Inclusive semileptonic decays from Lattice QCD

Recent developments: $D_s \mapsto X\ell\bar{\nu}$

Alessandro De Santis

June 24, 2025 - Charming Lattice QCD @ CERN 2025

Outline

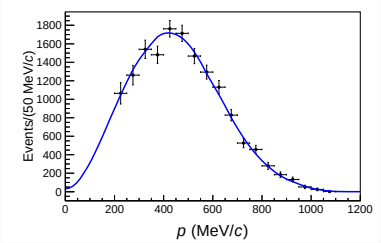
1. Introduction

2. Methods and pilot studies

3. Physics and phenomenology: $D_s \mapsto X\ell\bar{\nu}$

Introduction

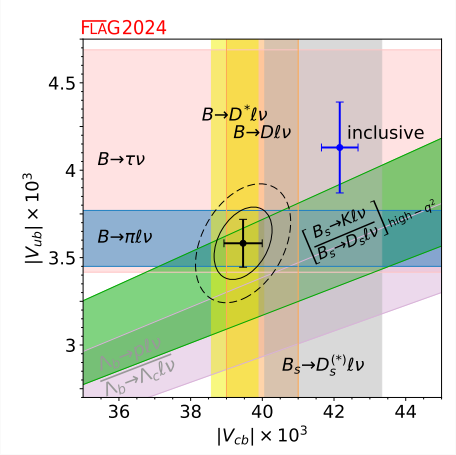
Motivation 1: phenomenology and methods testing



BES-III accuracy: 2.6%
CLEO accuracy: 6.4%

- The introduction of **new methods** has opened the possibility of performing Lattice QCD calculations of **inclusive processes**
- The inclusive semileptonic $D_s \mapsto X\ell\bar{\nu}$ offers the opportunity to do a **complete phenomenologically relevant calculation** with experimental accuracy within reach
- **Establishing the phenomenological relevance of a method** in an affordable setting is a mandatory checkpoint before applying to other processes
  $\mapsto$ Motivation 2

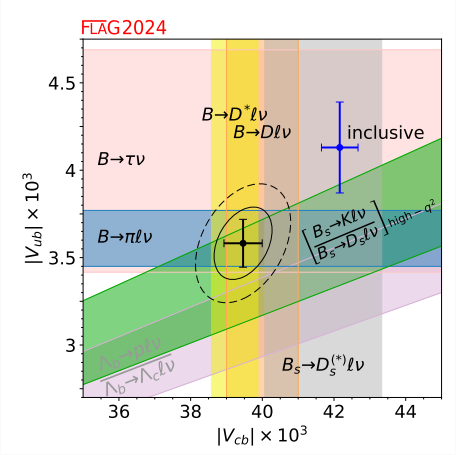
Motivation 2: $|V_{ub}|$ and $|V_{cb}|$ puzzle



- Exclusive relies on ab-initio **Lattice QCD** calculations $\mapsto$ **solid and mature framework**
- Inclusive = sum of all exclusive channels measured experimentally, or directly measured using tailored techniques $\mapsto$ **completely different systematics**

Is there a problem or a potential sign of physics BSM?

Motivation 2: $|V_{ub}|$ and $|V_{cb}|$ puzzle

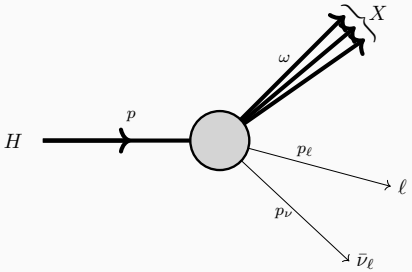


- EXCLUSIVE relies on ab-initio **Lattice QCD** calculations $\mapsto$ **solid and mature framework**
- INCLUSIVE = sum of all exclusive channels measured experimentally, or directly measured using tailored techniques $\mapsto$ **completely different systematics**

Is there a problem or a potential sign of physics BSM?

An inclusive determination of semileptonic decay $B_s \mapsto X_c \ell \bar{\nu}$ from Lattice QCD can answer this question.

Why so difficult?



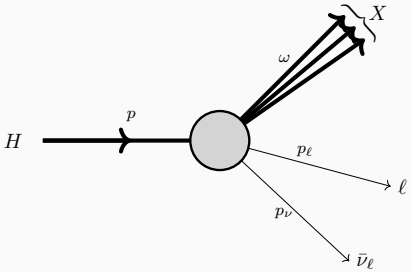
Hadronic decays are **non-perturbative** processes

- Compute each state and sum them all is unpractical

$$\omega = m_H(\omega_0, \boldsymbol{\omega})$$
$$p = (m_H, \mathbf{0})$$

mediated by the weak current

$$J^{\mu}_{\bar{f}g}(x) = \bar{\psi}_f(x)\gamma^{\mu}(1-\gamma^5)\psi_g(x)$$



$$\omega = m_H(\omega_0, \boldsymbol{\omega})$$
$$p = (m_H, \mathbf{0})$$

mediated by the weak current

$$J^\mu_{\bar{f}g}(x) = \bar{\psi}_f(x)\gamma^\mu(1-\gamma^5)\psi_g(x)$$

Hadronic decays are **non-perturbative** processes

- Compute each state and sum them all is unpractical
- The direct computation requires the knowledge of the hadronic tensor

$$\frac{\Gamma}{|V_{fg}|^2} = G_F^2 \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3} \frac{\mathrm{d}^3 p_\ell}{(2\pi)^3} \frac{L_{\mu\nu}(p_\ell, p_\nu)\ H^{\mu\nu}(\omega)}{4 m_H^2 e_\ell e_\nu}$$

$$H^{\mu\nu}(\omega) = \tfrac{(2\pi)^4}{2m_H} \langle H(0)|\, J^{\mu\dagger}(0)\delta^4(\mathbb{P}-\omega)J^\nu(0)\,|H(0)\rangle$$

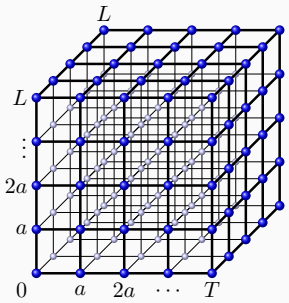
Why so difficult?

Lattice QCD is the only solid and improvable framework to perform non-perturbative calculations.

Why so difficult?

Lattice QCD is the only solid and improvable framework to perform non-perturbative calculations.

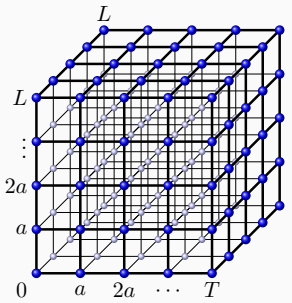
- discretize the spacetime introducing a **lattice spacing** a (UV regulator)
- put the physics inside a **finite box** of side L
- **tune the bare parameters** to reproduce the real world
- Repeat the simulation for different a and L



Why so difficult?

Lattice QCD is the only solid and improvable framework to perform non-perturbative calculations.

- discretize the spacetime introducing a **lattice spacing** a (UV regulator)
- put the physics inside a **finite box** of side L
- **tune the bare parameters** to reproduce the real world
- Repeat the simulation for different a and L



It gives access to

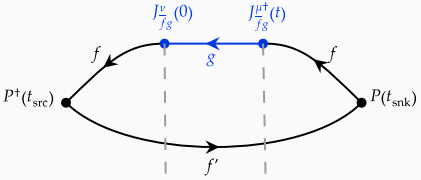
$$C(t) = \frac{\int \mathcal{D}[\bar{\psi}\psi U] e^{-S_{\mathrm{QCD}}[\bar{\psi}\psi U]} O(t)}{\int \mathcal{D}[\bar{\psi}\psi U] e^{-S_{\mathrm{QCD}}[\bar{\psi}\psi U]}}$$

$t = it_{\mathrm{M}}$ **Euclidean time**

Why so difficult?

- The hadronic tensor is a **spectral density** associated with a correlation function

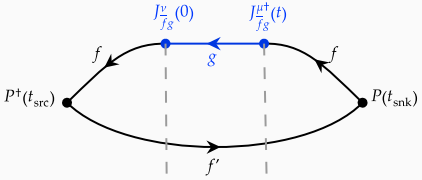
$$C^{\mu\nu}(t, \boldsymbol{\omega}) = \int_0^\infty \mathrm{d}\omega_0 \; e^{-(m_H \omega_0)t} \; H^{\mu\nu}(\omega_0, \boldsymbol{\omega})$$



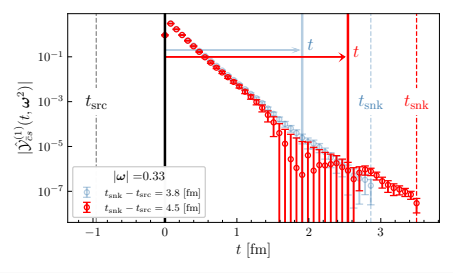
Why so difficult?

