx[1] over $\boldsymbol{n}$ and λ. This is justified as long as $\omega x \gg 1$; it must however be revisited for $\omega < x^{-1}$. Such frequencies are not just a theoretical construct; we will see that they are responsible for the gravitational-wave memory and other interesting things. The saddle-point eqs are

$$n^2 = 1 \quad 2\lambda\boldsymbol{n} = \omega\boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{n} = \pm\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \quad \lambda = \pm\frac{1}{2}\omega|\boldsymbol{x}| \ . \tag{56}$$

[1] $\int dz^n e^{if(z)} g(z) = (2\pi)^{n/2}\sum_{z_0} e^{if(z_0)} g(z_0) \frac{e^{i\pi/4\,\mathrm{sgn}(H(z_0)}}{\sqrt{\det H_{(z_0)}}}$

The two solutions depend on the retarded time $u = t - |\boldsymbol{x}|$ and the advanced time $v = t + |\boldsymbol{x}|$; assuming that the observer is in the future of the event, we can ignore the term depending on the advanced time.

So the metric fluctuation is the

$$\epsilon^{\mu\nu}\langle \mathbb{h}_{\mu\nu}\rangle = \langle\psi_{\rm out}|\epsilon^{\mu\nu}\mathbb{h}_{\mu\nu}|\psi_{\rm out}\rangle - \langle\psi_{\rm in}|\epsilon^{\mu\nu}\mathbb{h}_{\mu\nu}|\psi_{\rm in}\rangle \qquad |\psi_{\rm out}\rangle = \hat{S}|\psi_{\rm in}\rangle \quad \text{where} \quad \hat{S} = \mathbb{I} + i\hat{T}\ , \tag{57}$$

$$|\psi_{\rm in}\rangle = \int \mathrm{d}\Phi[p_1]\mathrm{d}\Phi[p_2]\phi(p_1)\phi(p_2)e^{ip_1\cdot b_1 + ip_2\cdot b_2}|p_1p_2\rangle \tag{58}$$

And from here we can read off the waveform. The precise polarization is not relevant, but needs to be chosen so that the real and imaginary parts are properly extracted.

$$W(x,t) = \frac{\kappa}{4\pi}\int d\omega e^{-iu\omega}\left[(-i)\langle\psi_{\rm in}|\hat{S}^\dagger\hat{a}_{hh}(k)\hat{S}|\psi_{\rm in}\rangle + \text{hc}\right] = \int[\text{wave packets}]\frac{\kappa}{4\pi}\int d\omega e^{-iu\omega}\left[\mathcal{B}_h + \text{hc}\right]$$
$$\mathcal{B}_h = \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\hat{\delta}(2\bar{p}_1\cdot q_1)\hat{\delta}(2\bar{p}_2\cdot q_2)e^{iq_1\cdot b}\ (-i)\langle p_1 - q_1, p_2 - q_2|\hat{S}^\dagger\hat{a}_{hh}(k)\hat{S}|p_1p_2\rangle \tag{59}$$

When taking the conjugate to construct "+hc" it is necessary to be careful not con conjugate the polarization tensor even if one chooses a helicity state. The reason is that one needs to pick the same combination of metric coefficients in both terms, and this is chosen by the polarization tensor. Alternatively, one may construct the waveform with open indices and project onto a particular real or complex polarization afterwards.

Before getting on with discussing some explicit aspects of the waveform, let's talk a little about what we might want from the perspective of integration. In general, both the time-domain and the frequency-domain waveforms are useful and, of course, related by Fourier transform. It turns out that, if one is aiming for the time-domain waveform, then all resulting integrals, including the Fourier transforms reduce to fairly standard Feynman integrals albeit sometimes with a fractional power of a propagator. One may therefore consider evaluating all integrals, including the loop integrals all at once, via the the usual methods that you learned last week. Let's see how this happens. Because of the classical limit, the each loop-order contribution to $\mathcal{B}_h(k)$ is homogeneous under rescaling k and q_i. So let's rescale them so that $k = \omega(1,\boldsymbol{n}) \to \tilde{k} = (1,\boldsymbol{n})$. Then the matrix element becomes

$$\mathcal{B}_h(k) = \sum_{L\geq 0}\mathcal{B}_{h,L}(k) = \sum_{L\geq 0}\omega^{L+(d-4)(L+1)}\widetilde{\mathcal{B}}_{h,L}(\tilde{k}) \tag{60}$$

$$\widetilde{\mathcal{B}}_{h,L} = \int \frac{\mathrm{d}^d\tilde{q}_1}{(2\pi)^d}\frac{\mathrm{d}^d\tilde{q}_2}{(2\pi)^d}\hat{\delta}(2\bar{p}_1\cdot\tilde{q}_1)\hat{\delta}(2\bar{p}_2\cdot\tilde{q}_2)e^{i\omega\tilde{q}_1\cdot b}(\Re_L + i\,\Im_L)$$
$$\Re_L + i\,\Im_L = (-i)\langle p_1 - \tilde{q}_1, p_2 - \tilde{q}_2|\hat{S}^\dagger\hat{a}_{hh}(\tilde{k})\hat{S}|p_1p_2\rangle\big|_{L\text{ loops}}\ , \tag{61}$$

$$\mathcal{J}_{n,L} \equiv \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(\tau - q_1\cdot b)}\omega^{2n}\left[\Theta(\omega)\omega^L(\Re_L + i\,\Im_L) + \Theta(-\omega)(-\omega)^L(\Re_L - i\,\Im_L)\right] \tag{62}$$
$$= \frac{i^{L+2n+1}}{2\pi}\Gamma(L+2n+1)\ \Re_L\left[\frac{1}{(\tau - q_1\cdot b + i0)^{L+2n+1}} - \frac{(-1)^{L+2n}}{(\tau - q_1\cdot b - i0)^{L+2n+1}}\right]$$
$$- \frac{i^{L+2n+2}}{2\pi}\Gamma(L+2n+1)\ \Im_L\left[\frac{1}{(\tau - q_1\cdot b + i0)^{L+2n+1}} + \frac{(-1)^{L+2n}}{(\tau - q_1\cdot b - i0)^{L+2n+1}}\right]\ ,$$

