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PHYSICS
GROUP

Go-Forward workshop, February 28, 2025

Constraint on initial conditions from non-linear causality

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Based on T. Hoshino, TH, Phys. Rev. C **111**, 014913 (2025), arXiv:2412.02405[nucl-th].

Outline

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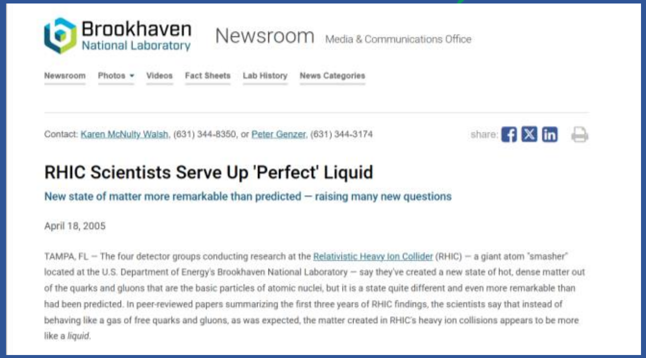
Introduction

Discovery of perfect fluidity announced in 2005

Precision QGP physics
spin/magneto hydrodynamics,
Bayesian analysis,
…

QGP fluids as thermal media
thermal photon/dilepton
jet quenching,
heavy quark(onium)

Validation of QGP fluidity



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Contact: Karen McNulty Walsh, (631) 344-8350, or Peter Genzer, (631) 344-3174 share:

RHIC Scientists Serve Up 'Perfect' Liquid

New state of matter more remarkable than predicted — raising many new questions

April 18, 2005

TAMPA, FL – The four detector groups conducting research at the Relativistic Heavy Ion Collider (RHIC) – a giant atom "smasher" located at the U.S. Department of Energy's Brookhaven National Laboratory – say they've created a new state of hot, dense matter out of the quarks and gluons that are the basic particles of atomic nuclei, but it is a state quite different and even more remarkable than had been predicted. In peer-reviewed papers summarizing the first three years of RHIC findings, the scientists say that instead of behaving like a gas of free quarks and gluons, as was expected, the matter created in RHIC's heavy ion collisions appears to be more like a *liquid*.

https://www.bnl.gov/newsroom/news.php?a=110303

Nonlinear causality of fluid dynamics

Linearized 2nd order hydro under static equilibrium background
$(\Pi = 0, \pi^{\mu\nu} = 0, u^{\mu} = 0)$

⇒ causality ✓

As long as large relaxation time

W.A. Hiscock, L. Lindblom, Annals of Physics **151**, 466 (1983).

2nd order dissipative hydrodynamics in nonlinear regime

⇒ causality ?

F.S. Bemfica *et al.*, Phys. Rev. Lett. **126**, 222301 (2021).

Characteristic equation

Hydro eqs. as quasi-linear PDE

$$A^{\alpha}(\Psi)\nabla_{\alpha}\Psi = F(\Psi)$$

$$\Psi = (e, u^{\mu}, \Pi, \pi^{0\mu}, \pi^{1\mu}, \pi^{2\mu}, \pi^{3\mu})^{\mathrm{T}}$$

$A^{\alpha}(\Psi)$: 22-by-22 coefficient matrix

$F(\Psi)$: 22nd order column vector without any derivatives

Characteristic eqs.

$$\det(A^{\alpha}\xi_{\alpha}) = 0,$$

where $\xi^{\alpha} = \nabla^{\alpha}\Phi(x)$

→ ξ^{α} regarded as gradient of characteristic hypersurface (next slide)

Characteristic velocity

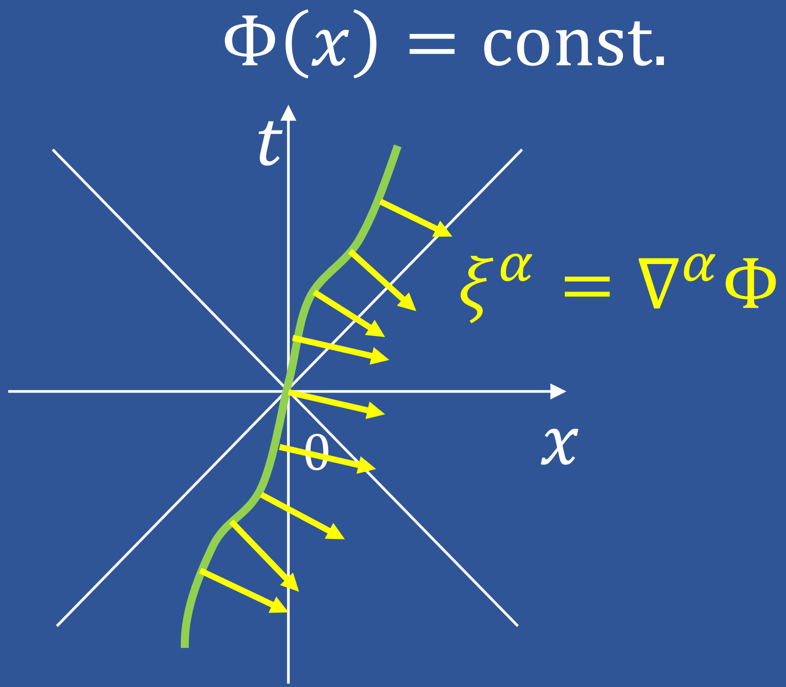
Normal vector of characteristic surface
→ (Light-like or) space-like vector

$$\xi^\alpha = bu^\alpha + a^\alpha, \quad \xi \cdot \xi = b^2 + a \cdot a \leq 0$$

time -like   space -like

$$0 \leq k(= -b^2/a \cdot a) \leq 1, \quad 0 \leq k = v_c^2 \leq 1$$

v_c^2 : Characteristic velocity



$^*g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

Speed of light: $c = 1$

W.A. Hiscock and T.S. Olson, Phys. Lett. A **141**, 125 (1989);
F.S. Bemfica *et al.*, Phys. Rev. Lett. **126**, 222301 (2021).

Conditions from nonlinear causality

Necessary conditions:

$$0 \le v_c^2 \le 1$$

v_c: characteristic velocity
(in a special case)

Sufficient conditions:

$$g(v_c^2 > 1) > 0$$
$$g(v_c^2 < 0) < 0$$

+ additional conditions

$g(v_c^2)$: third-degree polynomial

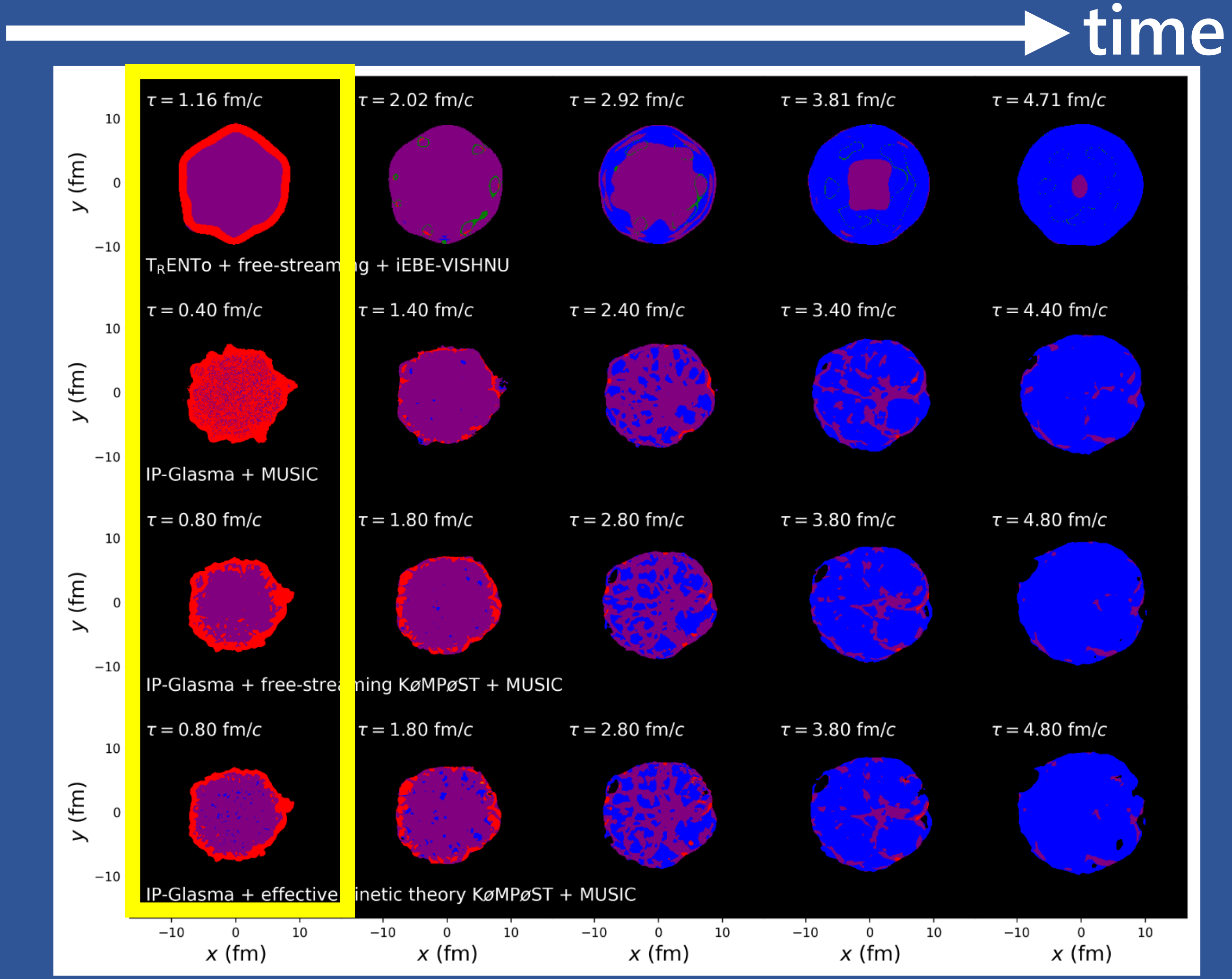
F.S. Bemfica *et al.*, Phys. Rev. Lett. **126**, 222301 (2021).