- The hadronic tensor is a **spectral density** associated with a correlation function

$$C^{\mu\nu}(t,\boldsymbol{\omega}) = \int_0^\infty \mathrm{d}\omega_0 \; e^{-(m_H\omega_0)t} \; H^{\mu\nu}(\omega_0,\boldsymbol{\omega})$$

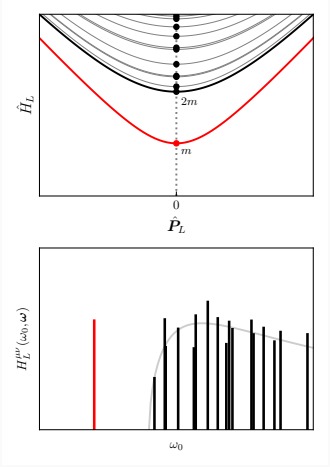


- The extraction of spectral densities from a Euclidean correlators is a **numerically ill-posed** **inverse Laplace transform**

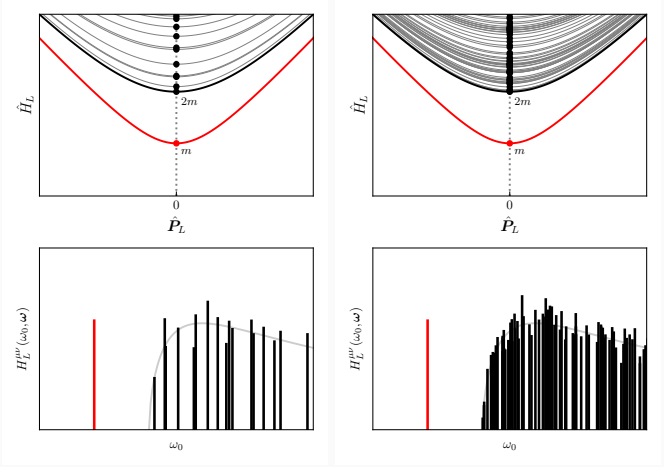


Hashimoto (2017) - Hansen, Meyer, Robaina (2017)

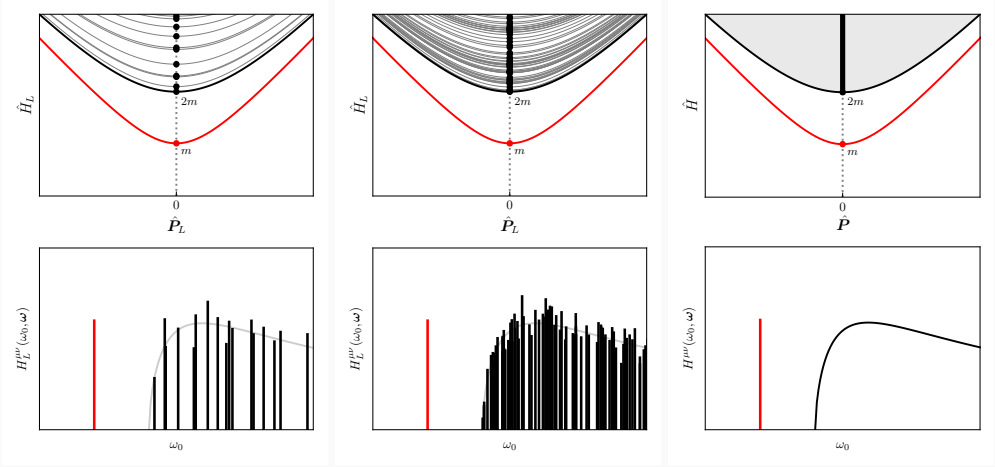
On top of that ...



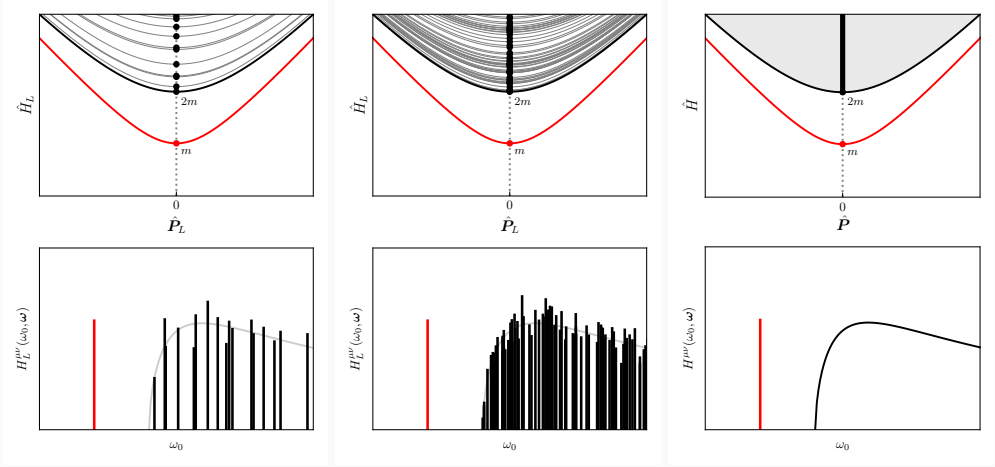
On top of that ...



On top of that ...



On top of that ...



Spectral densities (in finite volume) are (badly) behaving distributions, they **must be smeared**!

Smeared spectral functions

Choose as **smearing kernel** a resolution function

$$\lim_{\sigma \mapsto 0} \delta_\sigma(\bar{\omega}_0, \omega_0) = \delta(\bar{\omega}_0 - \omega_0)$$

$$H_L^{\mu\nu}[\delta](\omega_0, \sigma) = \int_0^\infty \mathrm{d}\bar{\omega}_0 \; \delta_\sigma(\bar{\omega}_0, \omega_0) \; H_L^{\mu\nu}(\bar{\omega}_0)$$

Smeared spectral functions

Choose as **smearing kernel** a resolution function

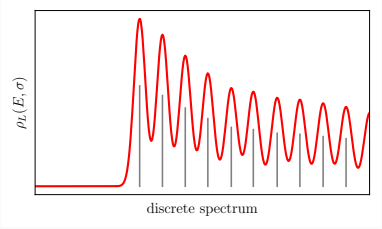
$$\lim_{\sigma \mapsto 0} \delta_\sigma(\bar{\omega}_0, \omega_0) = \delta(\bar{\omega}_0 - \omega_0)$$

$$H_L^{\mu\nu}[\delta](\omega_0, \sigma) = \int_0^\infty \mathrm{d}\bar{\omega}_0 \; \delta_\sigma(\bar{\omega}_0, \omega_0) \; H_L^{\mu\nu}(\bar{\omega}_0)$$

Then we can extract the spectral density

$$H^{\mu\nu}(\omega_0) = \lim_{\sigma \mapsto 0} \lim_{L \mapsto \infty} H_L^{\mu\nu}[\delta](\omega_0, \sigma)$$

and compute the decay rate or, more in general, S-matrix amplitudes!



Smeared spectral functions

Choose as **smearing kernel** a resolution function

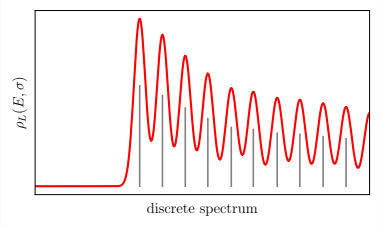
$$\lim_{\sigma \mapsto 0} \delta_\sigma(\bar{\omega}_0, \omega_0) = \delta(\bar{\omega}_0 - \omega_0)$$

$$H_L^{\mu\nu}[\delta](\omega_0, \sigma) = \int_0^\infty \mathrm{d}\bar{\omega}_0 \; \delta_\sigma(\bar{\omega}_0, \omega_0) \; H_L^{\mu\nu}(\bar{\omega}_0)$$

Then we can extract the spectral density

$$H^{\mu\nu}(\omega_0) = \lim_{\sigma \mapsto 0} \lim_{L \mapsto \infty} H_L^{\mu\nu}[\delta](\omega_0, \sigma)$$

and compute the decay rate or, more in general, S-matrix amplitudes!