Because of the $(d-4)$ power of ω, one may encounter Fourier transforms of logs of omega; they can be obtained by simply differentiating these formulae with respect to n. The square parenthesis

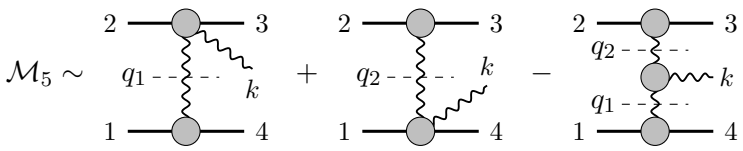

Figure 1: Diagrammatic representation of Eq. (63). The dashed lines across the graviton propagators indicate on-shell conditions and summation over physical states.

become either derivatives of δ functions, or Principal Values of propagator-like objects. The former are easy to integrate; the latter can be written as a regular linear propagator and a delta function.

One may also aim for the frequency-space waveform; the integrals then involve explicit Fourier-transforms. I won't discuss their computation, but you you should feel free to ask Stefano & David about how to evaluate them.

4.2 Leading order

At leading order only the single-T terms in Eq. (57) contribute, and they are the 5-point amplitude – 4scalar, 1 graviton. One may find this in a variety of ways: from Feynman rules, from double-copy starting from the 4-scalar-1-gluon amplitude (from dimensional reduction) upgraded to remove the dilaton-axion, or simply get it from the literature. Because we are in the classical limit so in particular we don't allow matter lines to cross each other, we can practice generalized unitarity methods and we construct the amplitude from tree-level unitarity. The relevant cuts are shown in Fig. 1.

The pieces assembly is

$$\mathcal{M}_5 = \sum_{\text{states}} \left(\mathcal{M}_3 \frac{1}{q_1^2} \mathcal{M}_4 + \mathcal{M}_4 \frac{1}{q_2^2} \mathcal{M}_3 - \mathcal{M}_3 \frac{1}{q_1^2} \mathcal{M}_3 \frac{1}{q_2^2} \mathcal{M}_3 + \text{contact terms} \right), \tag{63}$$

where the 3rd term removes a double-count. The d-dimensional 4-gluon amplitude in the form we discussed in the previous lecture together with the standard 3-gluon amplitude is sufficient to carry out this calculation. Don't forget that the factors are amplitudes so the on-shell conditions need to be religiously imposed. There is a check – gauge invariance.

Homework: do this calculation in QED. (the 3rd term is missing altogether; if you feel energetic, do the calculation in scalar-coupled YM theory)

In GR you should find:

$$\begin{aligned} M_{5,\text{tree}}^{\text{cl.GR}} = -\frac{\kappa^3}{4} \bar{m}_1^2 \bar{m}_2^2 \, \varepsilon^*(k)_{\mu\nu} \Bigg[& \frac{4 P_{12}^\mu P_{12}^\nu}{q_1^2 q_2^2} + \frac{2y}{q_1^2 q_2^2} \left(Q_{12}^\mu P_{12}^\nu + Q_{12}^\nu P_{12}^\mu \right) \\ & + \left(y^2 - \frac{1}{d-2} \right) \left(\frac{Q_{12}^\mu Q_{12}^\nu}{q_1^2 q_2^2} - \frac{P_{12}^\mu P_{12}^\nu}{(k \cdot \bar{u}_1)^2 (k \cdot \bar{u}_2)^2} \right) \Bigg] \,, \end{aligned} \tag{64}$$

$$P_{12}^\mu \equiv k \cdot \bar{u}_1 \, \bar{u}_2^\mu - k \cdot \bar{u}_2 \, \bar{u}_1^\mu \,, \quad Q_{12}^\mu \equiv (q_1 - q_2)^\mu - \frac{q_1^2}{k \cdot \bar{u}_1} \bar{u}_1^\mu + \frac{q_2^2}{k \cdot \bar{u}_2} \bar{u}_2^\mu, \tag{65}$$

$$\left. \begin{array}{ll} p_1^\mu = \bar{p}_1^\mu - q_1^\mu/2 & p_4^\mu = -\bar{p}_1^\mu - q_1^\mu/2 \\ p_2^\mu = \bar{p}_2^\mu - q_2^\mu/2 & p_3^\mu = -\bar{p}_2^\mu - q_2^\mu/2 \end{array} \right\} \implies \bar{p}_1 \cdot q_1 = \bar{p}_2 \cdot q_2 = 0 \,. \tag{66}$$

