We aim to numerically solve the differential equations that arise in the computation of scattering amplitudes in QFT.

Integrals that appear in gravitational contexts can also be represented as solutions of differential equations.

For the sake of clarity, we shall focus on the example of the "master integrals"

I = column vector of MI

$s, t, m^2, \dots$ kinematic variables

Eq's: $\frac{dI}{ds} = A_s I$; $\frac{dI}{dt} = A_t I$; $\frac{dI}{dm^2} = A_{m^2} I$; ..... DEs

$A_s(s,t,m^2,..,\varepsilon)$, $A_t(\,)$, ...

Rational matrices for simplicity (work also for algebraic coefficients)

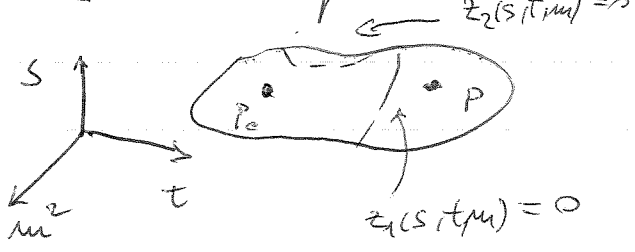
[! Fulfill: $\partial_s A_t - \partial_t A_s - [A_s, A_t] = 0$ integrability conditions. !]

BC: $I(s_0, t_0, m_0^2, \dots \varepsilon) = I_0$

Boundary condition at $P_0 = (s_0, t_0, m_0^2, \dots)$

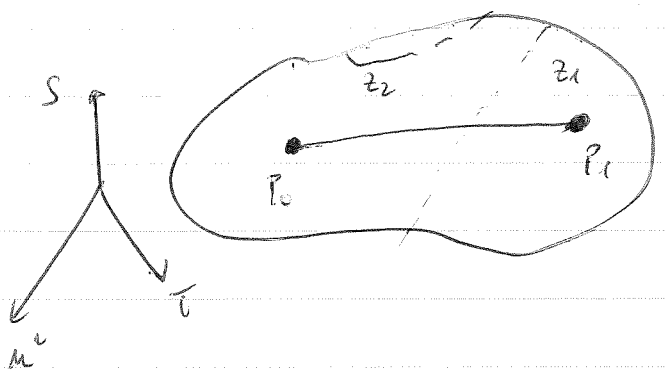
~~We want~~ we aim to evaluate I at the point P, distinct from P_0

- z_1, z_2 singularity of the DE



The first step is to introduce a path connecting the coordinates of the boundary point to those of the target point.

For example, a straight line: $\begin{cases} s_\eta = s_0 + \eta(s_1 - s_0) \\ t_\eta = t_0 + \eta(t_1 - t_0) \\ m^2 = m_0^2 + \eta(m_1^2 - m_0^2) \\ \dots \end{cases}$ $0 \le \eta \le 1$



$P_0 = B$

$P = T$

We then construct a univariate differential equation in the line parameter η by combining all — and only — the matrices corresponding to the kinematic variables that change along the path from B to T

$$\frac{d\,I(\eta,\varepsilon)}{d\eta} = \frac{dI}{ds}\frac{ds}{d\eta} + \frac{dI}{dt}\frac{dt}{d\eta} + \dots$$

$$= (s_1 - s_0)\,A_s(\eta,\varepsilon) + (t_1 - t_0)\,A_t(\eta,\varepsilon) + (m_1^2 - m_0^2)\,A_{m^2}(\eta,\varepsilon) + \dots$$

Still too complicated: 1) Expand in ε the equation.

The coefficients of the series solutions pertain to the specific propagator and do not carry intrinsic meaning.