$$F_i(e, p, \Pi, \pi^{\mu\nu}, \eta, \zeta, \cdots) \ge 0$$

Explicit form in backup

Validation of hydrodynamic description
from nonlinear causality

Example of causality violation



Red: Acausal

← Does not satisfy necessary conditions

Blue: Causal

← Satisfy sufficient conditions

- Violations of causality
  → in the early stage and/or near the edge
- Likely to violate causality far from equilibrium

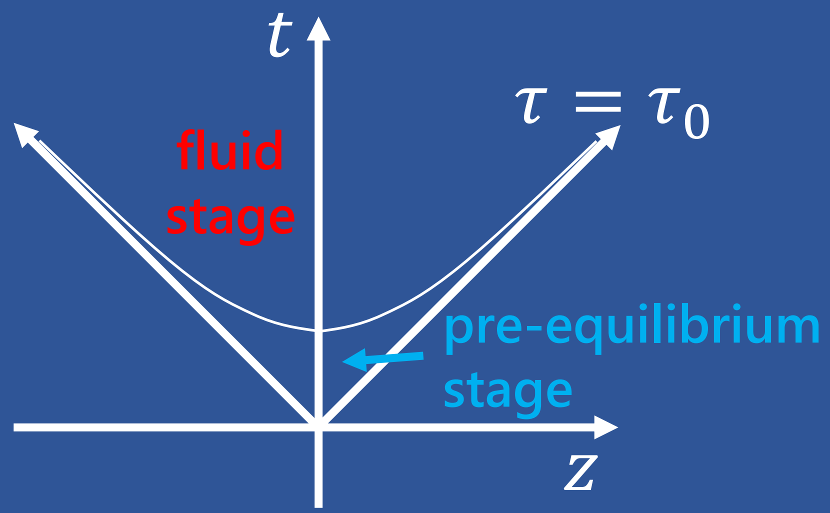
C. Plumberg *et al.*, Phys. Rev. C **105**, L061901 (2022).

Matching initial conditions

$T^{\mu\nu}_{\mathrm{init}}(\tau_0)$ ← Not always close to local equilibrium

Matching at initial time τ_0

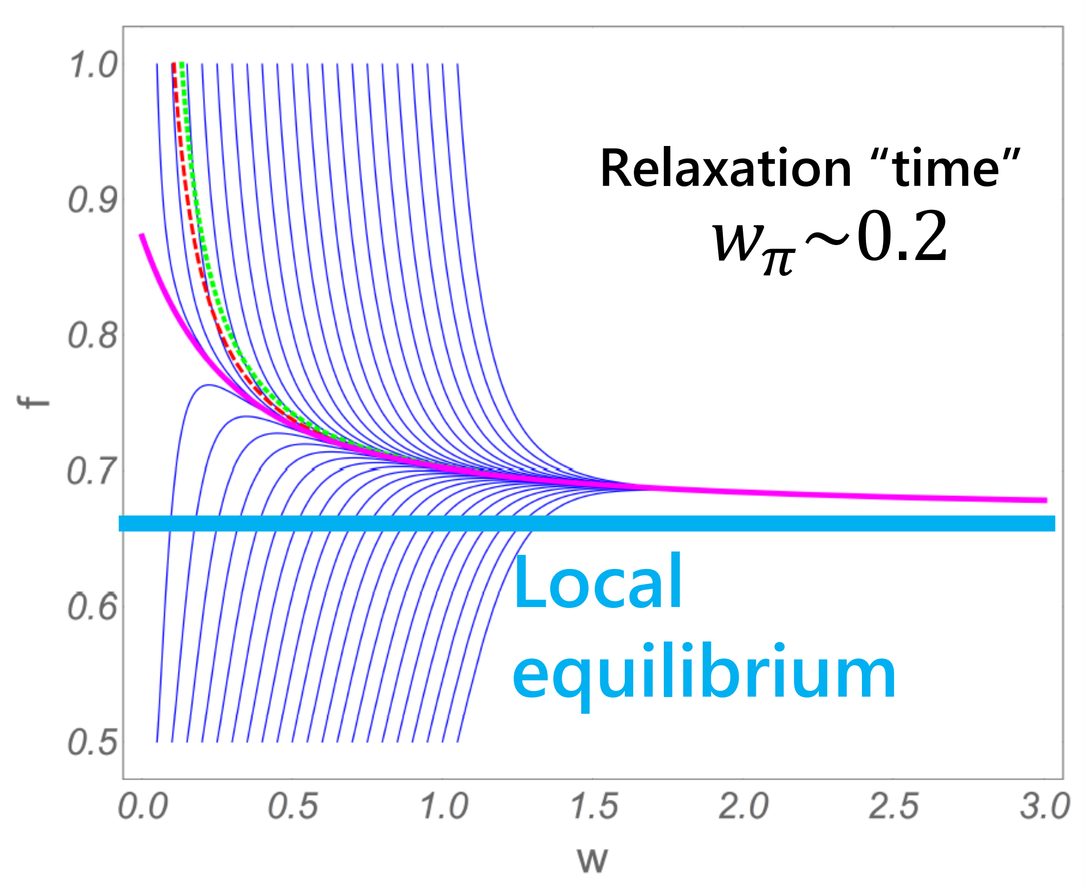
$$T^{\mu\nu}_{\mathrm{fluid}}(\tau_0) = eu^\mu u^\nu - p\Delta^{\mu\nu} + \pi^{\mu\nu}$$



$$Re^{-1} \approx \frac{|\pi^{\mu\nu}|}{e+p}$$

The inverse Reynolds number
← Sufficiently small !?

Is QGP fluid description valid after all?



M.P. Heller and M. Spaliński, Phys. Rev. Lett. **115**, 072501 (2015).

Hydrodynamic attractor solution

→ Is fluid dynamics far from equilibrium justified?

→ Are (almost) any initial conditions acceptable?

Purpose

Scrutinize validation of 1D expanding fluids from nonlinear causality

Model

Equation of motion in 1D expansion

Balance eq.+ BRSSS eq. with boost invariant solutions

R. Baier *et al.*, JHEP **04**, 100 (2008). J.D. Bjorken, Phys. Rev. D **27**, 140 (1983).

$$\tau \frac{d}{d\tau} e = -e - p(e) + \phi$$

$$\tau_\pi \frac{d}{d\tau}\phi = \frac{4\eta}{3\tau} - \phi - \frac{4\tau_\pi}{3\tau}\phi + \frac{\lambda}{2\eta^2}\phi^2$$

e: energy density
p: pressure
$\phi = \pi^{00} - \pi^{33}$: shear pressure

η: shear viscosity
τ_π: relaxation time
λ: 2nd order transport coefficient

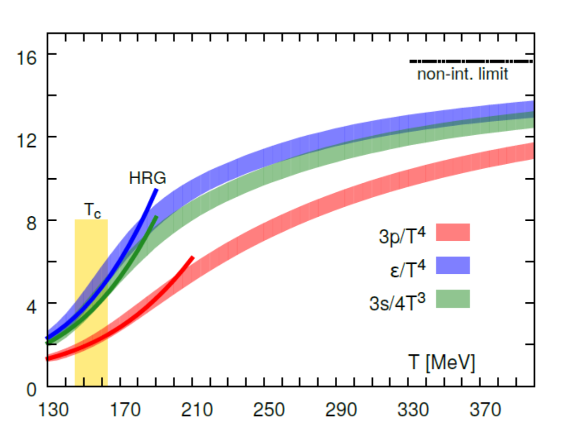
Equation of state and transport coefficient