After some straightforward organization, the time-domain waveform is

$$(\hat{h}^{(1)}_{+} + i\hat{h}^{(1)}_{\times})(\tau, \boldsymbol{n}) = -\frac{m_1 m_2}{16\mathsf{M}^2}\big(B_1\mathcal{J}^{0,0}_{1,0} + B_2\mathcal{J}^{1,0}_{1,0} + B_3\mathcal{J}^{0,1}_{1,0} + B_4\mathcal{J}^{0,0}_{0,1} + B_5\mathcal{J}^{1,0}_{0,1} + B_6\mathcal{J}^{0,1}_{0,1} + B_7\mathcal{J}^{0,0}_{1,1} + B_8\mathcal{J}^{1,0}_{1,1} + B_9\mathcal{J}^{2,0}_{1,1}\big)\,, \tag{67}$$

where the integrals $\mathcal{J}^{\alpha_1,\alpha_2}_{\beta_1,\beta_2}$ are defined as

$$\mathcal{J}^{\alpha_1,\alpha_2}_{\beta_1,\beta_2} = \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot(q_1-\tilde{k}))\delta(\tau/\sqrt{-b^2} - q_1\cdot\tilde{b})\frac{(\varepsilon_-\cdot q_1)^{\alpha_1}(\tilde{k}\cdot q_1)^{\alpha_2}}{(q_1^2)^{\beta_1}((q_1-\tilde{k})^2)^{\beta_2}}\,, \tag{68}$$

Because of the 3 delta functions, these are really 1-d integrals; the results are rational functions and square roots. Stefano discussed some of these integrals. Computed carefully, the result reproduces the memory we discussed earlier, the leading order scattering waveform originally obtained by Kovacs and Thorn, and also later obtained with worldline methods by Jakobsen, Plefka, Mogull, Steinhoff.

4.3 IR divergences; physical interpretation

Beyond leading order the calculations become somewhat more involved. At NLO, the amplitudes entering are trees, so you already have them. The amplitude part requires construction. We won't do it here beyond drawing the few cuts that need to be evaluated and merged. We can discuss this afterwards. Leaving aside the admittedly involved details of higher-order calculations, there is one thing that we should typically expect in amplitudes calculations with on-shell massless particles: IR divergences. Usually, IR divergences have a physical interpretation, and are a consequences of long-range interactions of massless particles. Here we have the opportunity to see this explicitly.

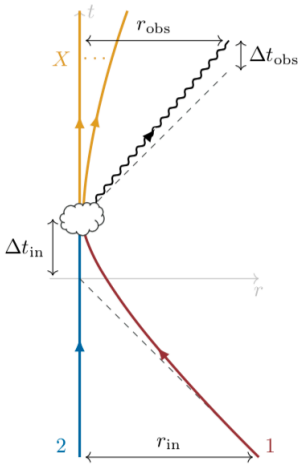

Figure 2: From 2308.02125 Caron-Huot et al.

For the outgoing trajectory – a null radial geodesic in Schwarzschild obeys

$$0 = -(1-2GE/r)dt^2 + (1-2GE/r)^{-1}dr^2 \Rightarrow dt \simeq= dr + 2GE\frac{dr}{r} \Rightarrow \Delta t_{\rm obs} = 2GE\ln\frac{r_{\rm obs}}{r_\mu} \tag{69}$$

For the incoming trajectory, the geodesic eq is

$$g_{\mu\nu}\ddot{x}^\nu + \left(\partial_\nu g_{\mu\rho} - \frac{1}{2}\partial_\mu g_{\nu\rho}\right)\dot{x}^\nu\dot{x}^\rho = 0 \tag{70}$$

Picking $\mu = r$, the gauge $t = \tau$ and $v_r = (dr/dt)/(1 - 2GM/r)$ this becomes

$$\frac{dv_r}{dt} = -\frac{GM}{r^2}(1 - v_r^2) + \mathcal{O}(r^{-3}) \Rightarrow t = -\frac{r}{v_\infty} + \frac{GM}{v_\infty^3}(3v_\infty^2 - 1)\ln\frac{r_{\rm start}}{r} + \mathcal{O}(r^{-1}) \tag{71}$$

We should therefore expect two types of divergence: one that is related to the masslessness of the external gravitons and another one that is not. In our calculation they come form

1) the amplitude, from soft gravitons between a matter particle and a massless particle (Weinberg 1965)

2) from the cut, from a soft loop momentum between matter particles next to a cut

There are other IR divergences that cancel between cut and amplitude. A computation reveals the first term in the expansion of

$$\begin{aligned}\langle p_1' p_2'|\hat{S}^\dagger \hat{a}_{hh}(k)\hat{S}|p_1 p_2\rangle = \exp\left[-iG(p_1\cdot k + p_2\cdot k)\left(1 + \frac{\sigma(\sigma^2-3/2)}{(\sigma^2-1)^{3/2}}\right)\left(\frac{1}{\epsilon} - \log\frac{\Lambda^2}{\mu^2}\right)\right] \\ \times \langle p_1' p_2'|\hat{S}^\dagger \hat{a}_{hh}(k)\hat{S}|p_1 p_2\rangle^0 \,,\end{aligned} \tag{72}$$

This agrees with the geodesic calculation with $\sigma = 1/\sqrt{1 - v_\infty^2}$.

Having surviving IR divergences is not good, so what is their fate? They can be removed via a redefinition of the retarded time in Eq. (59). This is easiest done in the COM frame, where $(p_1 \cdot k + p_2 \cdot k) = E\omega$. So

$$T = u - iGE\left(1 + \frac{\sigma(\sigma^2-3/2)}{(\sigma^2-1)^{3/2}}\right)\left(\frac{1}{\epsilon} - \log\frac{\Lambda^2}{\mu^2}\right) \,. \tag{73}$$

4.4 (Gravitational) memory

The (linear) gravitational wave memory is the permanent change in the space-time metric due to the passage of a gravitational wave. Denoting by $\mathcal{F}$ some construct of the metric fluctuation,

$$\mathcal{F}(\tau) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\widetilde{\mathcal{F}}(\omega)e^{-i\omega\tau} \,, \quad \mathcal{F} \in \{f_{\mu\nu\rho\sigma}, \widetilde{\Psi}_4^\infty, \tilde{h}_+ + i\tilde{h}_\times\} \,. \tag{74}$$

the memory is

$$\Delta\mathcal{F} \equiv \mathcal{F}(+\infty) - \mathcal{F}(-\infty) = \int_{-\infty}^{+\infty} dt\dot{\mathcal{F}}(t) = \int_{-\infty}^{+\infty} {\rm d}\omega(-i\omega)\widetilde{\mathcal{F}}(\omega)\delta(\omega) \,. \tag{75}$$

That is, it is entirely given by the leading order soft limit of the corresponding construct.