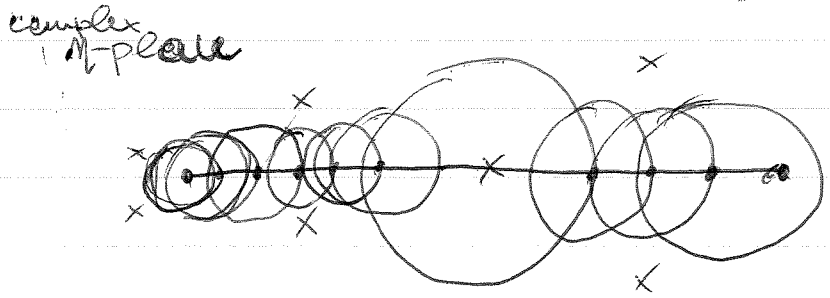
2) For numerical efficiency, it can be convenient to fix the dimensional parameter ε at several values, interpolate the results, and extract the Laurent coefficients of the solution. In this way, the analytic dependence on ε is effectively replaced by parallel computation over multiple ε values

(NOT A PROBLEM FOR DE WITHOUT ε) EXERCISE N. 2

CAVEAT: Solving multiple systems, each with a fixed value of ε, requires that the boundary conditions be known at every order in ε

What follows does not substantially depend on the chosen treatment of the ε-dependence; however, for simplicity, we will henceforth focus on univariate differential equations (x).

The path may pass near or encounter singularities, so the propagation must proceed step by step, taking into account the singularities of the differential equation



R R R R R R S R R R

9 Regular prop.

1 Singular prop.

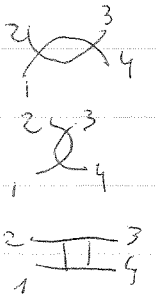
The DE we consider may exhibit singularities of different types:

(1) Physical singularities which originate from the physical cuts of Feynman integrals and require fixing the analytic continuation according to the Feynman prescription

(2) Spurious singularities, which are singularities of the differential equation that do not appear in the solution, which remains regular at those points. For these the series solution will not contain terms which require analytic continuation (constraint!)

(3) Non-physical singularities arising from the coefficients of a basis change.
Can be poles (no AC needed) or branch cuts.
In the latter case the choice of the sign of the infinitesimal imaginary part is arbitrary as the physical result is independent of this choice.

$$A_s = \begin{pmatrix} -\frac{\varepsilon}{s} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-2+4\varepsilon}{s+u} & \frac{2-4\varepsilon}{s+u} & \frac{\varepsilon t-u}{su} \end{pmatrix} \qquad A_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\varepsilon}{t} & 0 \\ \frac{t/2+4\varepsilon}{st+u} & \frac{-2+4\varepsilon}{t+u} & \frac{\varepsilon s-u}{tu} \end{pmatrix}$$



(1) : $s=0$; $t=0$

(2) : $u=0$

Computing the solution as a series expansion around a regular boundary point is relatively straightforward

problem: $\frac{d}{d\eta} I(\eta) = A(\eta)\, I(\eta)$; $I(\eta_{bc}) = I_{bc}$

we look for a solution as

$$I(\eta) = \sum_{j=0}^{\infty} I_j (\eta - \eta_{bc})^j$$

without loss of generality we simplify the notation by shifting the the boundary point at $\eta = 0$
$\eta \to \eta + \eta_{bc}$ in the matrix A

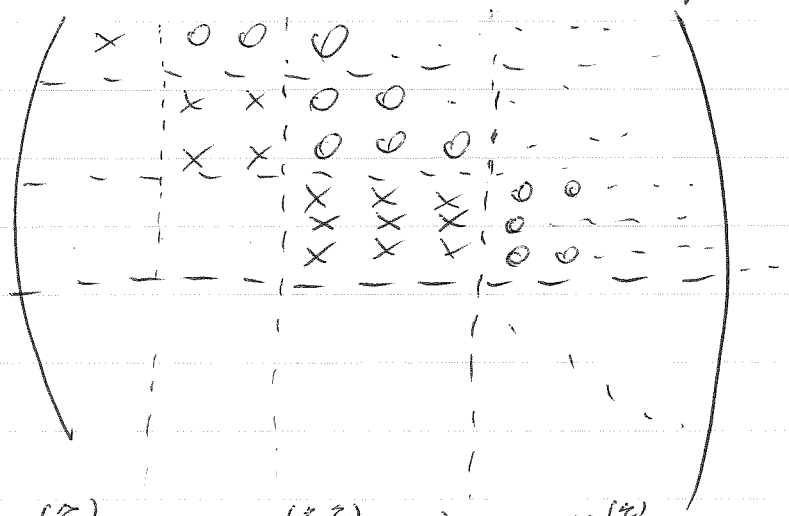
$$A(\eta) \to A(\eta) \equiv A(\eta + \eta_{bc})$$