In realistic LQCD getting $H^{\mu\nu}(\omega_0)$ (resonances, non-smooth thresholds ecc.) is challenging

Smearing? yes, but with a phase-space smearing kernel!

The exact spectral density is not really needed ...

$$\frac{\Gamma}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\rm max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\rm min}}^{\omega_0^{\rm max}} \mathrm{d}\omega_0 \; (\omega_0^{\rm max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

Smearing? yes, but with a phase-space smearing kernel!

The exact spectral density is not really needed ...

$$\frac{\Gamma}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\rm max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\rm min}}^{\omega_0^{\rm max}} \mathrm{d}\omega_0 \; (\omega_0^{\rm max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

- $\displaystyle\int_0^{|\boldsymbol{\omega}|^2_{\rm max}} \mathrm{d}\boldsymbol{\omega}^2$ done numerically by generating data for several $\boldsymbol{\omega} = \frac{\pi \boldsymbol{\theta}}{L}$

Smearing? yes, but with a phase-space smearing kernel!

The exact spectral density is not really needed ...

$$\frac{\Gamma}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\rm max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\rm min}}^{\omega_0^{\rm max}} \mathrm{d}\omega_0 \; (\omega_0^{\rm max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

- $\displaystyle\int_0^{|\boldsymbol{\omega}|^2_{\rm max}} \mathrm{d}\boldsymbol{\omega}^2$ done numerically by generating data for several $\boldsymbol{\omega} = \frac{\pi \boldsymbol{\theta}}{L}$
- $(\omega_0^{\rm max} - \omega_0)^p$ comes from phase-space kinematics

Smearing? yes, but with a phase-space smearing kernel!

The exact spectral density is not really needed ...

$$\frac{\Gamma}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\min}}^{\omega_0^{\max}} \mathrm{d}\omega_0 \; (\omega_0^{\max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

- $\displaystyle\int_0^{|\boldsymbol{\omega}|^2_{\max}} \mathrm{d}\boldsymbol{\omega}^2$ done numerically by generating data for several $\boldsymbol{\omega} = \frac{\pi \boldsymbol{\theta}}{L}$
- $(\omega_0^{\max} - \omega_0)^p$ comes from phase-space kinematics
- $Z^{(p)}(\omega_0, \boldsymbol{\omega}^2)$ are linear combinations of the initial hadronic tensor $H^{\mu\nu}$

Smearing? yes, but with a smearing kernel!

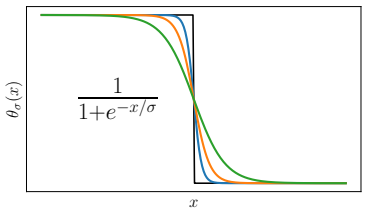
$$\frac{\Gamma(\sigma)}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \theta_\sigma(\omega_0^{\max} - \omega_0)(\omega_0^{\max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

Smearing? yes, but with a smearing kernel!

$$\frac{\Gamma(\sigma)}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \theta_\sigma(\omega_0^{\max} - \omega_0)(\omega_0^{\max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

$$\lim_{\sigma \mapsto 0} \theta_\sigma(x) = \theta(x)$$

- $\sigma > 0$ is still needed for mathematical and numerical purposes
- $\Gamma = \lim_{\sigma \mapsto 0} \lim_{L \mapsto \infty} \Gamma(\sigma)$

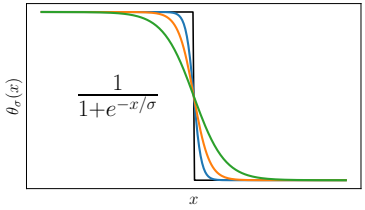


Smearing? yes, but with a smearing kernel!

$$\frac{\Gamma(\sigma)}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \theta_\sigma(\omega_0^{\max} - \omega_0)(\omega_0^{\max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

$$\lim_{\sigma \mapsto 0} \theta_\sigma(x) = \theta(x)$$

- $\sigma > 0$ is still needed for mathematical and numerical purposes
- $\Gamma = \lim_{\sigma \mapsto 0} \lim_{L \mapsto \infty} \Gamma(\sigma)$



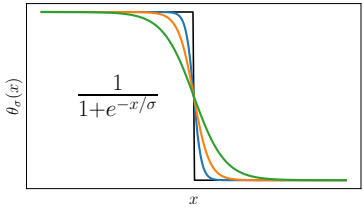
All we need is to compute the convolution of a spectral function with a **smooth smearing kernel**

Smearing? yes, but with a smearing kernel!

$$\frac{\Gamma(\sigma)}{|V_{fg}|^2} = \frac{m_H^5 G_F^2}{48\pi^4} \int_0^{|\boldsymbol{\omega}|^2_{\max}} \mathrm{d}\boldsymbol{\omega}^2 \left[\sum_{p=0}^{2} |\boldsymbol{\omega}|^{3-p} \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \theta_\sigma(\omega_0^{\max} - \omega_0)(\omega_0^{\max} - \omega_0)^p \; Z^{(p)}(\omega_0, \boldsymbol{\omega}^2) \right]$$

$$\lim_{\sigma \mapsto 0} \theta_\sigma(x) = \theta(x)$$

- $\sigma > 0$ is still needed for mathematical and numerical purposes
- $\Gamma = \lim_{\sigma \mapsto 0} \lim_{L \mapsto \infty} \Gamma(\sigma)$



All we need is to compute the convolution of a spectral function with a **smooth smearing kernel**

new methods!

Methods and pilot studies

Methods on the market

- Methods that provide regularization against noise and systematic error estimation
  - HLT Hansen, Lupo, Tantalo (2019) based on Backus, Gilbert (1968) - Pijpers, Thompson (1992)
  - Chebyshev polynomial approximation Bailas, Hashimoto, Ishikawa (2020) - Barata, Fredenhagen (1991)

- Other methods that address the extraction of smeared/unsmeared spectral functions
  - Bayesian inference Asakawa, Hatsuda, Nakahara (2000) - Burniern, Rothkopf (2013)
  - Machine Learning Fournier et al. (2018) - Kades et al. (2019) - Buzzicotti, De Santis, Tantalo (2023)
  - Gaussian processes Horak et al. (2021) (equivalent to HLT Del Debbio et. al (2024))
  - Others Bergamaschi, Jay, Oare (2023) - Bruno, Giusti, Saccardi (2024)

HLT

Hansen, Lupo, Tantalo (2019)

How to get the smeared spectral function from the Euclidean correlator?

$$\rho[K] = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; K(\omega_0) \, \rho(\omega_0)$$

$$C(an) = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; e^{-\omega_0 an} \, \rho(\omega_0) \qquad t = an$$

How to get the smeared spectral function from the Euclidean correlator?

$$\rho[K] = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; K(\omega_0) \; \rho(\omega_0)$$

$$C(an) = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; e^{-\omega_0 an} \; \rho(\omega_0) \qquad t = an$$

The **smooth kernel** can be exactly represented on the basis $e^{-\omega_0 t}$

$$K(\omega_0) = \lim_{N \mapsto \infty} \sum_{n=1}^{N} g_n(N) \; e^{-\omega_0 an} \qquad \Longrightarrow \qquad \rho[K] = \lim_{N \mapsto \infty} g_n(N) C(an)$$

How to get the smeared spectral function from the Euclidean correlator?

$$\rho[K] = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; K(\omega_0) \, \rho(\omega_0)$$

$$C(an) = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; e^{-\omega_0 an} \, \rho(\omega_0) \qquad t = an$$

The **smooth kernel** can be exactly represented on the basis $e^{-\omega_0 t}$

$$K(\omega_0) = \lim_{N \mapsto \infty} \sum_{n=1}^{N} g_n(N) \; e^{-\omega_0 an} \qquad \Longrightarrow \qquad \rho[K] = \lim_{N \mapsto \infty} g_n(N) C(an)$$

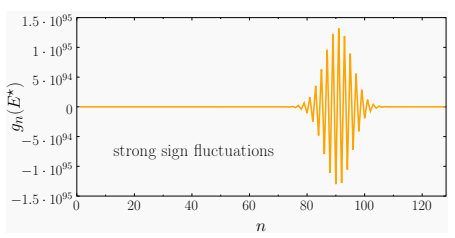
The number of time slices is limited by the size of the lattice $N \simeq \mathcal{O}(10 - 100) \mapsto$ **systematic error** induced by necessarily imperfect approximation of the kernel