We will derive an expression for this in a bit; before then though, I should mention that the soft expansion of the metric was analyzed by Sahoo, Sen and others. It has the general form

$$\varepsilon^\mu\varepsilon^\nu h_{\mu\nu} = \frac{i\,\mathcal{A}}{\omega} + \mathcal{B}\ln\omega + \mathcal{O}(\omega^1) \,,$$

The coefficients written are universal – in the sense that they have a process-independent expression. The one relevant for memory, $\mathcal{A}$, is for a 4-point process

$$\mathcal{A} = -\frac{(\varepsilon\cdot p_3)^2}{\tilde{k}\cdot p_3} - \frac{(\varepsilon\cdot p_4)^2}{\tilde{k}\cdot p_4} + \frac{(\varepsilon\cdot p_1)^2}{\tilde{k}\cdot p_1} + \frac{(\varepsilon\cdot p_2)^2}{\tilde{k}\cdot p_2} \,. \tag{76}$$

Importantly, the outgoing matter momenta are given in terms of the incoming momenta and the (classical) impulse; for any given gravitational theory, one needs to compute the impulse and then one has this coefficient. Through $\mathcal{O}(G^2)$ they are

$$p_4 = p_1 + \Delta p \quad p_3 = p_2 - \Delta p \quad \Delta p = G\Delta p^{(0)} + G^2\Delta p^{(1)} + \dots \,, \quad \mathcal{A} = G\mathcal{A}^{(0)} + G^2\mathcal{A}^{(1)} + \mathcal{O}(G^3) \tag{77}$$

Getting back to computing these in the observable-based formalism and using the amplitudes we discussed –

The definition of the memory applied to the asymptotic metric gives two terms –

$$\begin{aligned}\Delta(h^\infty_+ + ih^\infty_\times) = &\lim_{\omega\to 0^+}(-i\omega)\Theta(\omega)(-i)\langle\psi_{\rm in}|\hat{S}^\dagger\hat{a}_{--}(k)\hat{S}|\psi_{\rm in}\rangle^0\big|_{k=(\omega,\omega\boldsymbol{n})}\\ &+\lim_{\omega\to 0^-}(-i\omega)\Theta(-\omega)(+i)\langle\psi_{\rm in}|\hat{S}^\dagger\hat{a}^\dagger_{++}(k)\hat{S}|\psi_{\rm in}\rangle^0\big|_{k=(|\omega|,|\omega|\boldsymbol{n})}\,.\end{aligned} \tag{78}$$

Since each of them naturally splits into an amplitude contribution and a cut contribution, so does the memory:

$$\Delta(h^\infty_+ + ih^\infty_\times) = \Delta(h^\infty_+ + ih^\infty_\times)\Big|_{\rm amp} + \Delta(h^\infty_+ + ih^\infty_\times)\Big|_{\rm cut}\,. \tag{79}$$

The former starts at $\mathcal{O}(G)$, while the latter starts at $\mathcal{O}(G^2)$.

The first part is straightforward using the soft properties of gravitational amplitudes (see Martin's and Yu-tin's lectures)[2]

$$\lim_{k\to 0} M_5^{\rm gravity} = \frac{\kappa}{2}\left[\varepsilon^\mu\varepsilon^\nu\mathcal{S}(k,q)_{\mu\nu}\right]M_4^{\rm gravity} \tag{80}$$

$$\mathcal{S}(k,q)^{\mu\nu} = \sum_{i=1}^4\frac{p_i^\mu p_i^\nu}{p_i\cdot k} = -\sum_{i=1}^2\left[\frac{p_i^\mu q_i^\nu + p_i^\nu q_i^\mu}{p_i\cdot k} - \frac{p_i^\mu p_i^\nu q_i\cdot k}{(p_i\cdot k)^2}\right] + \mathcal{O}(\hbar) \qquad q_1 = -q_2 = q \tag{81}$$

Unlike gauge theories, gravitational soft factors are tree-level exact (by a combination of the equivalence principle and gauge invariance). [3]

At leading order this is all there is:

$$\Delta(h^\infty_+ + ih^\infty_\times)\Big|^{(0)}_{\rm amp} = -\frac{\kappa}{2}\left[\varepsilon_-^\mu\varepsilon_-^\nu\mathcal{S}(\tilde{k},\partial/\partial b)_{\mu\nu}\right]\int\frac{{\rm d}^d q}{(2\pi)^d}\hat{\delta}(2p_1\cdot q)\hat{\delta}(2p_2\cdot q)e^{iq\cdot b}\,{\rm Re}\left[M_4^{\rm tree}\right] \tag{82}$$

The real part doesn't do anything because the amplitude is real; we recognize the integral as the tree-level radial action or eikonal, so the derivative with respect to vector impact parameter yields the impulse.

$$\begin{aligned}\Delta(h^\infty_+ + ih^\infty_\times)\Big|^{(0)}_{\rm amp} &= -\frac{\kappa}{2}\left[\varepsilon_-^\mu\varepsilon_-^\nu\mathcal{S}(\tilde{k},\Delta p^{(0)})_{\mu\nu}\right] \qquad (83)\\ &= -\frac{\kappa}{2}\left[-\frac{2\varepsilon\cdot p_1\varepsilon\cdot\Delta p^{(0)}}{p_1\cdot\tilde{k}} + \frac{(\varepsilon\cdot p_1)^2\tilde{k}\cdot\Delta p^{(0)}}{(p_1\cdot\tilde{k})^2} + \frac{2\varepsilon\cdot p_2\varepsilon\cdot\Delta p^{(0)}}{p_2\cdot\tilde{k}} - \frac{(\varepsilon\cdot p_2)^2\tilde{k}\cdot p^{(1)}}{(p_2\cdot\tilde{k})^2}\right]\,.\end{aligned}$$

[2] This comes from Feynman diagrammatics: as the graviton becomes soft, the internal propagator right after the vertex goes on shell.