$$\frac{d}{d\eta} I(\eta) = A(\eta)\, I(\eta) \qquad I(0) = I_{bc}$$

sol. $I(\eta) = \sum_{j=0}^{\infty} I_j \eta^j$; $I_0 = I_{bc}$

can be efficiently solved block by block

Ordering vectors by increasing size the matrix has a block triangular form

$A =$ 

NOTE
For other problems it is needed to analyze the graf of the matrix to reorder the integrals

$$\frac{d}{dn} I^{(z)}(\eta) = A^{(z,z)} I^{(z)}(\eta) + J^{(z)}(\eta) \quad ; \quad J^{(z)}(\eta) = A^{(z,z_<)} I^{(z_<)}$$

scalar: $g^{(z)}(\eta) = LCD\left[A^{(z,z)}(\eta)\right] \Rightarrow g^{(z)}(\eta) = \sum_{n=0}^{\infty} g_n \eta^n$

$g_n = 0$ for $n > N$

matrix: $B^{(z)}(\eta) = g^{(z)}(\eta)\, A^{(z,z)}(\eta) \Rightarrow B^{(z)}(\eta) = \sum_{m=0}^{\infty} B_m \eta^m$

$B_m = 0$ for $m > M$

vector: $Y^{(z)}(\eta) = g^{(z)}(\eta)\, J^{(z)}(\eta) \Rightarrow Y^{(z)}(\eta) = \sum_{p=0}^{\infty} Y_p \eta^p$

Solution:

$$I^{(z)}_{k+1} = \frac{1}{(k+1) g_0}\left[Y^{(z)}_k - \sum_{n=a_1(k)}^{k} n\, g^{(z)}_{k+1-n} I^{(z)}_n + \sum_{n=a_2(k)}^{k} B^{(z)}_{k-n} I^{(z)}_n \right]$$

$a_1(k) = \max\{0, k+1-N\}$ $\quad$ $a_2(k) = \max\{0, k-M\}$

To construct a series solution in the neighborhood of a singular point, a few more steps are required

(1) Growth of Feynman Integrals in the vicinity of any singular point has to be power-like. This can be deduced by their parametric representation.

(2) A differential system is said to have a regular-singularity at $x = x_0 \neq \infty$ if x_0 is a singular point of $A(x)$ and all solutions of the system grow at most like a finite power of $x - x_0$ approaching x_0.

As for ∞ the definition applies after the transformation $x \to y = \frac{1}{x}$ has been performed.

A dll system is said to have a regular-singularity at $x = x_0 = \infty$ if $y = 0$ is a singular point of $\frac{A(1/y)}{y^2}$ and all solutions grow at most like a finite power of x

(1)+(2) It is natural to expect that all singular points of the DE for the MI are regular singularities

Different usage of the boundary

We construct the general solution of the differential equation in a neighborhood centered at the singularity and fit it by matching to the known value at a nearby regular point.

Before introducing the appropriate matrix and deriving the recurrence relations, it is necessary to NORMALIZE the matrix at the singular point.

Poincaré rank (p)

A differential system has Poincaré rank $p \geq 0$ at the singular point x_0 if the matrix $A(x)$ can be represented as

$$A(x) = \frac{M(x-x_0)}{(x-x_0)^{p+1}}$$

where
M is regular in x_0
and $M(0) \neq 0$

When the singularity is a regular singularity Poincaré rank can be reduced.

- In general $p=0$ at some point x_0 is a regular singularity of the system
- The opposite is not true but if a point x_0 is a regular singularity there exist a transformation that brings the system to an equivalent one with $p=0$ in x_0

- Consider the problem $\frac{dy}{dx} = A(x)\, y$ (1)

and an matrix $T(x)$ invertible in a neighborhood of x_0 such that $y = T(x)\, z(x)$

one has $\frac{dT(x)}{dx} z + T(x) \frac{dz}{dx} = A(x)\, T(x)\, z(x)$

and so

$$\frac{dz}{dx} = B z \qquad (2)$$

with $\left[B = T[A] = T^{-1} A T - T^{-1} \frac{dT}{dx} \right]$

Systems (1) and (2) and matrices A and B are said equivalent

EXERCISE N. 1

Lo: if x_0 is a regular singularity of a differential system it is possible to transform the matrix so to have $p=0$ (Fuchsian form)

and also "normalized" meaning that given (shifting the singularity in $x=0$ for easy of notation)

$$B(x) = \frac{1}{x} \sum_{i=0}^{\infty} B_i x^i$$

the eigenvalues of B_0 are all in $[0,1[$ interval.