The coefficients $\boldsymbol{g}$ are found by minimizing the weighted L_2-norm

$$A[\boldsymbol{g}] = \int_{\omega_0^{\min}-\epsilon}^{\infty} \mathrm{d}\omega_0 e^{\alpha\omega_0} \left[\sum_{n=1}^{N} g_n e^{-\omega_0 a n} - K(\omega_0) \right]^2$$

The coefficients $\boldsymbol{g}$ are found by minimizing the weighted L_2-norm

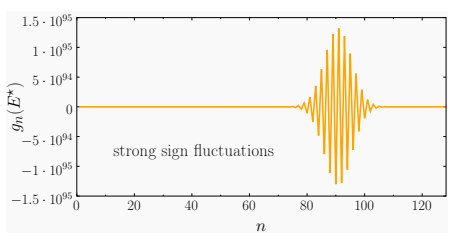
$$A[\boldsymbol{g}] = \int_{\omega_0^{\min}-\epsilon}^{\infty} \mathrm{d}\omega_0 e^{\alpha\omega_0} \left[\sum_{n=1}^{N} g_n e^{-\omega_0 a n} - K(\omega_0) \right]^2$$



Noise regulator from Backus, Gilbert (1968)

The coefficients $\boldsymbol{g}$ are found by minimizing the weighted L_2-norm

$$A[\boldsymbol{g}] = \int_{\omega_0^{\min}-\epsilon}^{\infty} \mathrm{d}\omega_0 e^{\alpha\omega_0} \left[\sum_{n=1}^{N} g_n e^{-\omega_0 a n} - K(\omega_0) \right]^2$$



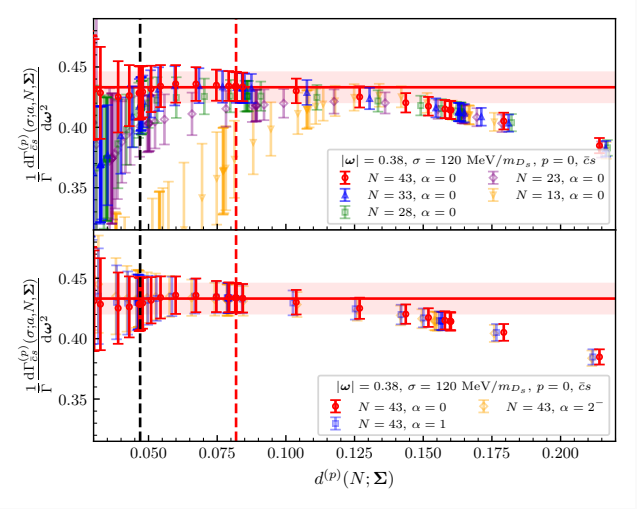
Add the noise regulator $B[\boldsymbol{g}] = \boldsymbol{g}^T \cdot \mathrm{Cov} \cdot \boldsymbol{g}$ and look for **optimal balance between statistical and systematic error**

$$W[\boldsymbol{g}] = (1-\lambda)A[\boldsymbol{g}] + \lambda B[\boldsymbol{g}]$$

Noise regulator from Backus, Gilbert (1968)

HLT: stability analysis

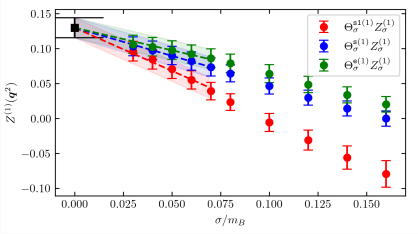
- The systematic error is gradually removed until reaching a plateau within statistical errors
- The saturation wrt $N \mapsto \infty$ is achieved
- Procedure tested in a controlled setting Bulava et al. (2021)

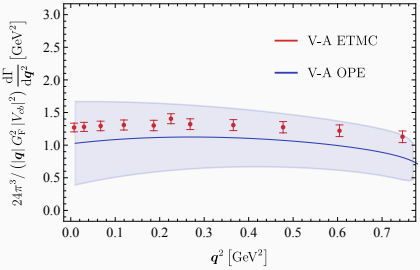


plot from De Santis et al. (2025)

Lattice QCD study of inclusive semileptonic decays of heavy mesons Gambino et al. (2022)

- $B_s \mapsto X_c \ell \bar{\nu}$ at **unphysical** pion ($M_\pi = 375$ MeV) and b ($M_{B_s} = 3.08$ GeV) **masses**
- **One lattice spacing** $a \simeq 0.081$ fm and **one (small) volume** $L \simeq 2.6$ fm (B32 ETMC ensemble)
- (wrong!) combined linear $\sigma \mapsto 0$ at fixed volume
- General good agreement with OPE





Chebyshev approach

Bailas, Hashimoto, Ishikawa (2020)

Chebyshev approach

$$\rho[K] = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; K(\omega_0) e^{\omega_0 a} \; e^{-\omega_0 a} \rho(\omega_0)$$

Chebyshev approach

$$\rho[K] = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; K(\omega_0) e^{\omega_0 a} \; e^{-\omega_0 a} \rho(\omega_0)$$

The smearing kernel is represented on the basis of **shifted Chebyshev polynomials**

$$K(\omega_0) e^{\omega_0 a} = \lim_{N \mapsto \infty} \sum_{n=0}^{N} c_n(N) \, \tilde{T}_n(\omega_0) = \lim_{N \mapsto \infty} \sum_{n=0}^{N} c_n(N) \sum_{k=0}^{n} \tilde{t}_k^{(n)} e^{-\omega_0 k a}$$

$$\rho[K] = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; K(\omega_0) e^{\omega_0 a} \; e^{-\omega_0 a} \rho(\omega_0)$$

The smearing kernel is represented on the basis of **shifted Chebyshev polynomials**

$$K(\omega_0) e^{\omega_0 a} = \lim_{N \mapsto \infty} \sum_{n=0}^{N} c_n(N) \, \tilde{T}_n(\omega_0) = \lim_{N \mapsto \infty} \sum_{n=0}^{N} c_n(N) \sum_{k=0}^{n} \tilde{t}_k^{(n)} e^{-\omega_0 k a}$$

Replacing

$$\rho[K] = C(a) \lim_{N \mapsto \infty} \sum_{n=0}^{N} c_n(N) \sum_{k=0}^{n} \tilde{t}_k^{(n)} \bar{C}(k) \qquad \bar{C}(k) \equiv \frac{C\big(a(k+1)\big)}{C(a)}$$

Chebyshev approach

$$X_n \equiv \sum_{k=0}^{n} \tilde{t}_k^{(n)} \bar{C}(k) = \frac{\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_n(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}}{\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_0(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}} \quad \Longrightarrow \quad |X_n| \leq 1$$

Chebyshev approach

$$X_n \equiv \sum_{k=0}^{n} \tilde{t}_k^{(n)} \bar{C}(k) = \frac{\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_n(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}}{\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_0(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}} \qquad \Longrightarrow \qquad |X_n| \leq 1$$

This mathematical bound can be violated when using $\bar{C}^{\mathrm{latt}}$ due to noise

Chebyshev approach

$$X_n \equiv \sum_{k=0}^{n} \tilde{t}_k^{(n)} \bar{C}(k) = \frac{\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_n(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}}{\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_0(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}} \qquad \Longrightarrow \qquad |X_n| \leq 1$$

This mathematical bound can be violated when using $\bar{C}^{\mathrm{latt}}$ due to noise

The strategy is to consider the inverse

$$\bar{C}(n) = \sum_{k=0}^{n} \tilde{a}_k^{(n)} X_k$$

Chebyshev approach

$$X_n \equiv \sum_{k=0}^{n} \tilde{t}_k^{(n)} \bar{C}(k) = \frac{\displaystyle\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_n(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}}{\displaystyle\int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \; \tilde{T}_0(\omega_0) \; \rho(\omega_0) e^{-\omega_0 a}} \qquad \Longrightarrow \qquad |X_n| \leq 1$$

This mathematical bound can be violated when using $\bar{C}^{\mathrm{latt}}$ due to noise