[3] The Equivalence Principle says that coupling is with stress tensor; gauge invariance fixes tensor structure; coupling is dimensionful, so one may allow some function $f(Gm^2)$ but this breaks the equivalence principle.

Because it is proportional to the 4-point amplitude, more care is needed beyond LO because of the presence of the classically-singular terms. The S matrix elements entering the cut part have a similar soft limit, so it is not difficult to intuit that, beyond leading order, the $\text{Re}\left[M_4^{\text{tree}}\right]$ gets replaces with the part of the loop amplitude with the classically-singular pieces removed. This was checked explicitly at 1 loop

$$\begin{aligned}\Delta(h_+^\infty + ih_\times^\infty)\Big|_{\text{amp}}^{(1)} &= -\frac{\kappa}{2}\left[\varepsilon_-^\mu \varepsilon_-^\nu \mathcal{S}(\tilde{k}, \Delta p^{(1)})_{\mu\nu}\right] \\ &= -\frac{\kappa}{2}\left[-\frac{2\varepsilon\cdot p_1 \varepsilon\cdot\Delta p^{(1)}}{p_1\cdot\tilde{k}} + \frac{(\varepsilon\cdot p_1)^2\tilde{k}\cdot\Delta p^{(1)}}{(p_1\cdot\tilde{k})^2} + \frac{2\varepsilon\cdot p_2\varepsilon\cdot\Delta p^{(1)}}{p_2\cdot\tilde{k}} - \frac{(\varepsilon\cdot p_2)^2\tilde{k}\cdot p^{(1)}}{(p_2\cdot\tilde{k})^2}\right].\end{aligned} \tag{84}$$

Unlike the impulse, where its contributions fully cancel against the amplitude contribution, the cut part of the waveform contributes finite terms as well. For starters, one should expect this because the expression we just derived does not fully reproduce the universal result given the tree-level and one-loop impulse. A direct calculation shows that

$$\begin{aligned}\Delta(\hat{h}_+^{(2)} + i\hat{h}_\times^{(2)})\big|_{\text{cut}} = -\frac{(2\sigma^2 - X)^2 m_1 m_2}{512\pi^2(\sigma^2-1)\mathsf{M}^3}\Bigg[&\frac{m_2(u_1\cdot\tilde{f}\cdot\tilde{b})^2}{(u_1\cdot\tilde{k})^3} + \frac{m_1(u_2\cdot\tilde{f}\cdot\tilde{b})^2}{(u_2\cdot\tilde{k})^3} \\ &- \frac{(u_1\cdot\tilde{f}\cdot u_2)^2}{2(\sigma^2-1)(u_1\cdot\tilde{k})(u_2\cdot\tilde{k})}\left(\frac{\sigma m_1 + m_2}{u_2\cdot\tilde{k}} + \frac{\sigma m_2 + m_1}{u_1\cdot\tilde{k}}\right)\Bigg]\end{aligned} \tag{85}$$

which together with the original contribution $\Delta(h_+^\infty + ih_\times^\infty)\Big|_{\text{amp}}^{(1)}$ restores order.

4.5 Large gauge transformations and BMS frame dependence of the waveform

We reviewed a little while ago the operator whose expectation value is the metric near null infinity, at leading order in $1/r$. That derivation makes use of the saddle-point approximation and, in particular, holds as stated if $\omega x \gg 1$. What of gravitons with tiny frequencies, $\omega < 1/x$? As I was mentioning, such modes are not an artificial construct; they represent metrics that vary extremely slowly with time. For example, is we have an initial state of 2 BH-s (or particles) almost at rest, the final state differ form the initial state by some metric that is almost static. The smaller the initial velocities, the weaker the time dependence is. Such modes are of course part of a continuum of states, e.g. in loops. There are however amplitudes that are supported only on these 0-energy modes – 3-point amplitudes in real kinematics, or more generally $1 \to 1+n$ amplitudes. Here are some contributions to observables that are supported on these modes:

<draw figures>

So the question is how to account for these kind of modes. There are several options:

- define the initial and final states at finite distance, so there is a natural frequency cutoff.
- push gravitons off-shell and approach null infinity from finite distance
- resum them (we'll pick this version)

q counting suggests that such amplitudes don't contribute is they are classical.