~ ○ ~ ○ ~ ○ ~ ○ ~

- starting with the problem

$$\frac{dI}{dx} = A(x) I(x) \qquad I(x_{bc}) = I_{bc}$$

- shifting the singularity in $\eta = 0$ for easy of notation

- and computing the transformation T such that

$$I = T \tilde{I} \quad \text{AND} \quad \frac{d\tilde{I}}{dx} = B(x) \tilde{I} \; ; \quad \tilde{I}_{bc} = \tilde{I}(x_{bc}) = T^{-1}(x_{bc}) I(x_{bc})$$

with $B = T^{-1} A T - T^{-1} \frac{dT}{dx}$ not B normalized Fuchsian

the series solution of the differential equation has the form

$$\tilde{I}(\eta) = \sum_{\lambda \in S} \eta^{\lambda} \sum_{k=0}^{K_\lambda} \frac{\log^k(\eta)}{k!} \sum_{q=0}^{\infty} \tilde{I}_{\lambda,k,q}\, \eta^q$$

λ's are the generalized eigenvalues of B_0.
By construction $\lambda \in [0, 1[$.
In fact B_0 can be defective.

K_λ is the lenght of the longest Jordan chain of the λ eigenvalue minus 1.

– We insert this ansatz in the form of a fundamental matrix solution H ($n \times n$ matrix)

$$\frac{d}{d\eta} H = B\, H$$

Recurrence need renormalization...

consider B_0 (in standard form)

$$B_0 = \begin{pmatrix} \lambda_1 & 0 & 1 & & & \\ 0 & \lambda_1 & 1 & 0 & & \\ & 0 & \lambda_1 & 0 & & \\ & & & \lambda_2 & & \\ & & & & \lambda_2 & \\ & & & & & \ddots \end{pmatrix}$$

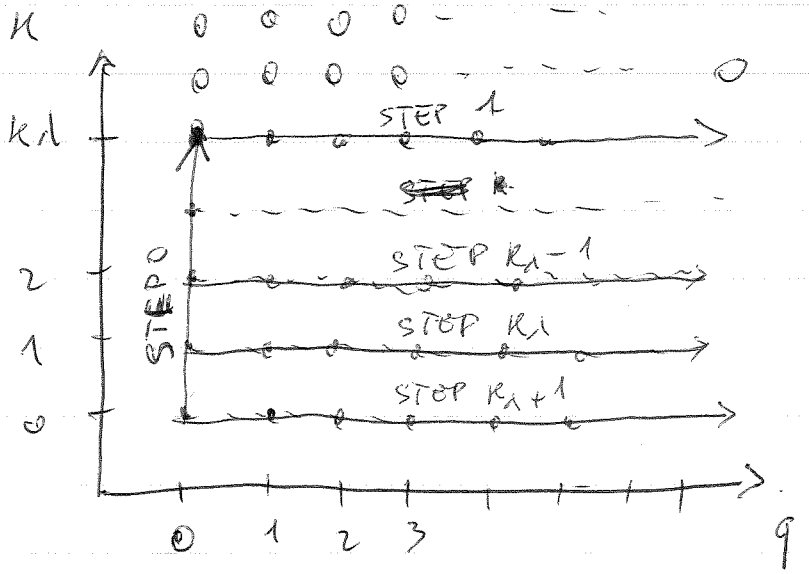
H matrix is

$$H(\eta,\zeta) = \sum_{\lambda \in S} \eta^{\lambda} \sum_{k=0}^{k_\lambda} \frac{\log^k(\eta)}{k!} \sum_{q=0}^{\infty} H_{\lambda,k,q}^{(i,j)} \eta^q$$

for each λ set $H_{\lambda,00} = \begin{cases} \delta_{ij} & \forall j : \lambda = \lambda_j \\ 0 & \text{elsewhere} \end{cases}$