The strategy is to consider the inverse

$$\bar{C}(n) = \sum_{k=0}^{n} \tilde{a}_k^{(n)} X_k$$

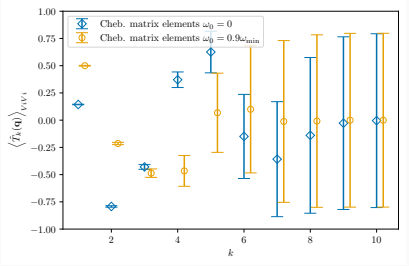
and determine X_k by fitting $\bar{C}^{\mathrm{latt}}$ while imposing the bound $|X_k| \leq 1$

$$\chi^2 = \sum_{i,j=1}^{N} \left(\bar{C}^{\mathrm{latt}}(i) - \sum_{k=1}^{i} \tilde{a}_k^{(i)} X_k \right) \mathrm{Cov}_{ij}^{-1} \left(\bar{C}^{\mathrm{latt}}(j) - \sum_{k=1}^{j} \tilde{a}_k^{(j)} X_k \right) + \chi^2_{\mathrm{priors}} \qquad \frac{\partial \chi^2}{\partial X_k} = 0 \implies X_k^{\mathrm{fit}}$$

Chebyshev approach: truncation error

$$\rho[K] = C(a) \lim_{N \mapsto \infty} \left[\sum_{n=0}^{N_{\mathrm{trunc}}} c_n(N) X_n^{\mathrm{fit}} + \sum_{n=N_{\mathrm{trunc}}+1}^{N} c_n(N) X_n \right]$$

X_n^{fit} is typically determined up to $N_{\mathrm{trunc}} \simeq 10$

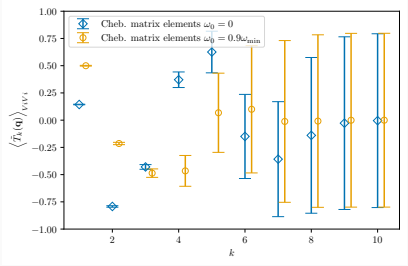


plots from Kellermann et al. (2025)

Chebyshev approach: truncation error

$$\rho[K] = C(a) \lim_{N \mapsto \infty} \left[\sum_{n=0}^{N_{\text{trunc}}} c_n(N) X_n^{\text{fit}} + \sum_{n=N_{\text{trunc}}+1}^{N} c_n(N) X_n \right]$$

X_n^{fit} is typically determined up to $N_{\text{trunc}} \simeq 10$



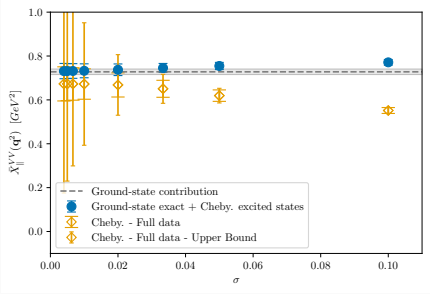
Truncation error

- mathematical upper bound

$$\sum_{n=N_{\text{trunc}}+1}^{\infty} c_n(N) X_n \leq \sum_{n=N_{\text{trunc}}+1}^{N_{\text{sat.}}} |c_n(N)|$$

- reasonable estimation

$$\sum_{n=N_{\text{trunc}}+1}^{\infty} c_n(N) X_n \simeq \sum_{n=N_{\text{trunc}}+1}^{N_{\text{sat.}}} c_n(N) \mathcal{N}(-1, 1)$$

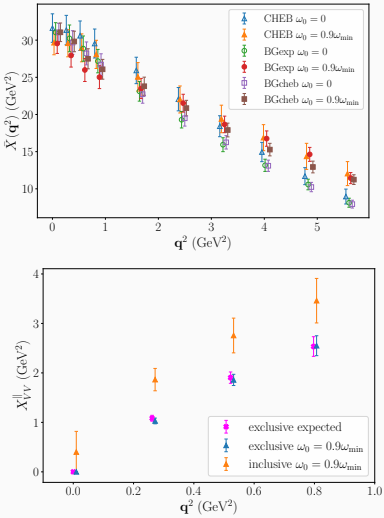


plots from Kellermann et al. (2025)

Chebyshev approximation: first application to semileptonic decays

Approaches to inclusive semileptonic $B_{(s)}$-meson decays from Lattice QCD Barone et al. (2023)

- $B_s \mapsto X_c \ell \bar{\nu}$ at **unphysical** pion ($M_\pi = 330$ MeV) but physical b (RHQ)
- **One lattice spacing** $a \simeq 0.11$ fm and **one (small) volume** $L \simeq 2.6$ fm (RBC/UKQCD ensemble)
- Systematic study of saturation wrt N_{trunc} and **comparison with HLT**
- Strong testing of the method by applying it to the exclusive channel



Physics and phenomenology: $D_s \mapsto X\ell\bar{\nu}$

The latest application

$$\Gamma = |V_{cs}|^2\Gamma_{cs} + |V_{cd}|^2\Gamma_{cd} + |V_{us}|^2\Gamma_{us}$$

$$\frac{\mathrm{d}M_n}{\mathrm{d}\boldsymbol{\omega}^2\mathrm{d}\omega_0\mathrm{d}e_\ell} = \frac{(m_{D_s}e_\ell)^n}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}\boldsymbol{\omega}^2\mathrm{d}\omega_0\mathrm{d}e_\ell}$$

The latest application

$$\Gamma = |V_{cs}|^2\Gamma_{cs} + |V_{cd}|^2\Gamma_{cd} + |V_{us}|^2\Gamma_{us}$$

$$\frac{\mathrm{d}M_n}{\mathrm{d}\boldsymbol{\omega}^2\mathrm{d}\omega_0\mathrm{d}e_\ell} = \frac{(m_{D_s}e_\ell)^n}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}\boldsymbol{\omega}^2\mathrm{d}\omega_0\mathrm{d}e_\ell}$$

- *Inclusive semileptonic decays of the D_s meson: A first-principles lattice QCD calculation* De Santis et al. (08/04/2025) - *Inclusive semileptonic decays of the D_s meson: Lattice QCD confronts experiments* De Santis et al. (08/04/2025)
  - HLT
  - All contributions evaluated at physical point + lepton-energy moments
  - $L \mapsto \infty$, $a \mapsto 0$, $\sigma \mapsto 0$ data-driven
  - bootstrap procedure to account for correlations
  - Confront with experiments BES-III, CLEO

- *Inclusive semileptonic decays from lattice QCD: analysis of systematic effects* Kellermann et. al (04/04/2025)
  - Chebyshev approach
  - Only $\bar{c}s$ channel at unphysical pion mass
  - Decomposition into $\Gamma_{||}^{VV}$, $\Gamma_{\perp}^{VV}$, $\Gamma_{||}^{AA}$, $\Gamma_{\perp}^{AA}$
  - Model-based study of $L \mapsto \infty$ effects
  - Extraction and subtraction of ground state

Relevant spectral functions

$$\frac{1}{|V_{fg}|^2}\frac{\mathrm{d}\Gamma_{fg}(\sigma)}{\mathrm{d}\boldsymbol{\omega}^2} = \bar{\Gamma}\sum_{p=0}^{2}|\boldsymbol{\omega}|^{3-p}\int_{\omega_0^{\min}}^{\infty}\mathrm{d}\omega_0\;\theta_\sigma(\omega_0^{\max}-\omega_0)(\omega_0^{\max}-\omega_0)^p\;Z^{(p)}(\omega_0,\boldsymbol{\omega}^2)$$

$$\bar{\Gamma} = \frac{m_{D_s}^5 G_F^2 S_{\mathrm{EW}}}{48\pi^4}$$

$$Z^{(0)} = \mathcal{Y}^{(2)} + \mathcal{Y}^{(3)} - 2\mathcal{Y}^{(4)}$$

$$Z^{(1)} = 2\big(\mathcal{Y}^{(3)} - 2\mathcal{Y}^{(1)} - \mathcal{Y}^{(4)}\big)$$

$$Z^{(2)} = \mathcal{Y}^{(3)} - 2\mathcal{Y}^{(1)}$$

$$\mathcal{Y}^{(1)} = -\frac{m_{D_s}}{2}\sum_{r=1}^{2}\sum_{i,j=1}^{3}\hat{n}_r^i\hat{n}_r^j H^{ij}$$

$$\mathcal{Y}^{(2)} = m_{D_s}H^{00}$$

$$\mathcal{Y}^{(3)} = m_{D_s}\sum_{i,j=1}^{3}\hat{\omega}^i\hat{\omega}^j H^{ij}$$