<do the counting; observe that only maximally-superclassical part contributes>

The relevant transition matrix elements can be written as,

$$T_{1\to 1+n} = \frac{1}{n!}\prod_{i=1}^{n}\sum_\eta \int d\Phi(k_i)\delta(2p\cdot k_i) A_3(p+k_i \to p, k_{i,\eta}) a_\eta^\dagger(k_i) \tag{86}$$

and can be resummed into an exponential:

$$U = e^{iQ_s} \qquad Q_s = \sum_\eta \int d\Phi(k_i)\delta(2p\cdot k_i)A_3(p+k_i\to p,k_{i,\eta})a^\dagger_\eta(k_i) \tag{87}$$

In words, Q_s attaches a 0-energy graviton to a matter particle, while the exponential creates a sort of coherent state of 0-energy gravitons:

$$|\psi\rangle \mapsto U|\psi\rangle \tag{88}$$

This new state has the nice features:

$$A_3^\eta = \kappa\epsilon^\eta_{\mu\nu}p^\mu p^\nu \tag{89}$$

$$e^{-iQ_s}\kappa\mathbb{h}_{\mu\nu}e^{+iQ_s} = \kappa\mathbb{h}_{\mu\nu} + \kappa\frac{1}{4\pi|\boldsymbol{x}|}\sum_\eta\int d\Phi(k)d\omega'\delta(2p\cdot k)A_3^\eta(k)\epsilon^\rho_{\mu\nu}[a^\dagger_\eta(k),a_\rho(k')] \tag{90}$$

$$= \kappa\mathbb{h}_{\mu\nu} + \frac{\kappa}{4\pi|\boldsymbol{x}|}\sum_\eta\int d\Phi(k)d\omega'\delta(2p\cdot k)A_3^\eta(k)\epsilon^\rho_{\mu\nu}2\omega(k)(2\pi)^3\delta^3(k-k') \tag{91}$$

$$= \kappa\mathbb{h}_{\mu\nu} + \frac{4G}{|\boldsymbol{x}|}\frac{\mathcal{P}^{ab}_{\mu\nu}p_ap_b}{p\cdot n} = \kappa\mathbb{h}_{\mu\nu} + \kappa h^{\text{Schwarzschild}}_{\mu\nu} \tag{92}$$

This is the asymptotic linearized Schwarzschild solution is the black hole moves with velocity $v = p/m$. That is, the coherent state creates the asymptotic field of a free particle. Variants of the object we defined Q_s appeared before in the discussions of the BMS symmetry of S-matrix elements.

Assume that there exists an operator Q_h with the following properties – such an operator does exist:

$$Q_h|\psi\rangle = 0 \qquad [Q_s,Q_h] = 0 \qquad [Q,Q_s+Q_h] = 0 \tag{93}$$

With such an operator in hand, then we can inspect the consequences of the resummation of 0-energy gravitons on observables without much calculation. Of course, some calculation – which we are not going to do explicitly but we will sketch – is necessary to figure out what Q_h is.

$${}_{\text{in}}\langle\psi|S^\dagger\mathcal{O}S|\psi\rangle_{\text{in}} = {}_{\text{in}}\langle\psi|e^{+iQ_s}S^\dagger\mathcal{O}Se^{+iQ_s}|\psi\rangle_{\text{in}} \to {}_{\text{in}}\langle\psi|e^{-i(Q_s+Q_h)}S^\dagger\mathcal{O}Se^{+i(Q_s+Q_h)}|\psi\rangle_{\text{in}} \tag{94}$$

$$= {}_{\text{in}}\langle\psi|S^\dagger e^{-i(Q_s+Q_h)}\mathcal{O}e^{+i(Q_s+Q_h)}S|\psi\rangle_{\text{in}} \tag{95}$$

So, if $\mathcal{O} = \mathbb{h}_{\mu\nu}$, then

$$e^{-i(Q_s+Q_h)}\mathbb{h}_{\mu\nu}e^{+i(Q_s+Q_h)} = e^{-iQ_h}\mathbb{h}_{\mu\nu}e^{+iQ_h} + h^{\text{Schwarzschild}}_{\mu\nu}\ . \tag{96}$$

To know what this means, we need to know Q_hm so we need $[S,Q_s]$. Since Q_s is linear in a_k, we don't need multiple commutators. Also, we need only the bit of the S matrix that is linear in gravitons, and even more, we need it in the soft limit because Q_s is localized on the 0-energy gravitons. Using the soft limit we discussed before, the S matrix in this regime can be written as

$$S = -i\sum_\eta\int \mathrm{d}\Phi(k,P,P')\frac{H(P,P')\mathcal{A}(P,k_\eta|P+k)}{2P\cdot k+i\varepsilon}a_\eta(k) \tag{97}$$

$$S = -i\sum_\eta\int \mathrm{d}\Phi(k,P,P')\frac{H(P,P')\mathcal{A}(P'-k,k_\eta|P')}{-2P'\cdot k+i\varepsilon}a_\eta(k). \tag{98}$$

if P is incoming or outgoing, and it is either matter of graviton state.

The commutator of S with Q_s is very simple: it is H up to a multiplicative factor. So the commutator can be written as another commutator – with a number operator counting the external lines to which the soft graviton can attach to.

$$[S, Q_s] \sim [S, \int \text{stuff} a^\dagger a] \ , \tag{99}$$

So

$$Q_h \sim \int \text{stuff}\ a^\dagger a \ . \tag{100}$$

Explicitly, and focusing on P being an outgoing graviton – call it K. Then the commutator is

$$C_0 = -\int \mathrm{d}\Phi(K) i H(K) \int \mathrm{d}\Phi(k) \hat{\delta}(2p\cdot k) \frac{\kappa^2 (K\cdot p)^2}{2K\cdot k + i\varepsilon} \, . \tag{101}$$

The relevant integral requires some regularization beyond dim.reg. It can be computed by pushing K off shell, of by pushing p off-shell. The result is

$$C_0 = -\int \mathrm{d}\Phi(K) i H(K) \int \mathrm{d}\Phi(k) \hat{\delta}(2p\cdot k) \frac{\kappa^2 (K\cdot p)^2}{2K\cdot k + i\varepsilon} \, . \tag{102}$$