EoS 1: Conformal EoS (default)

$$p(e) = \frac{1}{3}e$$

EoS 2: Lattice EoS

A. Bazavov *et al.*, Phys. Rev. D **90**, 094503 (2014).



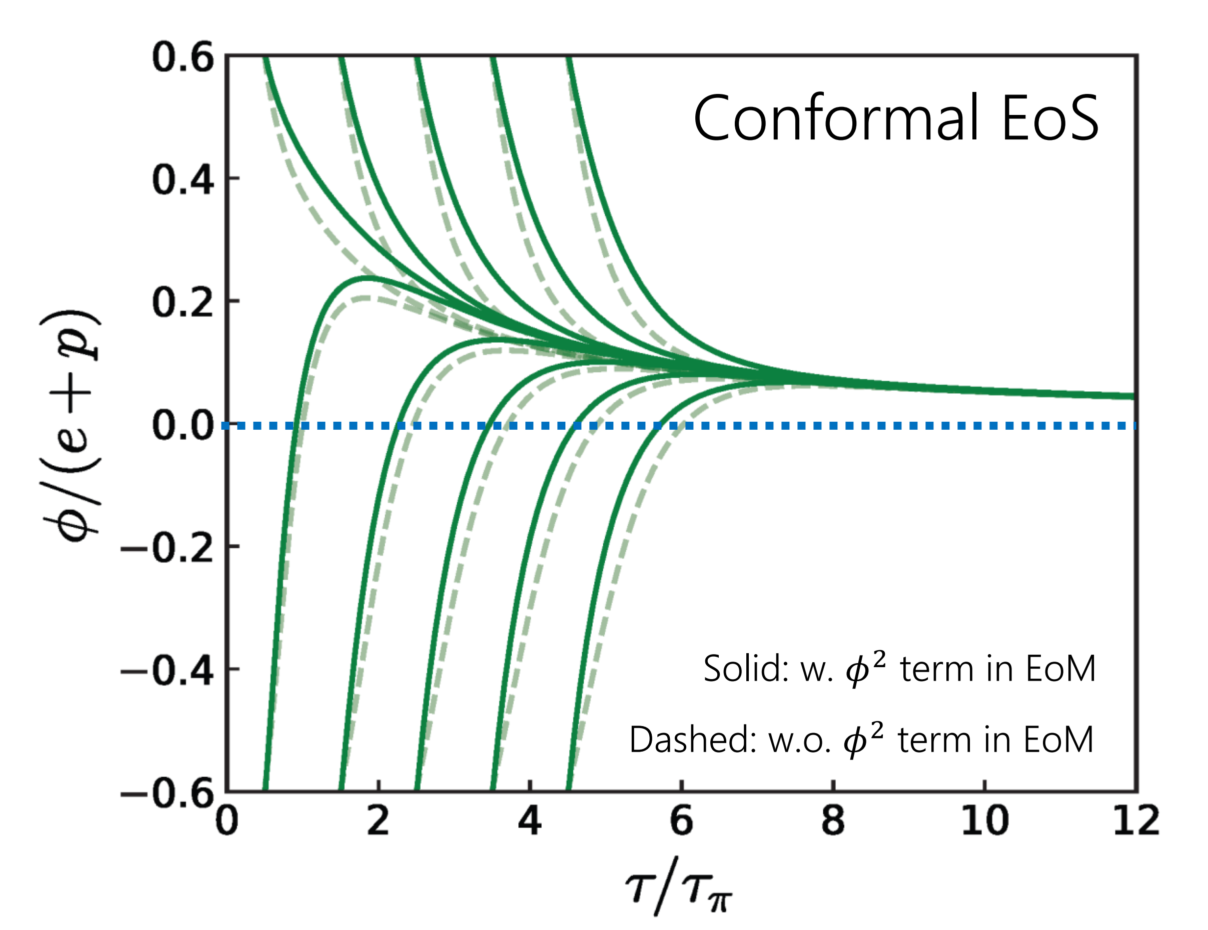
Transport coefficients (AdS/CFT)

*No bulk viscosity in this study

P. Kovtun *et al.*, Phys. Rev. Lett. **94**, 111601 (2005): R. Baier *et al.*, JHEP **04**, 100 (2008).

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau_\pi T = \frac{2 - \ln 2}{2\pi}, \quad \frac{\lambda T}{\eta} = \frac{1}{2\pi}$$

Behavior of solutions



"Any" initial conditions

Hydrodynamization

Attractor solution

Local equilibrium ($\phi = 0$)

Acceleration of hydrodynamization due to ϕ^2 term in EoM

Results

Necessary conditions in 1D expansion

$$\eta \geq 0 \qquad \frac{\eta}{\tau_\pi} \geq 0 \qquad e + p - \frac{\eta}{\tau_\pi} \geq 0$$

$$e + p - \frac{\phi}{2} - \frac{\eta}{\tau_\pi} \geq 0 \qquad e + p + \phi - \frac{\eta}{\tau_\pi} \geq 0$$

$$\left(e \ + p - \frac{\phi}{2}\right) c_s^2 + \frac{4}{3}\left(-\frac{\phi}{2}\right) + \frac{4\eta}{3\tau_\pi} \geq 0 \qquad \left(e \ + p - \frac{\phi}{2}\right)(1 - c_s^2) + \frac{2}{3}\phi - \frac{4\eta}{3\tau_\pi} \geq 0$$

$$(e \ + p + \phi)c_s^2 + \frac{4}{3}\phi + \frac{4\eta}{3\tau_\pi} \geq 0 \qquad (e \ + p + \phi)(1 - c_s^2) - \frac{4}{3}\phi - \frac{4\eta}{3\tau_\pi} \geq 0$$

Sufficient conditions in 1D expansion

$\phi > 0$ case

$$e + p - \frac{\phi}{2} - \frac{\eta}{\tau_\pi} \geq 0 \qquad \left(e + p - \frac{\phi}{2}\right) c_s^2 - \frac{2}{3}\phi + \frac{\eta}{3\tau_\pi} \geq 0$$

$$\frac{4}{3}\left(\phi + \frac{\eta}{\tau_\pi}\right) + \left(\frac{1}{2} + c_s^2\right)\phi + \frac{3c_s^2\phi^2}{e + p - \frac{\phi}{2} - \frac{\eta}{\tau_\pi}} \leq (e + p)(1 - c_s^2)$$

$$\left(\frac{\eta}{\tau_\pi}\right)^2 - 3c_s^2\phi^2 \geq 0$$

$$\left(e + p - \frac{\phi}{2}\right) c_s^2 + \frac{4}{3}\left(-\frac{\phi}{2} + \frac{\eta}{\tau_\pi}\right) \geq \frac{(e + p + \phi)^2 \left(e + p - \frac{\phi}{2} + \frac{2\eta}{\tau_\pi}\right)}{3\left(e + p - \frac{\phi}{2}\right)^2}$$

Sufficient conditions in 1D expansion

$\phi < 0$ case

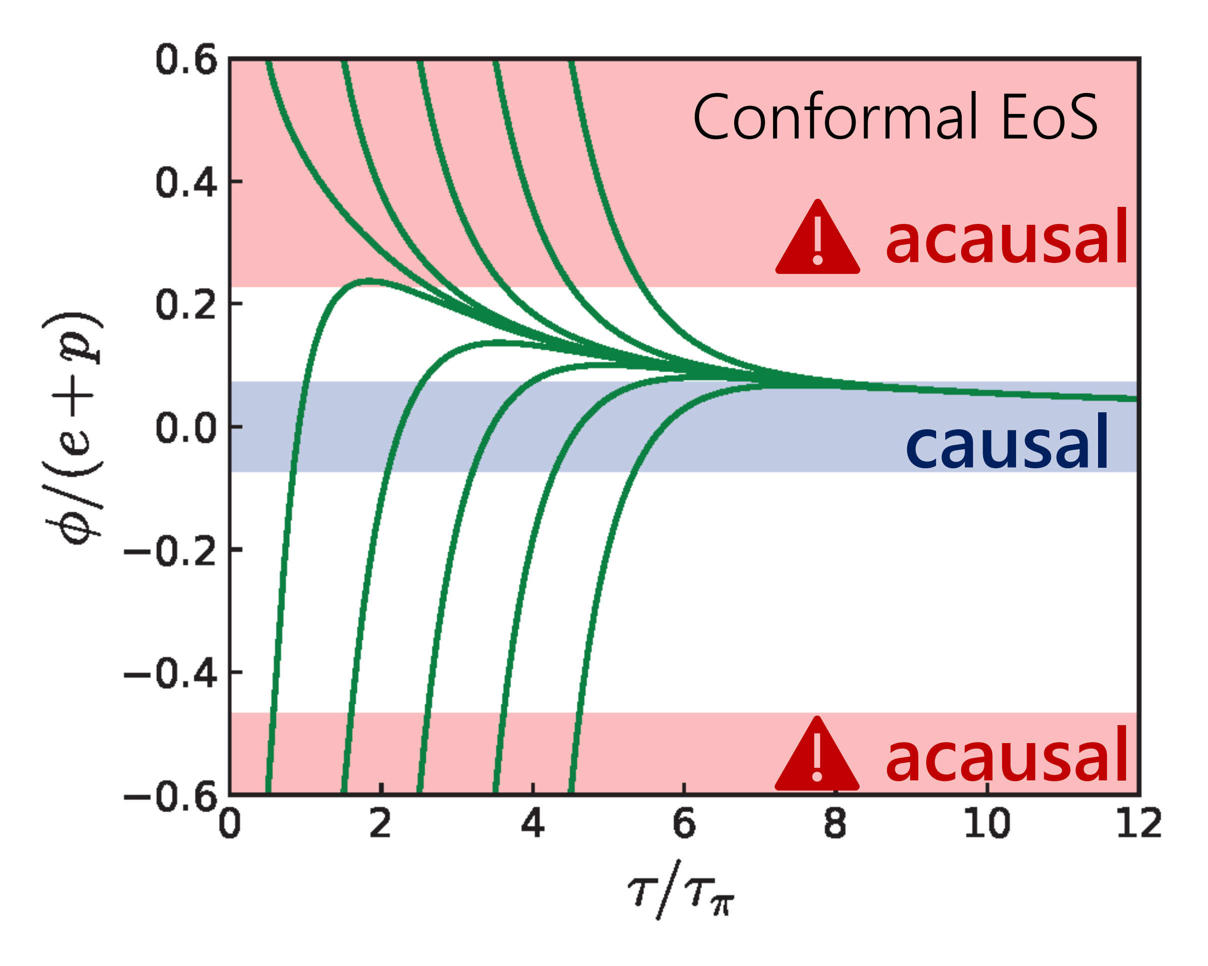
$$e + p - \phi - \frac{\eta}{\tau_\pi} \geq 0 \qquad (e + p + \phi)c_s^2 + \frac{4}{3}\phi + \frac{\eta}{3\tau_\pi} \geq 0$$