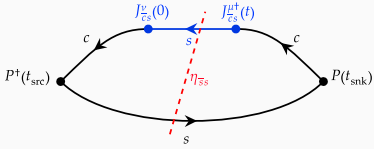
$$\mathcal{Y}^{(4)} = -m_{D_s}\sum_{i=1}^{3}\hat{\omega}^i H^{0i}$$

$$\mathcal{Y}^{(5)} = -\frac{im_{D_s}}{2}\sum_{i,j,k=1}^{3}\varepsilon^{ijk}\hat{\omega}^k H^{ij}$$

The Euclidean correlation functions

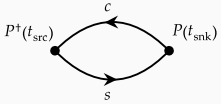
- 4-point function

$$C^{\mathrm{4pt}}_{\mu\nu}(t_{\mathrm{snk}},t,t_{\mathrm{src}},\boldsymbol{\omega}^2) = \sum_{\boldsymbol{x}_{\mathrm{src}},\boldsymbol{x}_{\mathrm{snk}},\boldsymbol{x}} e^{im_{D_s}\boldsymbol{\omega}\cdot\boldsymbol{x}} \times T\,\langle 0|\,P(x_{\mathrm{snk}})J^\dagger_\mu(x)J_\nu(0)P^\dagger(x_{\mathrm{src}})\,|0\rangle$$



- 2-point function

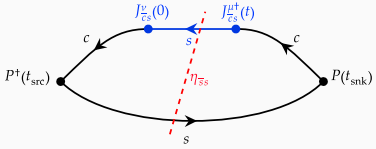
$$C^{\mathrm{2pt}}(t) = \sum_{\boldsymbol{x}} T\,\langle 0|\,P(t,\boldsymbol{x})P^\dagger(0)\,|0\rangle$$



The Euclidean correlation functions

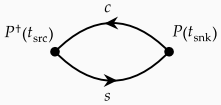
- 4-point function

$$C^{4\mathrm{pt}}_{\mu\nu}(t_{\mathrm{snk}},t,t_{\mathrm{src}},\boldsymbol{\omega}^2) = \sum_{\boldsymbol{x}_{\mathrm{src}},\boldsymbol{x}_{\mathrm{snk}},\boldsymbol{x}} e^{im_{D_s}\boldsymbol{\omega}\cdot\boldsymbol{x}} \times T\,\langle 0|\, P(x_{\mathrm{snk}})J^\dagger_\mu(x)J_\nu(0)P^\dagger(x_{\mathrm{src}})\,|0\rangle$$



- 2-point function

$$C^{2\mathrm{pt}}(t) = \sum_{\boldsymbol{x}} T\,\langle 0|\, P(t,\boldsymbol{x})P^\dagger(0)\,|0\rangle$$



$$\hat{\mathcal{Y}}_{\mu\nu}(t,\boldsymbol{\omega}^2) = \lim_{\substack{t_{\mathrm{snk}}\mapsto+\infty \\ t_{\mathrm{src}}\mapsto-\infty}} \frac{C^{4\mathrm{pt}}_{\mu\nu}(t_{\mathrm{snk}},t,t_{\mathrm{src}},\boldsymbol{\omega}^2)}{C^{2\mathrm{pt}}(t_{\mathrm{snk}}-t)C^{2\mathrm{pt}}(0-t_{\mathrm{src}})}$$

The Euclidean correlation functions

- 4-point function

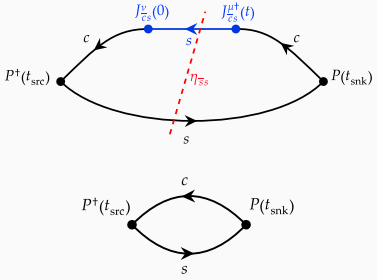
$$C^{\mathrm{4pt}}_{\mu\nu}(t_{\mathrm{snk}},t,t_{\mathrm{src}},\boldsymbol{\omega}^2) = \sum_{\boldsymbol{x}_{\mathrm{src}},\boldsymbol{x}_{\mathrm{snk}},\boldsymbol{x}} e^{im_{D_s}\boldsymbol{\omega}\cdot\boldsymbol{x}} \times T\,\langle 0|\,P(x_{\mathrm{snk}})J^{\dagger}_{\mu}(x)J_{\nu}(0)P^{\dagger}(x_{\mathrm{src}})\,|0\rangle$$

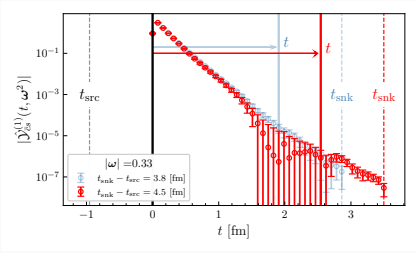
- 2-point function

$$C^{\mathrm{2pt}}(t) = \sum_{\boldsymbol{x}} T\,\langle 0|\,P(t,\boldsymbol{x})P^{\dagger}(0)\,|0\rangle$$

$$\hat{\mathcal{Y}}_{\mu\nu}(t,\boldsymbol{\omega}^2) = \lim_{\substack{t_{\mathrm{snk}}\mapsto+\infty \\ t_{\mathrm{src}}\mapsto-\infty}} \frac{C^{\mathrm{4pt}}_{\mu\nu}(t_{\mathrm{snk}},t,t_{\mathrm{src}},\boldsymbol{\omega}^2)}{C^{\mathrm{2pt}}(t_{\mathrm{snk}}-t)C^{\mathrm{2pt}}(0-t_{\mathrm{src}})}$$

$$\hat{\mathcal{Y}}_{\mu\nu}(t,\boldsymbol{\omega}^2) = \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 e^{-\omega_0 m_{D_s} t} H_{\mu\nu}(\omega_0,\boldsymbol{\omega}^2)$$





ETMC ensembles

Simulated on ensembles produced by Extended Twisted Mass Collaboration

ID	$L^3 \times T$	a [fm]	$1/a$ [GeV]	L [fm]	m_π [MeV]
B48	$48^3 \times 96$	0.080	2.5	3.82	140
B64	$64^3 \times 128$	0.080	2.5	5.09	140
B96	$96^3 \times 192$	0.080	2.5	7.63	140
C80	$80^3 \times 160$	0.068	2.9	5.46	137
D96	$96^3 \times 192$	0.057	3.5	5.46	141
E112	$112^3 \times 224$	0.049	4.0	5.48	136

- Finite Volume Effects (FVE), $L \mapsto \infty$
- Continuum limit, $a \mapsto 0$
- Tuned very close to real world

Final hadron phase-space

$$\omega_X = m_{D_s}(\omega_0, \boldsymbol{\omega})$$

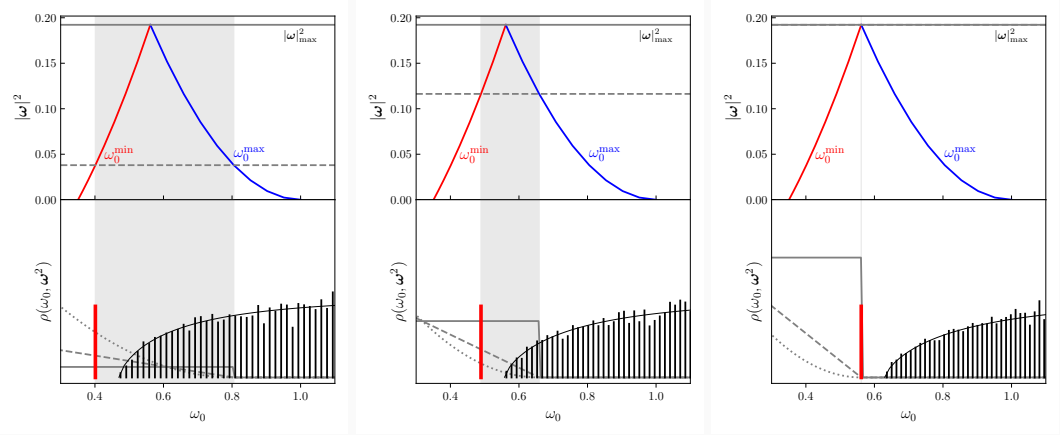
$$|\boldsymbol{\omega}|^{\min} = 0 \qquad \omega_0^{\min} = \sqrt{r^2 + |\boldsymbol{\omega}|^2}$$

$$|\boldsymbol{\omega}|^{\max} = \frac{1 - r^2}{2} \qquad \omega_0^{\max} = 1 - |\boldsymbol{\omega}|$$