So

$$\begin{aligned} Q_{h,0} = -2G \sum_\eta \int \mathrm{d}\Phi(K)\, p\cdot K \\ \times \left[\log\left(\frac{p\cdot K}{mK^0}\right) + Y(\beta,\epsilon) \right] a^\dagger_\eta(K) a_\eta(K) \, , \end{aligned} \tag{103}$$

So now that we know the Q_h that talks to gravitons, we can go back to see its effect on $\mathbb{h}_{\mu\nu}$:

$$e^{-iQ_h} \mathbb{h}_{\mu\nu} e^{+iQ_h} = \sum_\eta \frac{-i}{4\pi|\boldsymbol{x}|} \int_0^\infty \hat{\mathrm{d}}\omega \, e^\eta_{\mu\nu}(k) e^{-iQ_h} a_\eta(k) e^{+iQ_h} e^{-i\omega u}|_{k=\omega n} + \text{h.c.} \tag{104}$$

$$= \sum_\eta \frac{-i}{4\pi|\boldsymbol{x}|} \int_0^\infty \hat{\mathrm{d}}\omega \, e^\eta_{\mu\nu}(k) e^{-iQ_h} a_\eta(k) e^{+iQ_h} e^{-i\omega(u - 2G\sum_p p\cdot\hat{k}(\ln u\cdot\hat{k} + Y)}|_{k=\omega n} + \text{h.c.} \tag{105}$$

where $\hat{k} = k/\omega$ and $u = p/m$.

So we note that this resummation of the 0-energy graviton contributions is equivalent to a shift of the retarded time by some angle-dependent quantity. This is a BMS transformation, that is a large coordinate transformation; these ones, shifting the retarded time coordinate, are known as a supertranslations. While it is a coordinate transformation, this one has observable effects:

- it shifts the origin (of time) so it affects the angular momentum of the field

- it turns out that it shifts the on-shell 3-point amplitudes (basically the one we derived sets them to 0)

- it in principle affects the observed waveform, though with effectively coincident detectors it is probably not possible to make a determination as it all amounts to a constant shift of the time.

- the discussion here refers to a specific transformation which, like I said, sets to zero the on-shell 3-point amplitudes. There are infinitely many of them, and understanding their implications for amplitudes and asymptotic observables is currently under investigation.

5 Massive particle EFT and generating function of observables from scattering

Since we know how to compute amplitudes, we want to solve the inverse scattering problem: given an S matrix, what is the Hamiltonian producing it. The simplest place to look for this is NRQM.

5.1 Hamiltonian construction from NRQM scattering

In NRQM, we usually talk about scattering off potentials. Assume that you are handed an NRQM scattering amplitude which is a function of some parameter/coupling g,

$$f_k(\theta,\phi)=\sum_n g^n f_k^{(n)}(\theta,\phi) \tag{106}$$

and you are asked to determine the potential that yields it.

Scattering in NRQM: solve Schrödinger's eq with prescribed boundary conditions ($E=k^2/2n$):

$$(-\nabla^2-k^2+U(r))\psi(\boldsymbol{r})=0\Longrightarrow\psi(r)=\psi_0(r)+\int d^3r' G(r-r')U(r')\psi(\boldsymbol{r'}) \tag{107}$$

$$(-\nabla^2-k^2)\psi_0(\boldsymbol{r})=0 \qquad (\nabla^2+k^2)G(\boldsymbol{r}-\boldsymbol{r'})=\delta^3(r-r') \tag{108}$$

$$G_\pm(r)=-\frac{1}{4\pi}\frac{e^{\pm ikr}}{r} \qquad \psi_0(\boldsymbol{r})=e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}=e^{ikz} \tag{109}$$

$$\psi(\mathrm{r})\Big|_{\mathrm{r}\to\infty}=e^{ikz}+f_k(\theta,\phi)\frac{e^{ikr}}{r} \tag{110}$$

Assuming range of potential is much smaller than distance to observer:

$$|\boldsymbol{r}-\boldsymbol{r'}|\simeq r-\boldsymbol{u}\cdot\boldsymbol{r'} \tag{111}$$

$$G_+(r-r')=-\frac{1}{4\pi}\frac{e^{\pm ik|\boldsymbol{r}-\boldsymbol{r'}|}}{|\boldsymbol{r}-\boldsymbol{r'}|}\simeq-\frac{1}{4\pi}\frac{e^{\pm ikr}}{r}e^{-iku\cdot\boldsymbol{r'}} \tag{112}$$

$$k\boldsymbol{u}=\boldsymbol{k}_{\mathrm{out}} \qquad k\boldsymbol{e}_z=\boldsymbol{k}_{\mathrm{in}} \qquad \boldsymbol{k}_{\mathrm{out}}-\boldsymbol{k}_{\mathrm{in}}=\boldsymbol{q} \tag{113}$$