$$\frac{4}{3}\left(-\frac{\phi}{2} + \frac{\eta}{\tau_\pi}\right) - \left(1 + \frac{1}{2}c_s^2\right)\phi + \frac{3c_s^2\phi^2}{e + p + \phi - \frac{\eta}{\tau_\pi}} \leq (e + p)(1 - c_s^2)$$

$$\left(\frac{\eta}{\tau_\pi}\right)^2 - 3c_s^2\phi^2 \geq 0$$

$$(e + p + \phi)c_s^2 + \frac{4}{3}\left(\phi + \frac{\eta}{\tau_\pi}\right) \geq \frac{\left(e + p - \frac{\phi}{2}\right)\left(e + p + \phi + \frac{2\eta}{\tau_\pi}\right)}{3(e + p + \phi)}$$

Constraint on inverse Reynolds number



Necessary conditions

$$-0.47 \leq \frac{\phi}{e+p} \leq 0.23$$

Sufficient conditions

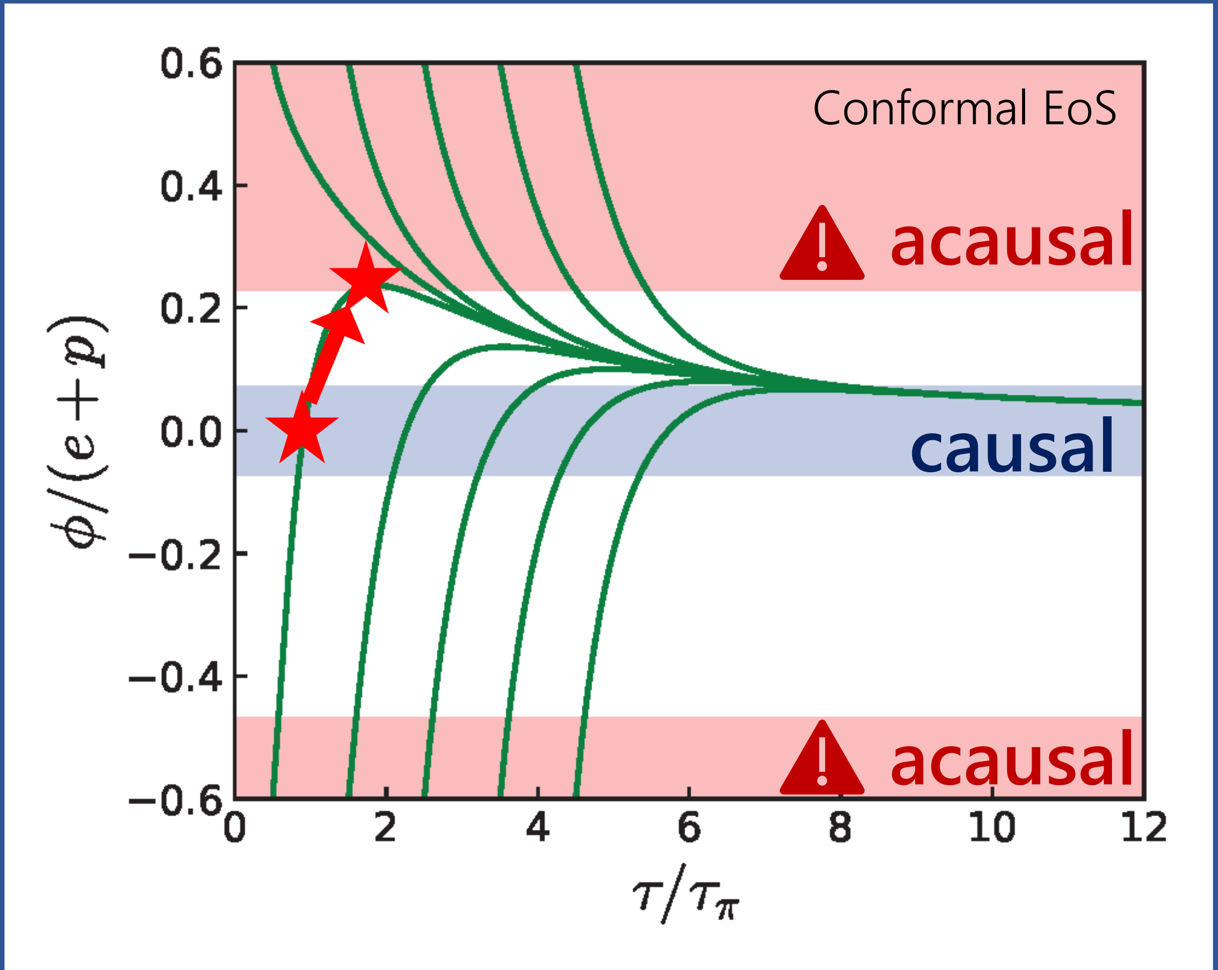
$$-0.07 \leq \frac{\phi}{e+p} \leq 0.07$$

Inverse Reynolds number

$$Re^{-1} \equiv \frac{|\phi|}{e+p}$$

← constrained from nonlinear causality

Dynamical violation of causality



Local equilibrium state

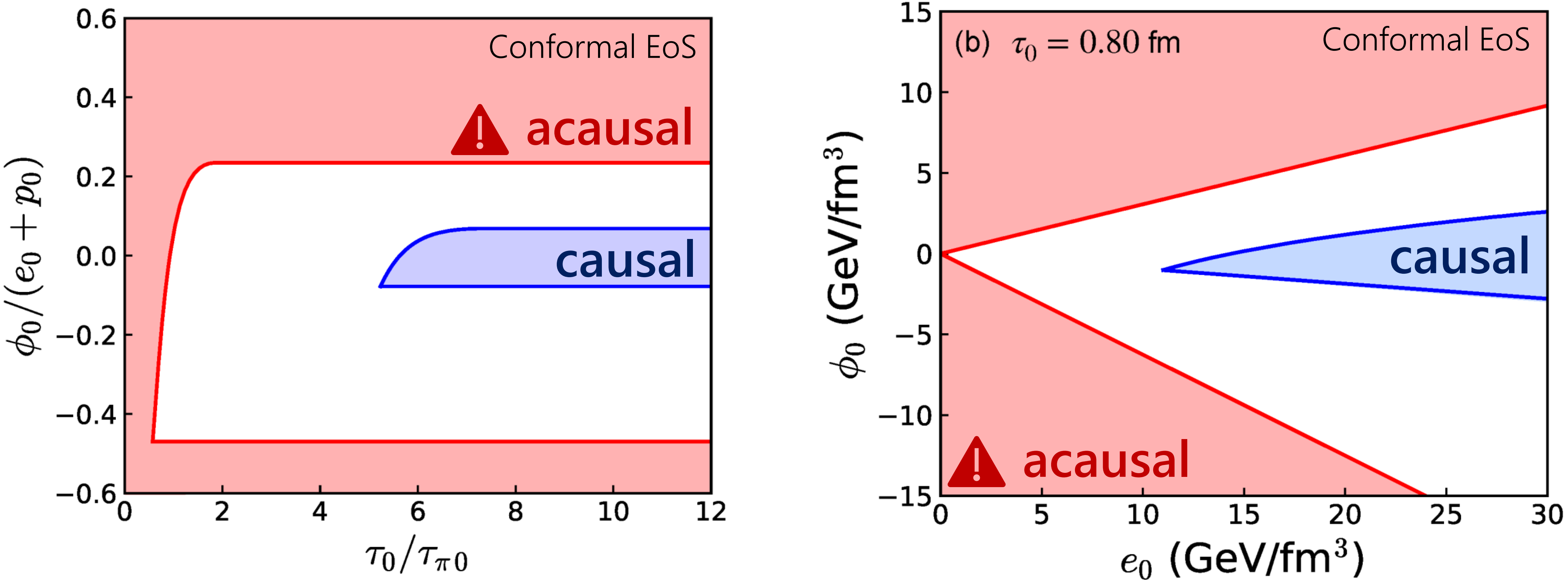
Large expansion rate $\theta_{\mathrm{Bj}} = 1/\tau$