$\bar{f}g$	contraction	Ground State	$m_{\rm GS}$ [MeV]	$r = \frac{m_{\rm GS}}{m_{D_s}}$	$\lvert\boldsymbol{\omega}\rvert^{\max}$
$\bar{c}s$	conn.	$\eta_{\bar{s}s}$	700	0.36	0.44
$\bar{c}s$	disconn.	π	135	0.07	0.50
$\bar{c}d$	conn.	K	500	0.25	0.47
$\bar{u}s$	conn.	D	1860	0.94	0.05

- $m_{D_s} < m_D + m_\pi$ $\quad\Longrightarrow\quad$ the inclusive $\bar{u}s$ channel coincides with the exclusive $D_s \mapsto D\ell\bar{\nu}$

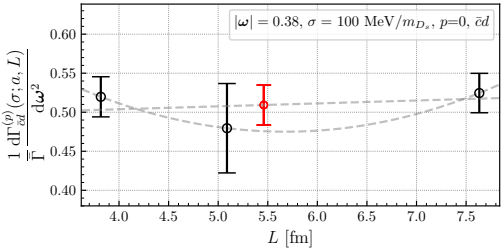
Final hadron phase-space



As $|\boldsymbol{\omega}| \mapsto |\boldsymbol{\omega}|_{\max}$ only the ground state contributes to the decay rate

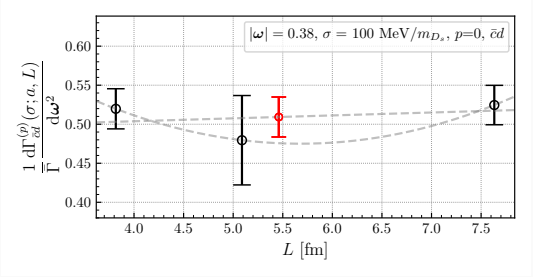
De Santis et al. (2025) - data-driven

- Data-driven estimation using 3 volumes
- FVEs are found subdominant wrt statistical errors
- FVEs expected $\mathcal{O}\left(e^{-\sigma L}\right)$



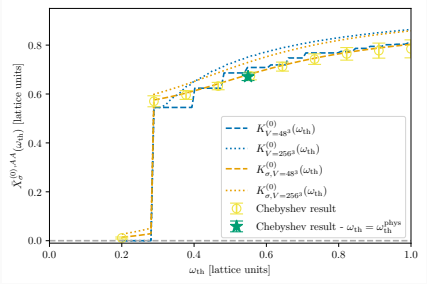
De Santis et al. (2025) - data-driven

- Data-driven estimation using 3 volumes
- FVEs are found subdominant wrt statistical errors
- FVEs expected $\mathcal{O}\left(e^{-\sigma L}\right)$



Kellermann et al. (2025) - model-based

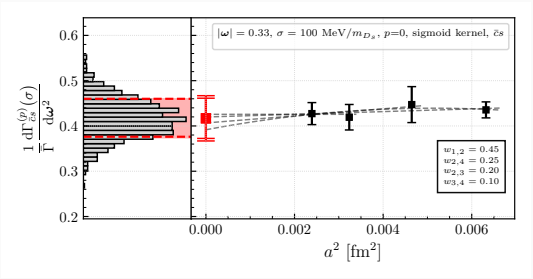
- model based on $D_s \mapsto K\bar{K}\ell\bar{\nu}$ in finite volume
- FVEs are predicted to be small
- Could be a good idea to check $\mathcal{O}\left(e^{-\sigma L}\right)$?

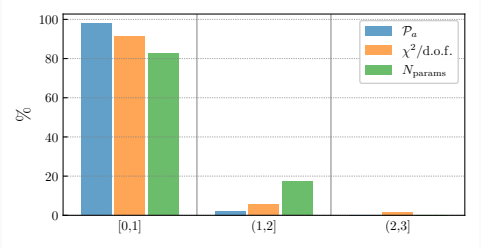


- several fit Ansaetze combine with BAIC Model Average

$$w_k \propto \exp\left[-(\chi^2_k + 2N^k_{\text{params}} - N^k_{\text{points}})/2\right]$$

- $\chi^2/\text{dof} < 1$ in most of the cases and very mild dependence on a
- cut-off effects under control!



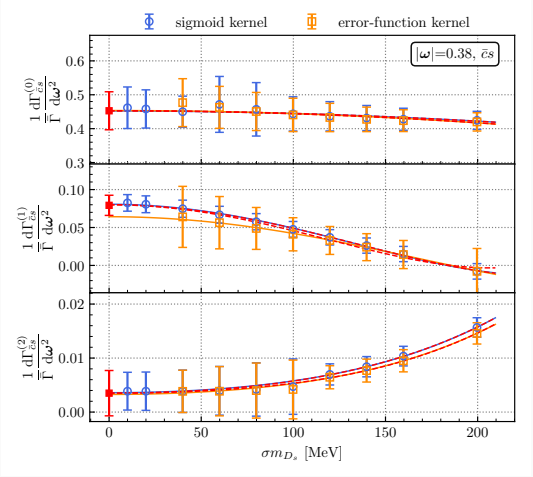


- **Two smoothed version of** $\theta(x)$ are used to control the $\sigma \mapsto 0$ extrapolation
- Theoretical ansatz based on **asymptotic expansion of** $x^p\theta_\sigma(x)$ **for** $\sigma \simeq 0$

$$\frac{\mathrm{d}\Gamma^{(p=0,1)}(\sigma)}{\mathrm{d}\boldsymbol{\omega}^2} = \frac{\mathrm{d}\Gamma^{(p=0,1)}}{\mathrm{d}\boldsymbol{\omega}^2} + \mathcal{O}(\sigma^2)$$

$$\frac{\mathrm{d}\Gamma^{(p=2)}(\sigma)}{\mathrm{d}\boldsymbol{\omega}^2} = \frac{\mathrm{d}\Gamma^{(p=2)}}{\mathrm{d}\boldsymbol{\omega}^2} + \mathcal{O}(\sigma^4)$$

- Expected behaviour well reproduced by data, not linear at all!



$$\theta_\sigma(x) = \frac{1}{1+e^{-x/\sigma}} \qquad \theta_\sigma(x) = \frac{1+\mathrm{erf}(x/s\sigma)}{2}$$

Asymptotic expansion firstly derived in Evangelista et al. (2023) based on Bulava et al. (2021)

The Ground State (GS) and sub-dominant contributions

The Ground State (GS) contribution

Why?

- **Phenomenological implications**
  - Comparison with exclusive channels measured in experiments
  - How much does it contribute to the total inclusive rate? Is $\text{INCLUSIVE} \simeq \text{EXCLUSIVE}$ a good approximation?

The Ground State (GS) contribution

Why?

- **Phenomenological implications**
  - Comparison with exclusive channels measured in experiments
  - How much does it contribute to the total inclusive rate? Is $\text{INCLUSIVE} \simeq \text{EXCLUSIVE}$ a good approximation?

- **Practical implications**
  - The exclusive contribution can be calculated avoiding the $\sigma \mapsto 0$ limit

$$\frac{\mathrm{d}\Gamma^{\text{excl.}}_{\bar{f}g}}{\mathrm{d}\boldsymbol{\omega}^2} = \frac{m_{D_s}^5}{24\pi^3}\frac{1}{\omega_0^{\min}}|\boldsymbol{\omega}|^3 \left| f^+_{\bar{f}g}(\boldsymbol{\omega}^2) \right|^2$$

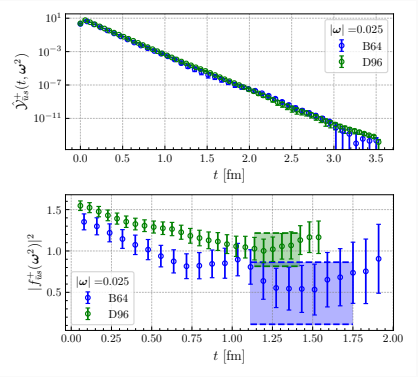
$$\langle \mathrm{GS}| J^\mu_{\bar{f}g}(0) |D_s\rangle = (\omega_0^{\min} + p)^\mu \, f^+_{\bar{f}g}(\boldsymbol{\omega}^2) + (\omega_0^{\min} - p)^\mu f^-_{\bar{f}g}(\boldsymbol{\omega}^2)$$

  - Subtract it and apply the methods only to the excited states to reduce the systematic errors