Solving iteratively for the wave function:

$$\psi(\boldsymbol{r})=e^{i\boldsymbol{k}_{\mathrm{in}}\cdot\boldsymbol{r}}+\int d^3r_1 G_+(\boldsymbol{r}-\boldsymbol{r}_1)U(\boldsymbol{r}_1)e^{i\boldsymbol{k}_{\mathrm{in}}\cdot\boldsymbol{r}_1} \tag{114}$$

$$+\int d^3r_1d^3r_2 G_+(\boldsymbol{r}-\boldsymbol{r}_1)U(\boldsymbol{r}_1)G_+(\boldsymbol{r}_1-\boldsymbol{r}_2)U(\boldsymbol{r}_2)e^{i\boldsymbol{k}_{\mathrm{in}}\cdot\boldsymbol{r}_2}+\dots \tag{115}$$

$$f_k(\theta,\phi)=-\frac{1}{4\pi}\int d^3r_1 e^{-i\boldsymbol{k}_{\mathrm{out}}\cdot\boldsymbol{r}_1}U(\boldsymbol{r}_1)e^{i\boldsymbol{k}_{\mathrm{in}}\cdot\boldsymbol{r}_1} \tag{116}$$

$$-\frac{1}{4\pi}\int d^3r_1d^3r_2 e^{-i\boldsymbol{k}_{\mathrm{out}}\cdot\boldsymbol{r}_1}U(\boldsymbol{r}_1)G_+(\boldsymbol{r}_1-\boldsymbol{r}_2)U(\boldsymbol{r}_2)e^{i\boldsymbol{k}_{\mathrm{in}}\cdot\boldsymbol{r}_2}+\dots \tag{117}$$

Now, to answer the question, we make an ansatz for the potential,

$$U(r)=\sum g^n U^{(n)}(r) \tag{118}$$

plug both it and the formal amplitude (106) in the solution to Schrödinger's equation, and identify powers of the coupling g.

LO:

$$f_k^{(1)} = -\frac{1}{4\pi}\int d^3 r_1 e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_1} U^{(1)}(\boldsymbol{r}_1) \tag{119}$$

So the leading order potential is the Fourier transform of the leading order amplitude:

$$U^{(1)}(\boldsymbol{r}_1) = -\frac{1}{2\pi^2}\int d^3 q e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_1} f_k^{(1)} \tag{120}$$

NLO:

$$f_k^{(2)} = -\frac{1}{4\pi}\int d^3 r_1 e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_1} U^{(2)}(\boldsymbol{r}_1) - \frac{1}{4\pi}\int d^3 r_1 d^3 r_2 e^{-i\boldsymbol{k}_{\rm out}\cdot\boldsymbol{r}_1} U^{(1)}(\boldsymbol{r}_1) G_+(\boldsymbol{r}_1-\boldsymbol{r}_2) U^{(1)}(\boldsymbol{r}_2) e^{i\boldsymbol{k}_{\rm in}\cdot\boldsymbol{r}_2} \tag{121}$$

So the NLO potential is the Fourier transform of the NLO amplitude from which we subtract some double interaction with the leading order potential.

$$U^{(2)}(\boldsymbol{r}_1) = -\frac{1}{2\pi^2}\int d^3 q e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_1} f_k^{(2)} - \text{bilinear in the LO amplitude} \tag{122}$$

Exercise: Massage the second term so that it looks like the iteration of a leading-order amplitude

5.2 QFT version

The same general structure exists also in a relativistic field theory. The potential can be extracted by matching the amplitudes with the amplitudes coming from Lagrangian/Hamiltonian

$$\mathcal{L} = \int \frac{d^{D-1}\boldsymbol{k}}{(2\pi)^{D-1}}\,\phi_1^\dagger(-\boldsymbol{k})\left(i\partial_t - \sqrt{\boldsymbol{k}^2+m_1^2}\right)\phi_1(\boldsymbol{k}) + \int \frac{d^{D-1}\boldsymbol{k}}{(2\pi)^{D-1}}\,\phi_2^\dagger(-\boldsymbol{k})\left(i\partial_t - \sqrt{\boldsymbol{k}^2+m_2^2}\right)\phi_2(\boldsymbol{k}) \tag{123}$$

$$-\int \frac{d^{D-1}\boldsymbol{k}}{(2\pi)^{D-1}}\frac{d^{D-1}\boldsymbol{k}'}{(2\pi)^{D-1}}\,V(\boldsymbol{k},\boldsymbol{k}')\,\phi_1^\dagger(\boldsymbol{k}')\phi_1(\boldsymbol{k})\phi_2^\dagger(-\boldsymbol{k}')\phi_2(-\boldsymbol{k})\,, \tag{124}$$

One may intuitively see that V is the interaction potential between the two fields ϕ_i or one may derive it using H to constrct a Schrödinger's equation and read the potential from there.

We will not discuss here the construction of V by matching the amplitudes from $\mathcal{L}$ with the amplitudes from GR coupled with scalar fields, but you may read about it in 1808.02489 and 1908.01493.

6 Micro-primer on BMS supertranslations

Consider a spacetime which is a solution of Einstein's equations which at infinity is Minkowski + perturbation, and is in the gauge

$$g_{rr} = 0 \ , \quad g_{rA} = 0 \ , \quad \partial_r \det \gamma_{AB} = 0 \ . \tag{125}$$

To leading order in $1/r$ the metric is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m}{r} + \dots\right) du^2 - 2(1 + \dots) du dr + r^2 \left(\Omega_{AB} + \frac{1}{r} f_{AB} + \dots\right) (d\theta^A + \mathcal{U}^A du)(d\theta^B + \mathcal{U}^B du) \\ \mathcal{U}^A &= -\frac{1/2}{r^2} D^B f_{AB} + \dots \end{aligned} \tag{126}$$

The BMS symmetry is the set of coordinate transformations that preserve the asymptotic Minkowski and the gauge condition. (a good reference is 1510.03386) One of these transformations is

$$u \longrightarrow u + \alpha(\theta^A) \tag{127}$$

$$\delta f_{AB} \sim \alpha \, \partial_u f_{AB} + \left(\Omega_{AB} D^2 \alpha - 2 D_A D_B \alpha\right) \quad \delta m = \alpha \, \dot{m} + \frac{1}{4} D_A D_B \alpha + \frac{1}{2} D_A \alpha \, D_B \partial_u f^{AB} \tag{128}$$