NOTE. DE WILL NOT MIX COEFFS RELATED TO DIFFERENT λ's

For each λ DE provides recurrence relations



ALL COLUMNS ARE SO FILLED IN $H(i,s)$
(UP TO A CERTAIN ORDER)

The sought solution is the linear combination of the columns of H (H_j) that reproduces the boundary

given $\tilde{I}(n) = \sum_j c_j H_j(n)$ (constants)

we impose

$$T(n_{bc})\,\tilde{I}(n_{bc}) = I_{bc}$$

that give rise to a linear system for the c_j's

- The solution is now known and we can compute the limit for $\eta \to 0$ if this is a useful/meaningful target
residual divergences have to be discarded - related to AC E, limits do not commute
or evaluate the solution in the next point of a path if desired.

(13)

Analytic continuation for η^λ with λ rational and $\log(\eta)$. Remember we have shifted the singularity at $\eta=0$ so we will match at a negative value of η and evaluate at $\eta=0$ or a positive value of η.

Auxiliary variable associated to the branch cut

$z = s - 4m^2$ for example will become

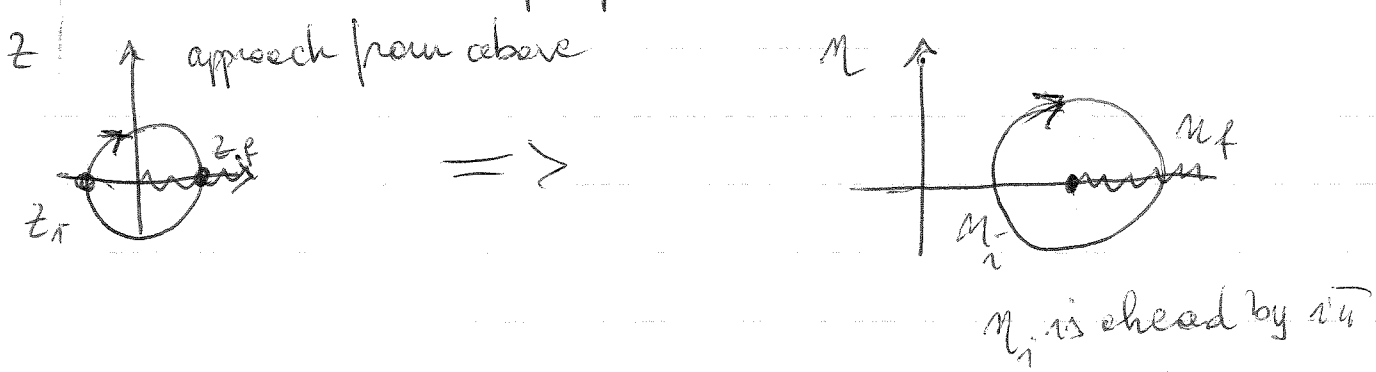
$z(\eta)$. Feynmen prescription tells

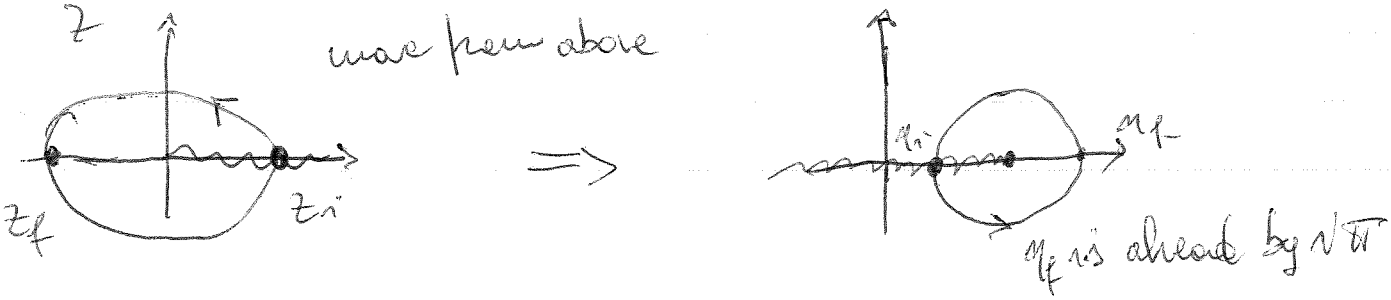
that the branch cut is at $z > 0$

the $z(\eta)$ relation

By inverting we will have a branch cut in the η plane and the prescription to add a small imaginary part to η.