Violation of causality

→ Limited available initial proper time

→ Necessity of nonequilibrium description

Constraint on initial conditions



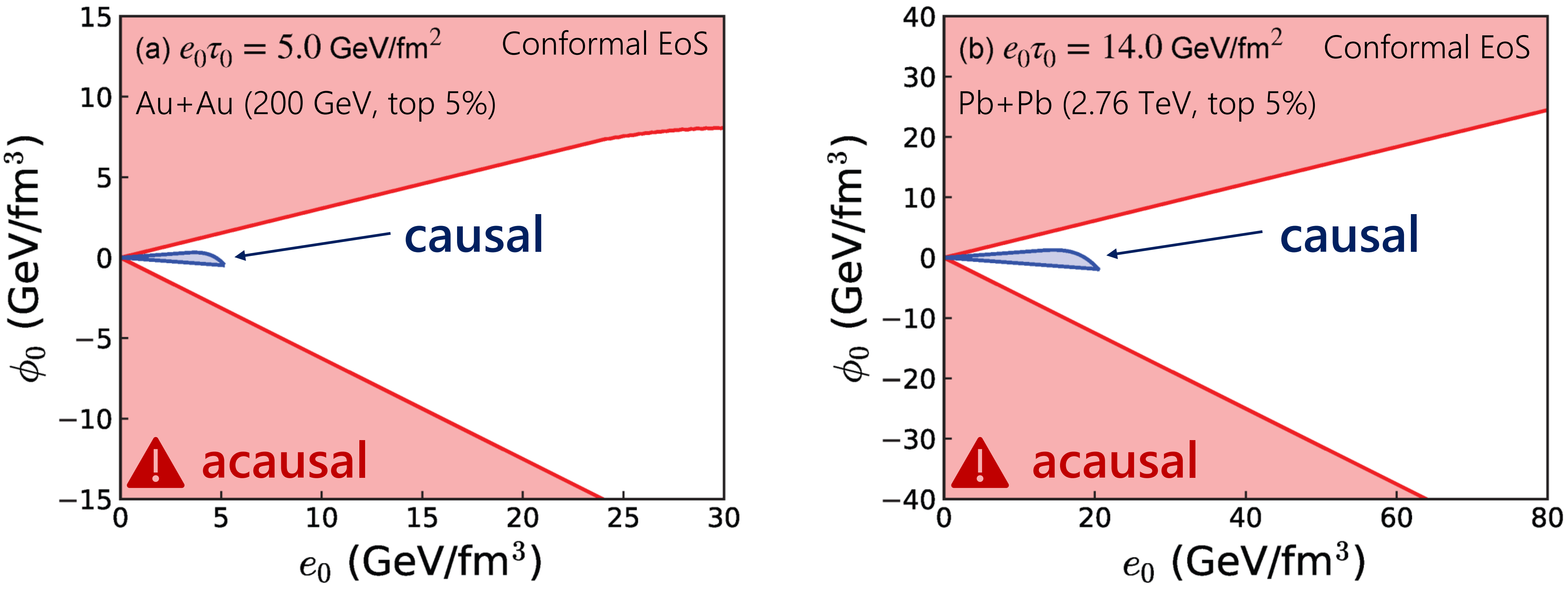
$^{\exists}\tau_{0,\mathrm{min}} \approx 5.3\tau_{\pi 0}$

$^{\exists}e_{0,\mathrm{min}} \sim 10\ \mathrm{GeV/fm^3}\ (\tau_0 = 0.8\ \mathrm{fm})$

Constraint from measurement

J. Adams *et al.* (STAR), Phys. Rev. C **70**, 054907 (2004).

S. Chatrchyan *et al.* (CMS), Phys. Rev. Lett. **109**, 152303 (2012).



$\tau_{0,\text{min}} \sim 1$ fm, $e_{0,\text{max}} \sim 5$ GeV/fm^3

$\tau_{0,\text{min}} \sim 0.7$ fm, $e_{0,\text{max}} \sim 20$ GeV/fm^3

Pocket formulae of minimum initial time and maximum energy density

$$\tau_0 e_0 \equiv E_0 = \frac{1}{S}\frac{dE_T}{dy}\ (\mathrm{GeV/fm^2})$$

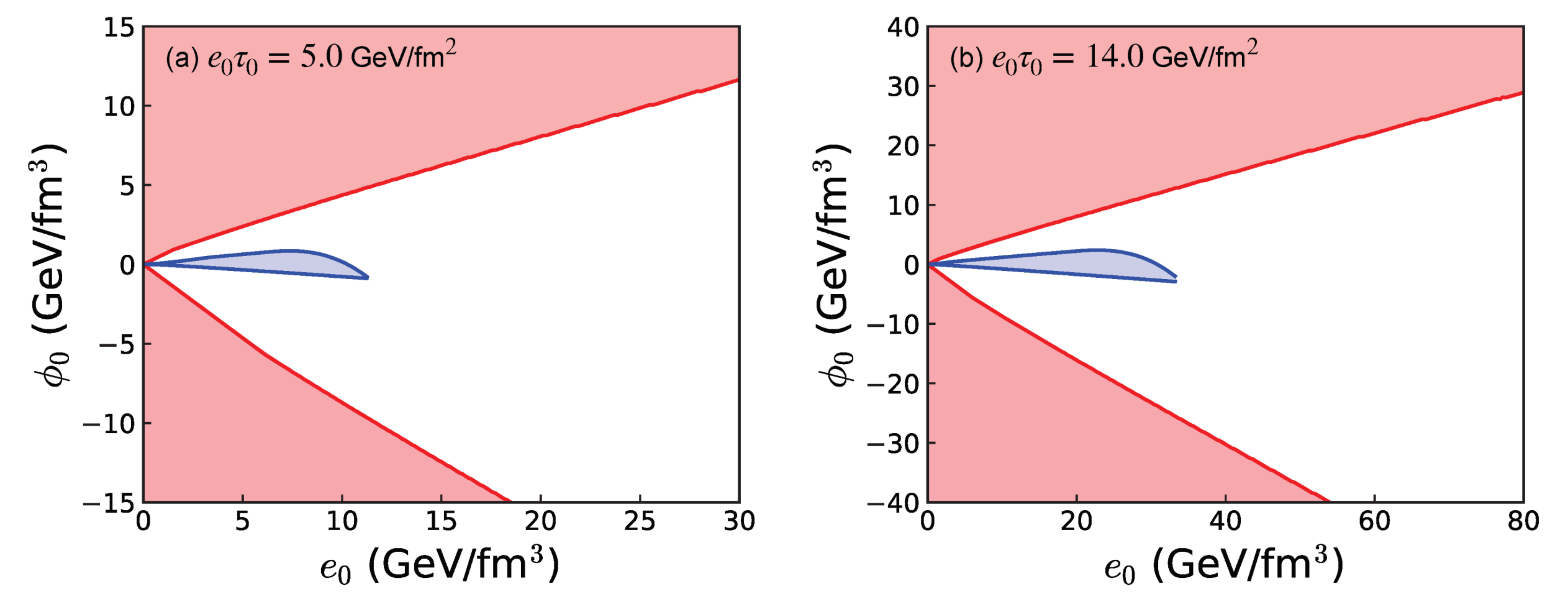
Glauber estimation of transverse area

Measurement of transverse energy

$$\tau_{0,\mathrm{min}} \sim 1.6\, E_0^{-\frac{1}{3}}\ (\mathrm{fm}) \qquad e_{0,\mathrm{max}} \sim 0.63\, E_0^{\frac{4}{3}}\ (\mathrm{GeV/fm^3})$$

(*Conformal EoS with $N_f = 3$ case)

Results with lattice EoS



Softer EoS → Smaller sound velocity → Tend to satisfy conditions

Summary

Summary

We scrutinized the initial conditions in 1D expansion from nonlinear causality.