The linear map preserves circular orientation





MOSER 1960
BARKATOU-PFLÜGEL 2007

EXERCISE N. 1

Consider the differential system with matrix

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i \qquad p = \text{Poincaré rank}$$

def. generalized rank $\quad m(A) = p + \frac{r}{n}$

r = rank of A_0
n = dimension of A

Moser proved that if $m(A) > 0$ and

$$\Theta(A,\lambda) = \lim_{x \to 0} \left\{ x^r \det\left(\frac{A_0}{x} + A_1 - \lambda I \right) \right\} \equiv 0$$

identically zero for every λ, then A is reducible, in the sense that there exist a transformation T such that $m(B) < m(A)$ with

$$B = T[A] = T^{-1} A T - T^{-1} \frac{dT}{dx}$$

We will arrive at $p=0$ through a sequence of transformation, accumulating them

$$\frac{dy}{dx} = Ay \quad \rightarrow \quad y = T_1 z \qquad \frac{dz}{dx} = Bz \qquad B = T_1[A]$$

$$\rightarrow \quad z = T_2 w \qquad \frac{dw}{dx} = Cw \qquad C = T_2[B]$$

$T = T_1 T_2$ is the cumulative transformation that reproduces the same matrix of the full sequence of transformations. $C = T_1 T_2 [A]$

In fact

$$T_1 T_2 [A] = T_2^{-1} T_1^{-1} A T_1 T_2 - T_2^{-1} T_1^{-1} \frac{d}{dx}(T_1 T_2)$$

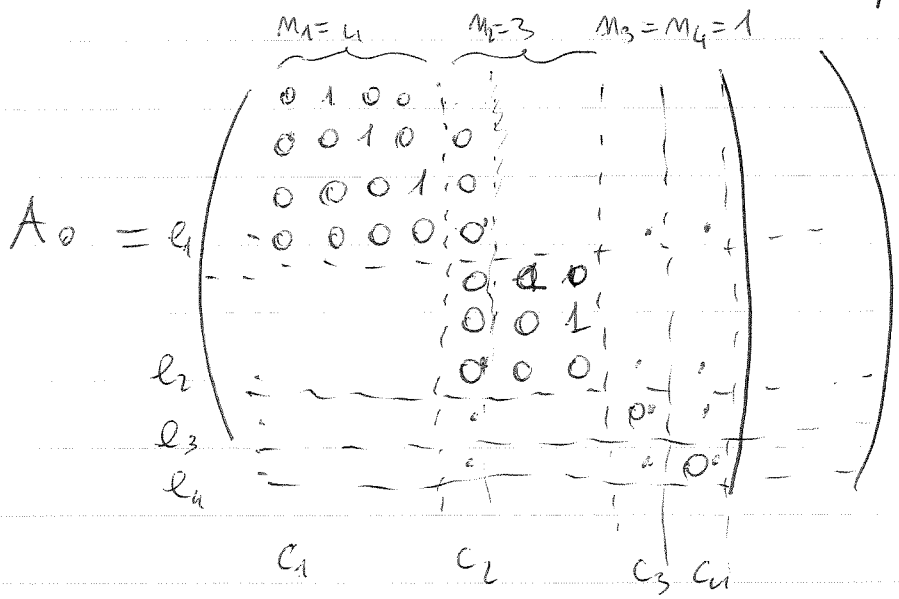
$$= T_2^{-1} T_1^{-1} A T_1 T_2 - T_2^{-1} T_1^{-1} \left(\frac{d}{dx} T_1\right) T_2 - T_2^{-1} T_1^{-1} T_1 \frac{d}{dx} T_2$$

$$= T_2^{-1} \underbrace{\left(T_1^{-1} A T_1 - T_1^{-1} \frac{d}{dx} T_1\right)}_{B = T_1[A]} T_2 - T_2^{-1} \frac{d}{dx} T_2$$

and so on for multiple transformations.

The algorithm consists of the iteration of a basic procedure to compute a transformation matrix that lowers the generalized rank $\left(p + \frac{z}{n}\right)$

Step 1: Find a constant transformation that brings A_0 in Jordan form (T_1)



in general:
d blocks with $n_i \geq 2$
s blocks with $n_i = 1$

$d+s$ lines ℓ_i
$d+s$ columns c_j

Now build the $(d+s) \times (d+s)$ matrix

$$L(A,\lambda) = A_1(\ell_i, c_j) - d_-(0_d, \lambda I_s)$$

~~note that~~
~~$\det L(A,\lambda) = (-1)^z \vartheta(A,\lambda)$~~

One has the $\det\left(L(A,\lambda)\right) = (-1)^z \vartheta(A,\lambda)$

So: $\vartheta \equiv 0 \Rightarrow \det\left(L(A,0)\right) = 0$

(16')

$$L = \overbrace{\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ \cdots & \cdots & \cdots & \cdots \\ x & x & x-\lambda & x \\ x & x & x & x-\lambda \end{pmatrix}}^{d} \Bigg\} s = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

$\underbrace{\qquad\qquad}_{s}$

now

or (1) $\operatorname{rank}\begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} + s = \operatorname{rank}\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$

or (2) one of the s columns is dependent.

If (1) is true go to the next step

otherwise find the dependent columns (put as the last) and eliminate it with a constant transformation

$$L(\tilde{A}, \lambda) = \begin{pmatrix} L_{11} & L_{12} & 0 \\ L_{21} & L_{22}-\lambda & 0 \\ L_{31} & L_{32} & \underbrace{L_{33}-\lambda}_{q} \end{pmatrix}$$

transformation has to make L_{33} upper triangular.

now

or (1) $\operatorname{rank}\begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} + s - q = \operatorname{rank}\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$

or (2) some other column among the $s-q$ has to be dependent. Repeat elimination until (1) is verified.

Note that all these constant transformations extended to the full A matrix will not change A_0 by construction.

Finally, we apply the following diagonal transformation (and cumulate)

$$S = \text{diag}\left(I_{m_1-1}, x, I_{m_2-1}, x, \dots, I_{m_{w-1}}, x, x I_{s-q}, I_q \right)$$

and ..... $m(S[A]) < m(A)$!!!

This procedure is repeated until $p = 0$ so that the differential system is fuchsianized.

Next we normalize the system, we bring all eigenvalues of A_0 in the $[0,1[$ interval.

We obtain this repeating the following 2 steps

1) put A_0 in Jordan form (and cumulate the transformation)
2) Apply a so called shearing transformation

$$T_{SH} = \text{diag}\left(1, 1, \dots, 1, \underbrace{\eta^{-s}, \dots, \eta^{-s}}_{\text{block } b}, 1, \dots 1\right)$$

to shift eigenvalue of the Jordan block b by $s = \pm 1$

For matrices in block Triangular form (as the ones for MI) one can efficiently proceede by fuchsianizing the diagonal and normalizing the diagonal block first.

Transformations on a diagonal block will not affect the others but will affect of course sub-diagonal blocks.

Proceeding block by block we currently have

$$\tilde{A}^{(a,a)} = \frac{1}{\eta} \sum_{k=0}^{n} A^{(a,a)}_k \eta^k \quad \text{and}$$

$$T = \begin{pmatrix} T^{11} & 0 & \cdots & 0 \\ 0 & T^{22} & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & T^{BB} \end{pmatrix}$$

For the off-diagonal blocks we build a transformation via the form of

$$T_{off} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ T^{21} & 1 & 0 & \cdots 0 \\ T^{31} & T^{32} & 1 & \cdots \\ \vdots & & & \\ T^{B1} & T^{B2} & \cdots & T^{BB} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots \\ T_{21} & 1 & \cdots 0 \\ 0 & 0 & 1 \cdots \\ \vdots & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & T_{32} & 1 \cdots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \\ T_{31} & 0 & 1 & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

We proceed line by line top down and column by column from right to left

Although the final T matrix does not depend by the sequence we multiply a set of matrices like these (lower triangular) the final A matrix will!

Order matter and to avoid that fuchsmentation of a block is ruined we proceed in this way. In fact each of these transformation will touch only subcolumns and sub-row blocks.