- Nonlinear causality constrains the inverse Reynolds number

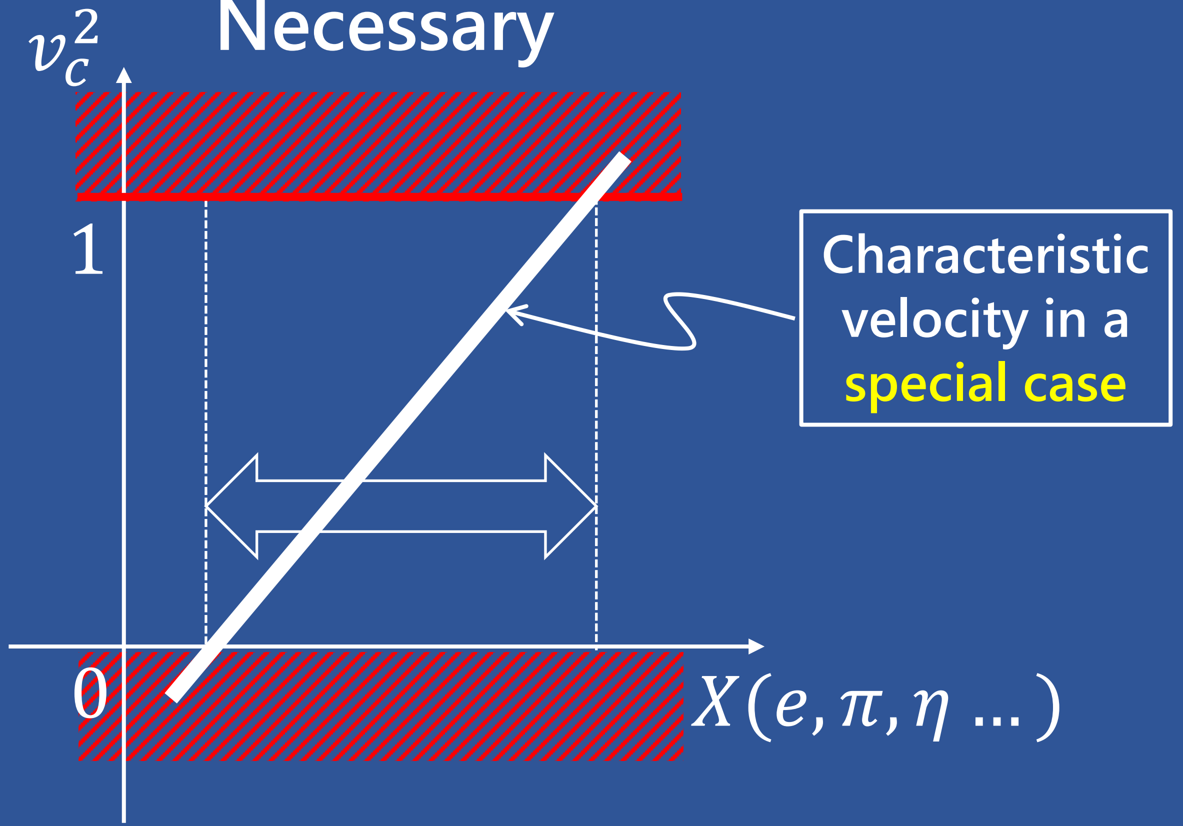
  $Re^{-1} < 0.23$ From necessary conditions
  $Re^{-1} < 0.07$ From sufficient conditions

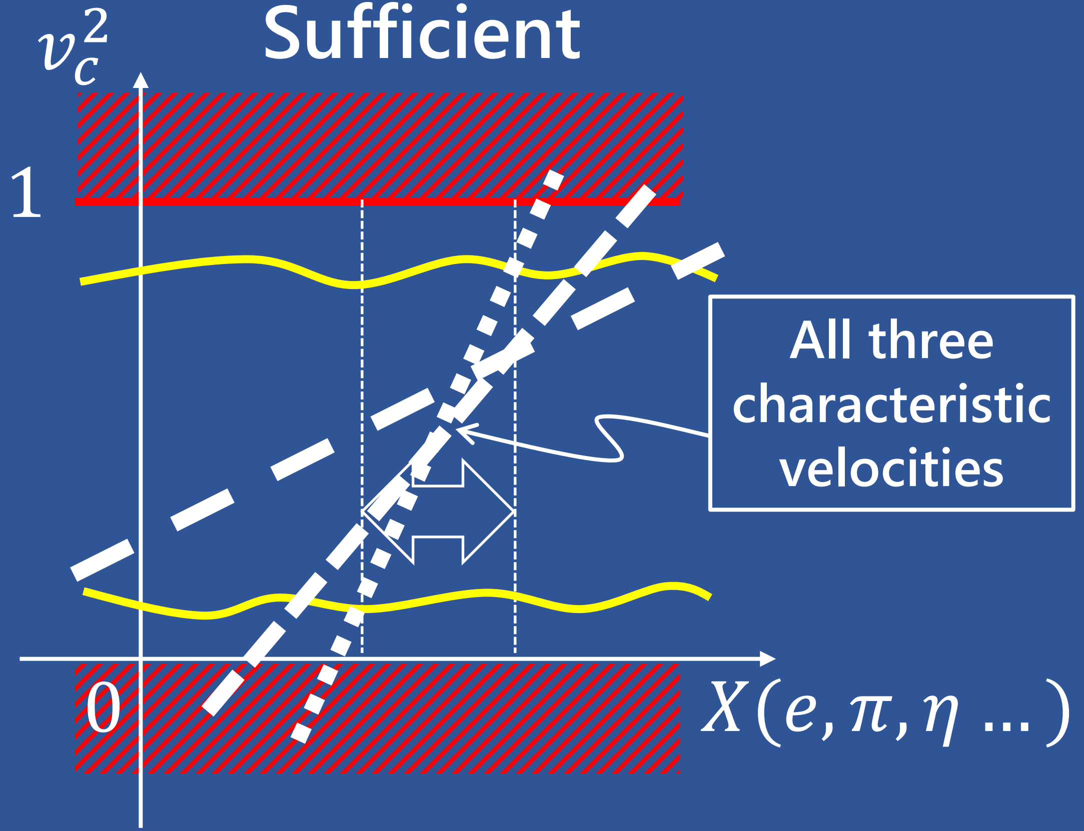
- Available regions of initial conditions from nonlinear causality
  - No hope for hydrodynamization
    → Need nonequilibrium description
  - Insufficient to start from local equilibrium at early time
  - Existence of minimum time and maximum energy density with a help of Bjorken energy density

Necessary and sufficient conditions

ALL characteristic velocities must satisfy $0 \leq v_c^2(X) \leq 1$

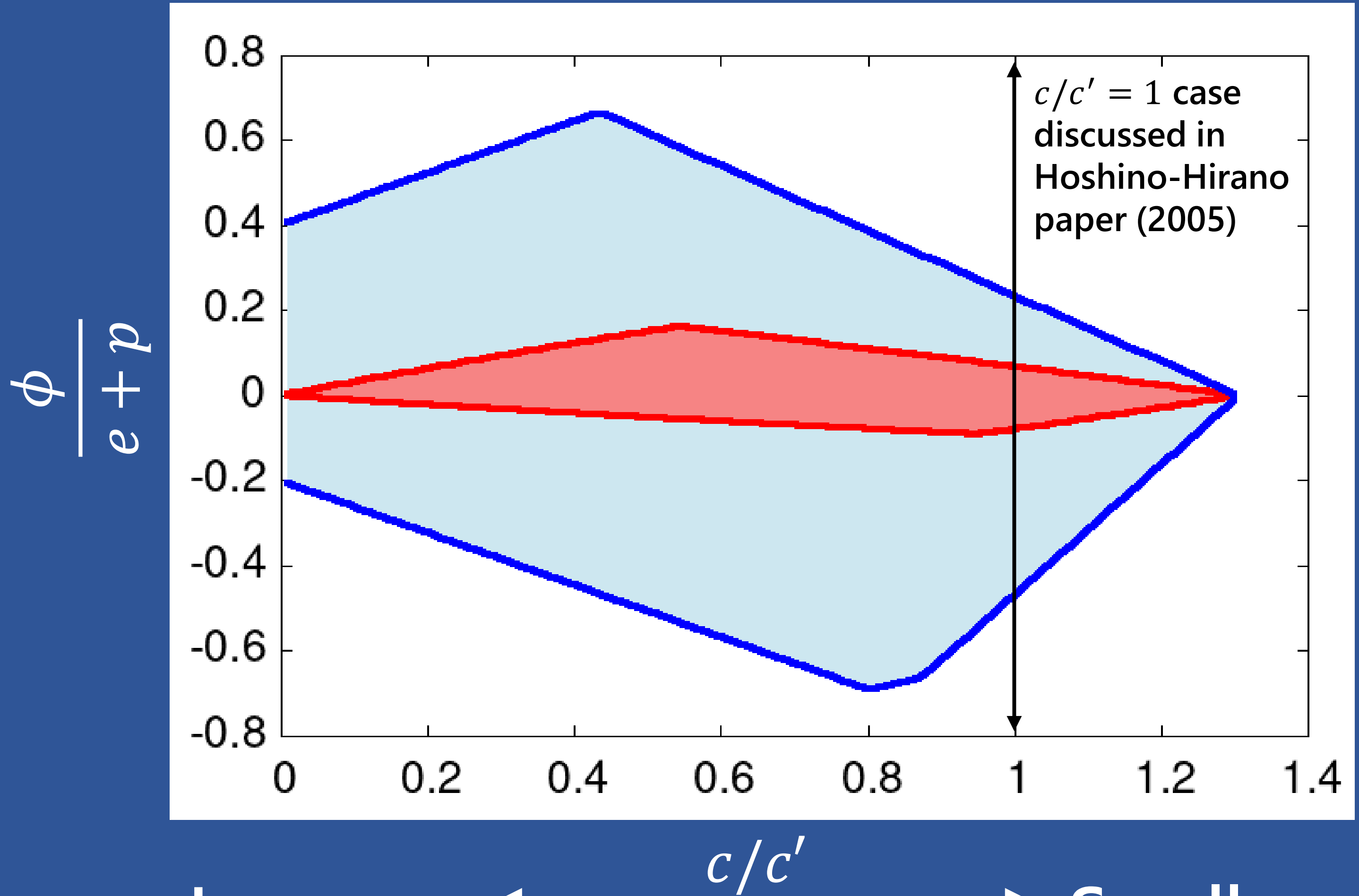
Difficult to solve characteristic equation to obtain $v_c^2(X)$





*Schematic interpretation

Transport coefficient dependence in 1D



Large τ_π ⟷ Small τ_π

$$\tau_\pi = c' \frac{2 - \ln 2}{2\pi T}$$

$$\eta = c \frac{s}{4\pi}$$

Violation of causality in Landau frame in the 2nd order theory when $\tau_\pi \to 0$

Y. Suichi, bachelor thesis, Sophia University (2025).

Transport coefficient dependence of conditions

(1) $$\frac{\phi}{e+p} \geq -1 + \frac{c}{c'}\frac{1}{2(2-\ln 2)}$$

(2) $$\frac{\phi}{e+p} \leq 2 - \frac{c}{c'}\frac{3}{2-\ln 2}$$

(3) $$\frac{\phi}{e+p} \leq \frac{2}{5} + \frac{c}{c'}\frac{4}{5(2-\ln 2)}$$

(4) $$\frac{\phi}{e+p} \geq -\frac{1}{5} - \frac{c}{c'}\frac{2}{5(2-\ln 2)}$$

(5) $$\frac{\phi}{e+p} \geq -2 + \frac{c}{c'}\frac{2}{2-\ln 2}$$

(6) $$\frac{\phi}{e+p} \leq 1 - \frac{c}{c'}\frac{1}{2-\ln 2}$$

Y. Suichi, bachelor thesis, Sophia University (2025).

Acausality of the first order relativistic dissipative equations

The "first" order theories (a.la. Eckart/Landau-Lifshitz)
→ Entropy current with the first order terms in dissipative currents ($s^\mu = s_0 u^\mu + q^\mu / T$)

Dispersion relation against linear perturbation

$$\boxed{\text{E.g.) Transverse mode } (\boldsymbol{k} \perp \boldsymbol{v}): \quad \omega = -i \frac{\eta}{e+P} k^2}$$

Diffusive → Infinite characteristic speed → Acausal!

"Density frame!?" ← No need for Landau matching condition???

F.S. Bemfica *et al.*, Phys. Rev. D **98**, 104064 (2018), P. Kovtun, JHEP **10**, 034 (2019).

Conformal fluids in Bjorken expansion

Balance eq. (Landau frame) and EoS

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = eu^\mu u^\nu - P(g^{\mu\nu} - u^\mu u^\nu) + \pi^{\mu\nu}, \qquad P = e/3$$

Constitutive eq. (BRSSS eq. with relevant terms in Bjorken expansion

$$\tau_\pi D\pi^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} - \frac{4}{3}\tau_\pi\theta\pi^{\mu\nu} + \frac{\lambda_1}{\eta^2}\pi^{\langle\mu}{}_\rho\pi^{\nu\rangle\rho}$$

R. Baier *et al.*, JHEP 0804, 100 (2008).

Boost invariant flow $u^\mu = (\cosh\eta_s, 0,0, \sinh\eta_s)$

J.D. Bjorken, Phys. Rev. D 27, 140 (1983).

$$\frac{d}{d\tau}e = -\frac{4}{3\tau}e + \frac{1}{\tau}\phi, \qquad \left(1 + \tau_\pi\frac{d}{d\tau}\right)\phi = -\frac{4\tau_\pi}{3\tau}\phi + \frac{4\eta}{3\tau},$$

$$\phi = \pi^{00} - \pi^{33}$$

$$\Rightarrow P_L = \frac{e}{3} - \phi$$

*Ignore ϕ^2 term for the moment by putting $\lambda_1 = 0$

Variable transformation

"Conformal time": $w = \tau T$

"Equilibrium measure": $f = \frac{3}{2}\tau \frac{1}{w}\frac{dw}{d\tau}$

$$C_{\tau\pi} w f \frac{df}{dw} + 4C_{\tau\pi} f^2 + \left(\frac{2}{3}w - \frac{32}{9}C_{\tau\pi}\right) f - C_\eta + 4C_{\tau\pi} - \frac{3}{2}w = 0$$

Transport coefficients: $\eta = C_\eta s, \quad \tau_\pi = \frac{C_{\tau\pi}}{T}$

M.P. Heller and M. Spaliński, Phys. Rev. Lett. **115**, 072501 (2015).

Note 1: In ideal hydrodynamics, $w \propto \tau^{2/3}$ from $T \propto \tau^{-1/3}$

Note 2: Different normalization employed for f

Derivation of equilibrium measure in conformal + boost invariant flow

$$\boxed{f} = \frac{3}{2}\tau\frac{1}{w}\frac{dw}{d\tau}$$

Conformality $e \propto T^4$

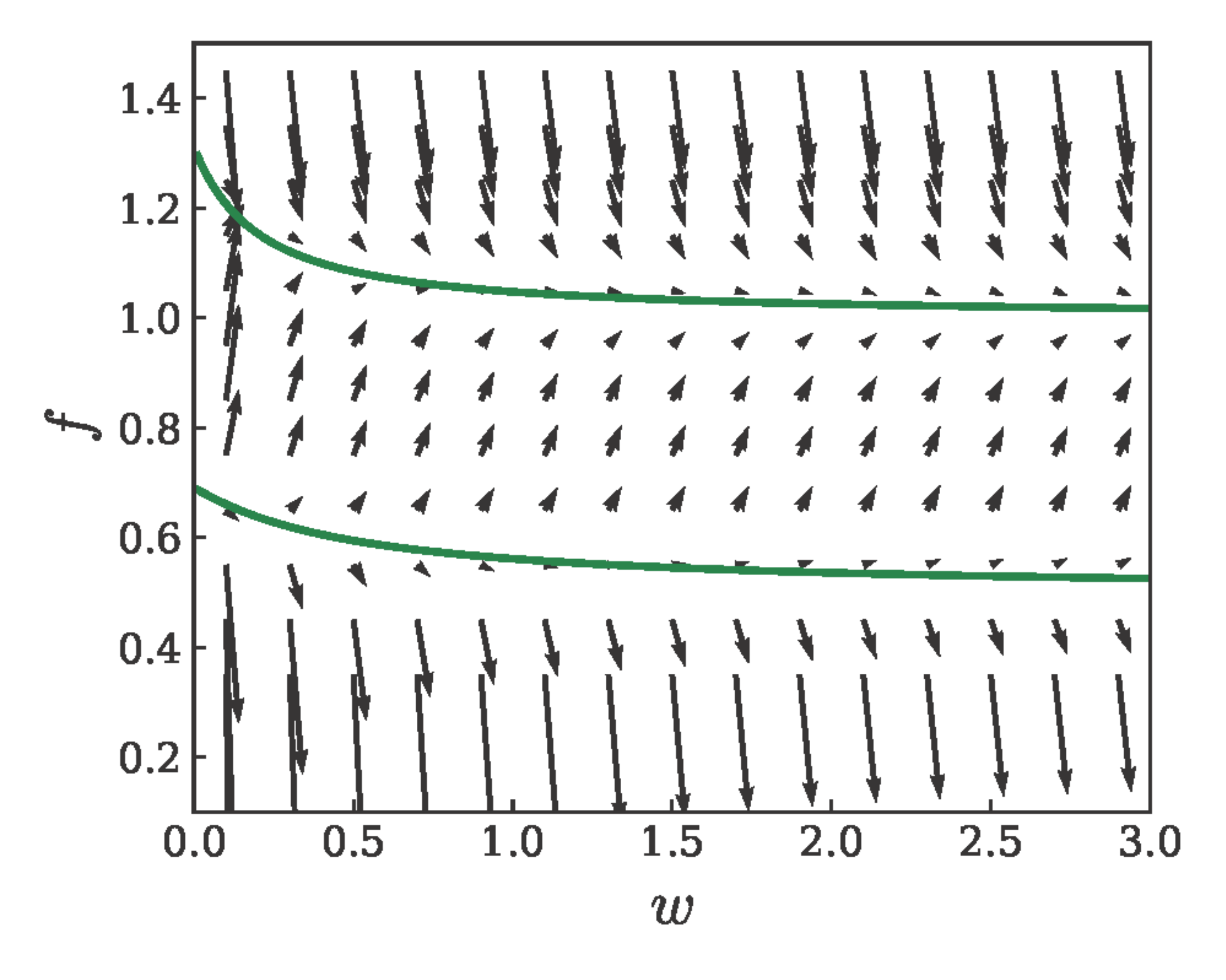
$$= \frac{3}{2}\left(1 + \tau\frac{1}{T}\frac{dT}{d\tau}\right) = \frac{3}{2}\left(1 + \tau\frac{1}{4e}\frac{de}{d\tau}\right)$$