To lower rank p of the block $A^{(r,s)}$ with $r>s$ we look for

$$T^{(r,s)} = \frac{G^{(r,s)}}{\eta^p} \quad \text{with } G^{(r,s)} \text{ a constant matrix}$$

By imposing cancellation of ~~$A^{(r,s)}_{p+1}$~~ the leading ~~term~~ order $1/\eta^{p+1}$

$$0 = A^{r,s}_{-p} - G^{rs} \cdot A^{ss}_0 + A^{rr}_0 G^{rs} + p G^{rs}$$

which is a linear system for the $L_r \times L_s$ matrix elements of G^{rs}. One has to iterate until $p=0$ and then proceede to the next block

EXERCISE N. 2 : Boundary conditions with the 'AMFlow method

- Introduce an auxiliary mass (squared) parameter (η) in all denominators
- Find the extended list of Master Integrals for the new topology. At fixed kinematic it's a one scale problem (η).
- Compute the DE in η

$$\frac{d}{d\eta} I(\epsilon, \bar{s}, \eta) = A(\epsilon, \bar{s}, \eta)\, I(\epsilon, \bar{s}, \eta)$$

It is relatively easy to have a boundary condition for this equation considering the $\eta \to \infty$ limit

Expansion by region predict:

$$I_i(\epsilon, \bar{s}, \eta) \xrightarrow{\eta \to \infty} \eta^{\alpha_i} \left[I_{0,i}(\epsilon, 0, 1) + \mathcal{O}(\eta) \right]$$

(VANISHING KINEMATICS)

!! Fundamental eq. no $\log s(\eta)$, no other eigenvalues giving contributions

with $\alpha_i = \frac{dL}{2} - \sum_j \nu_j$

(ν_j: exponents of the denominator factors)

$I_1(\varepsilon, 0, 1)$ are uniform unit mass bedpoles.

- At 1 Loop for examples, all integrals reduce to a combination of (tree reduction)

$$= (-1)^{\nu} \frac{\Gamma(\nu - 2 + \varepsilon)}{\Gamma(\nu)}$$

- At 2 Loop all integrals reduce to a combination of the 1loop bedpole squared and

$$= (-1)^{\nu} \left[\frac{\Gamma(\cdot)\Gamma(\cdot)}{\Gamma(\cdot)\Gamma(\cdot)} \, {}_4F_3(\dots; \tfrac{1}{u}) + \frac{\Gamma(\cdot)\Gamma(\cdot)\Gamma(\cdot)\Gamma(\cdot)}{\Gamma(\cdot)\Gamma(\cdot)\Gamma(\cdot)\Gamma(\cdot)} \, {}_4F_3(\dots; \tfrac{1}{u}) \right]$$

IN PRACTICE :

- consider the limit $\eta \to -i\infty$

- transform $\eta \to \frac{1}{\eta}$ to study the $\eta \to 0$ limit

$$I_\infty(\varepsilon, \eta) = I(\varepsilon, \tfrac{1}{\eta})$$

$$A_\infty(\varepsilon, \eta) = -\frac{A(\varepsilon, \frac{1}{\eta})}{\eta^2}$$

$$\frac{d I_\infty}{d\eta} = A_\infty I_\infty$$

One can also factor out the leading η behaviour $\quad \mathrm{diag}\,\eta_\mu^{-e} = (\eta^{-a_1}, \eta^{-a_2}, \dots)$

$$\tilde{I}_\omega = \mathrm{diag}\,\eta_\mu^{-a}\, I_\omega \qquad \tilde{A}_\omega(\epsilon,\eta) = \mathrm{diag}\,\eta^{a}\left(A_\omega(\epsilon,\eta) + \frac{\mathrm{diag}\, a}{\eta}\right)\mathrm{diag}\,\eta^{-a}$$

$$\frac{d\tilde{I}_\omega}{d\eta} = \tilde{A}_\omega(\epsilon,\eta)\,\tilde{I}_\omega$$

then solve for Taylor series $\tilde{I}_\omega(\epsilon,\eta)$

using the BC to initialize recurrence relations.

The solution is used to compute MI to the first finite value of η and proceede along the path from singularity through regular points up to a point sufficiently close to $\eta = 0$.

point $\eta = \infty$ can be regular or regular singular and we have seen already how to compute the expansion around ~~both kind of~~ a regular sing. points. In the latter case one has to take the limit $\eta \to \infty$ to get the result for the physical master integrals.