Bjorken equation

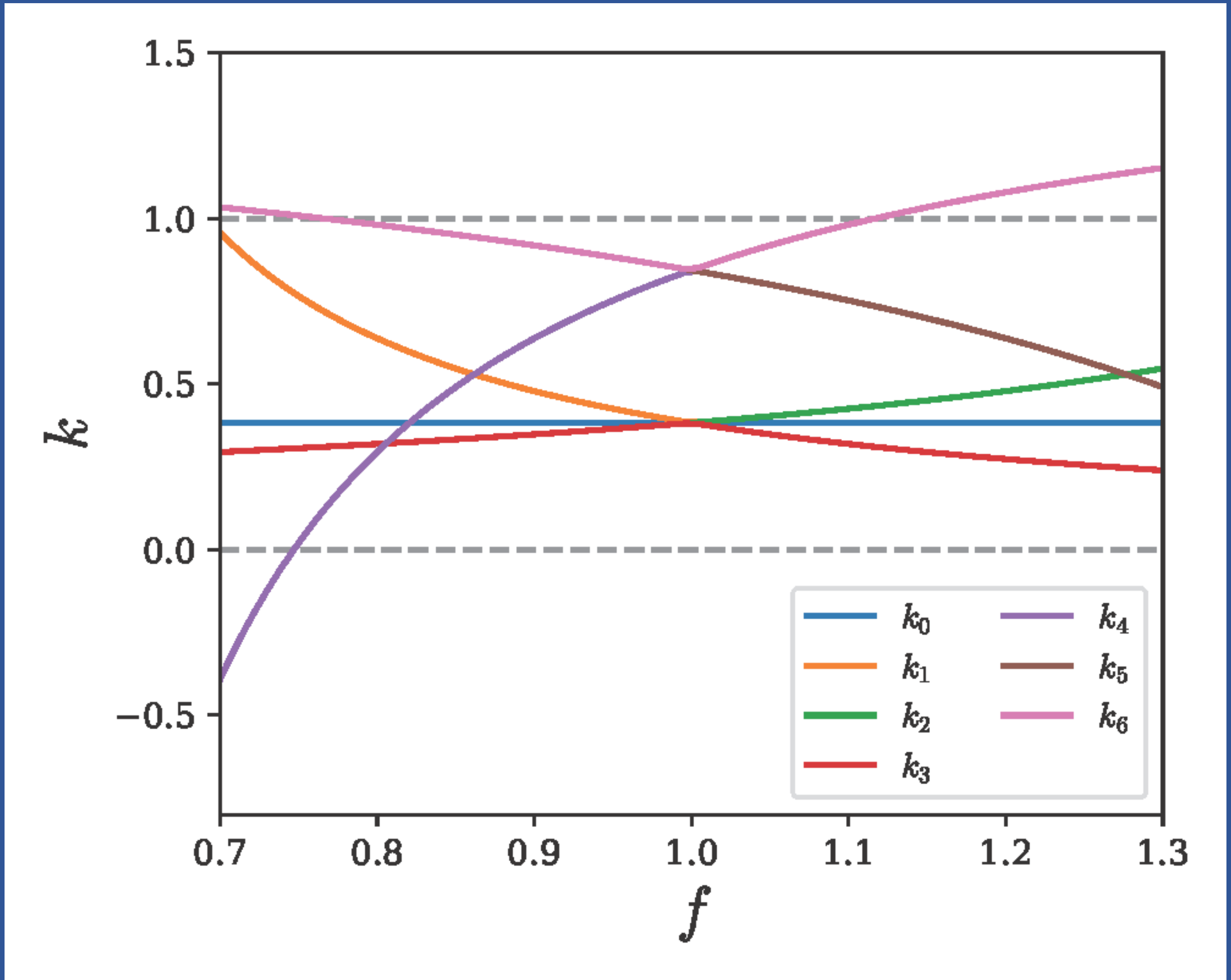
$$= \frac{3}{2}\left[1 + \tau\frac{1}{4e}\left(-\frac{e + P - \phi}{\tau}\right)\right]$$

$$= \frac{3}{2} + \frac{3}{8e}\left(-\frac{4}{3}e + \phi\right) = \boxed{1 + \frac{3\phi}{8e}}$$

Attractor and repulsive line



Square of characteristic velocity



Causality of fluid dynamics

Causality of second order hydrodynamics under static equilibrium background in linear perturbation

$$\Pi = 0, \qquad \pi^{\mu\nu} = 0, \qquad u^{\mu} = 0$$

bulk pressure shear stress four velocity

See, e.g., W.A. Hiscock, L. Lindblom, Annals of Physics **151**, 466 (1983).

→ Effects of transport coefficients in modern second order constitutive eqs. ?

$$\delta_{\pi\pi}\pi^{\mu\nu}\theta, \qquad \tau_{\pi\pi}\pi^{\langle\mu}{}_{\alpha}\sigma^{\nu\rangle\alpha}, \qquad \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}$$

→ Need to go beyond linear regime to capture full non-linearity of relativistic dissipative hydrodynamic equation

Conditions for non-linear causality

Quasi-linear PDE

$$A^{\alpha}(\Psi)\nabla_{\alpha}\Psi = F$$

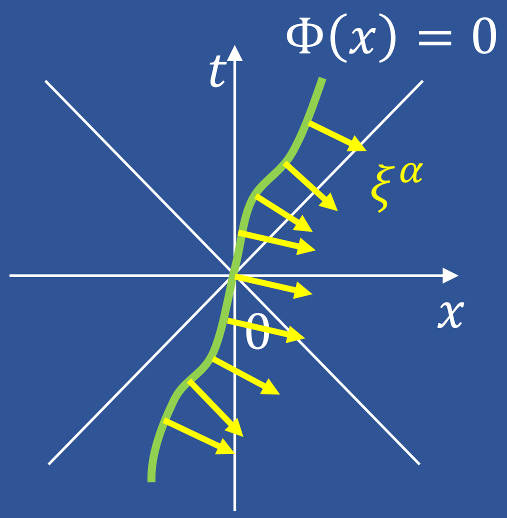
Characteristic eqs.

$$\det(A^{\alpha}\xi_{\alpha}) = 0, \quad \xi^{\alpha} = \nabla^{\alpha}\Phi(x)$$

The system is causal if*

Condition 1: The roots of characteristic equations $\xi^0 = \xi^0(\xi^i)$ are real.

Condition 2: The normal vector ξ^{α} of a characteristic surface is space-like (or light-like) so that the surface $\Phi(x) = \text{const.}$ is time-like.



$$\xi^{\alpha} = bu^{\alpha} + a^{\alpha}, \quad \xi\cdot\xi = b^2 + a\cdot a \le 0, \quad 0 \le k(= -b^2/a\cdot a) \le 1$$

*There exists a mathematically rigorous definition of causality.

F.S. Bemfica *et al.*, Phys. Rev. Lett. **126**, 222301 (2021).

Necessary conditions in DNMR

$$(2\eta + \lambda_{\pi\Pi}) - \frac{1}{2}\tau_{\pi\pi}|\Lambda_1| \geq 0$$

$$\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_a + \Lambda_d) \geq 0$$

$$e + P_s + \Pi - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\Lambda_3 \geq 0$$

$$e + P_s + \Pi + \Lambda_a - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_d + \Lambda_a)$$

$$\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d + \frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\pi} + (e + P_s + \Pi + \Lambda_d)c_s^2 \geq 0$$

$$e + P_s + \Pi + \Lambda_d - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d - \frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] - \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\pi} - (e + P_s + \Pi + \Lambda_d)c_s^2 \geq 0$$

F.S. Bemfica *et al.*, Phys. Rev. Lett. **126**, 222301 (2021).

Sufficient conditions in DNMR

$$\tau_{\pi\pi} \leq 6\delta_{\pi\pi}$$

$$(e + P_s + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 \geq 0$$

$$(2\eta + \lambda_{\pi\Pi}\Pi) - \tau_{\pi\pi}|\Lambda_1| > 0 \qquad \frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi} \geq 0$$

$$1 \geq \frac{\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3 + |\Lambda_1|)^2}{\left[\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right]^2}$$

$$\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (\tau_{\pi\pi} - 6\delta_{\pi\pi})|\Lambda_1|] + \frac{\zeta + \delta_{\Pi\Pi}\Pi - \lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi} + (e + P_s + \Pi - |\Lambda_1|)c_s^2 \geq 0$$

$$\frac{1}{3\tau_\pi}[4\eta + 2\lambda_{\pi\Pi}\Pi + (3\delta_{\pi\pi})\Lambda_3] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_3}{\tau_\pi} + |\Lambda_1| + \Lambda_3 c_s^2 + \frac{\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3 + |\Lambda_1|)^2}{e + P_s + \Pi - |\Lambda_1| - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3} \leq (e + P_s + \Pi)(1 - c_s^2)$$

$$\frac{1}{3\tau_\pi}[4\eta + 2\lambda_{\pi\Pi}\Pi - (3\delta_{\pi\pi} + \tau_{\pi\pi}|\Lambda_1|)] + \frac{\zeta + \delta_{\Pi\Pi}\Pi - \lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi} + (e + P_s + \Pi - |\Lambda_1|)c_s^2 \geq \frac{(e + P_s + \Pi + \Lambda_2)(e + P_s + \Pi + \Lambda_3)}{3(e + P_s + \Pi - |\Lambda_1|)}\left\{1 + \frac{2\left[\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]}{e + P_s + \Pi - |\Lambda_1|}\right\}$$

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