Extraction of the form factor

1) Extraction from the asymptotic behaviour of 4-point function made of $\hat{\mathcal{Y}}^{(i)}$

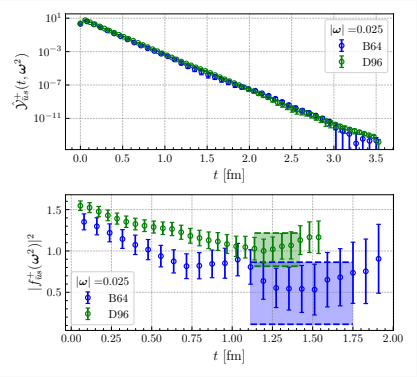
$$\hat{\mathcal{Y}}^{+}(t,\boldsymbol{\omega}^2) = \frac{2\pi}{\omega_0^{\min}} |f^{+}(\boldsymbol{\omega}^2)|^2 e^{-\omega_0^{\min} t} + \cdots$$



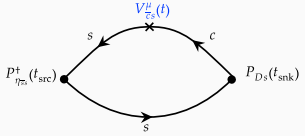
Extraction of the form factor

1) Extraction from the asymptotic behaviour of 4-point function made of $\hat{\mathcal{Y}}^{(i)}$

$$\hat{\mathcal{Y}}^{+}(t,\boldsymbol{\omega}^2) = \frac{2\pi}{\omega_0^{\min}} |f^{+}(\boldsymbol{\omega}^2)|^2 e^{-\omega_0^{\min} t} + \cdots$$



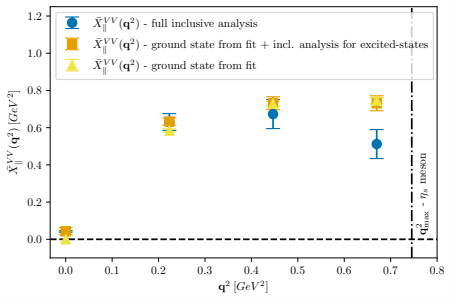
2) Extraction from explicit computation of 3-point functions associated with e.g. $D_s \mapsto \eta_{\bar{s}s} \ell \bar{\nu}$



Exclusive contribution to the decay rate

Kellermann et al. (2025)

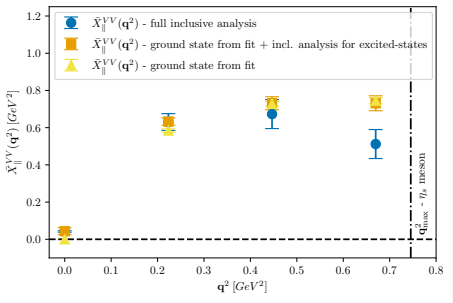
- Subtraction of ground state is beneficial to keep under control systematic associated with $\sigma \mapsto 0$
- The $\bar{c}s\ \Gamma_{||}^{VV}$ is almost entirely saturated by the exclusive channel
- What is the benefit if this is not the case?



Exclusive contribution to the decay rate

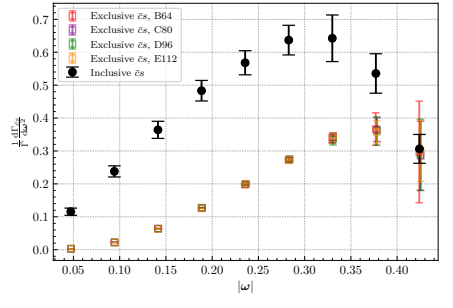
Kellermann et al. (2025)

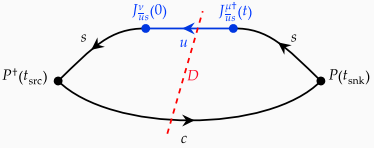
- Subtraction of ground state is beneficial to keep under control systematic associated with $\sigma \mapsto 0$
- The $\bar{c}s\ \Gamma_{||}^{VV}$ is almost entirely saturated by the exclusive channel
- What is the benefit if this is not the case?

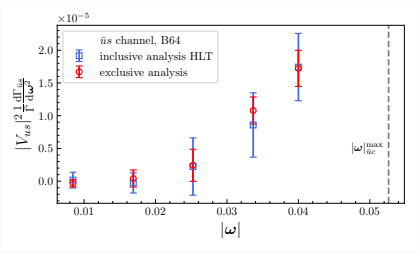


De Santis et al. (2025)

- $(V-A)\ \bar{c}s$ contribution not subtracted
- The exclusive contribution does not saturate the decay rate!
- Detailed study of ground state subtraction from 3- or 4-pt function in future

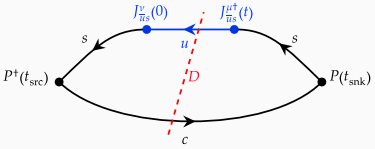




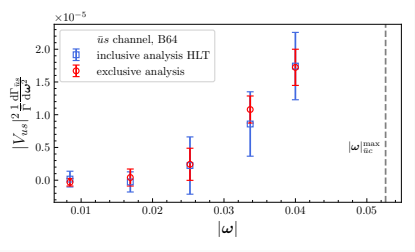


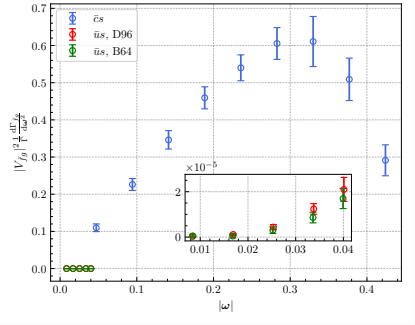
- Cross-check between exclusive analysis and inclusive HLT applied to exclusive-saturated channel

- Suppressed by small phase-space and $|V_{us}| \simeq 0.23$ vs $|V_{cs}| \simeq 0.97$

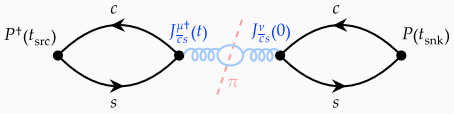


- Cross-check between exclusive analysis and inclusive HLT applied to exclusive-saturated channel
- Suppressed by small phase-space and $|V_{us}| \simeq 0.23$ vs $|V_{cs}| \simeq 0.97$

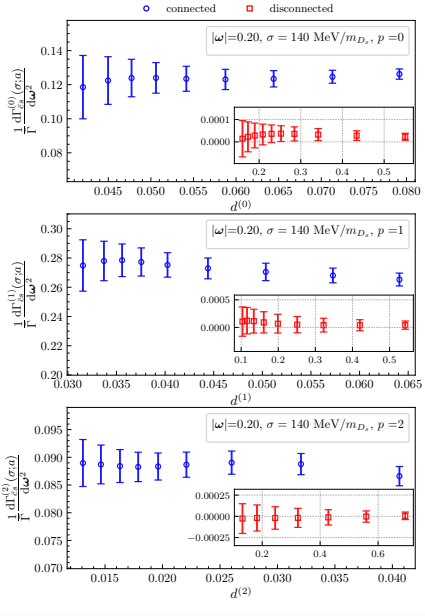




$\bar{c}s$ disconnected diagram (aka Weak-Annihilation)

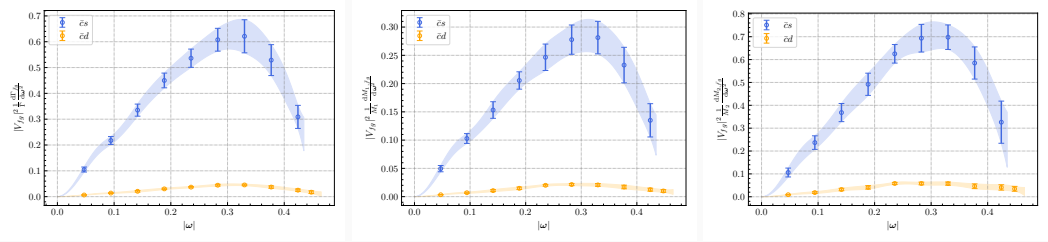


- Explicitly computed for $|\boldsymbol{\omega}| = 0.10$ and $|\boldsymbol{\omega}| = 0.20$ on B64 ensemble with good signal on $\hat{\mathcal{Y}}^{(i)}(t)$
- Decay rate compatible with 0 and negligible compared to $\bar{c}s$ channel
- This was an open question



Final results

Differential decay rate and first two lepton-energy moments



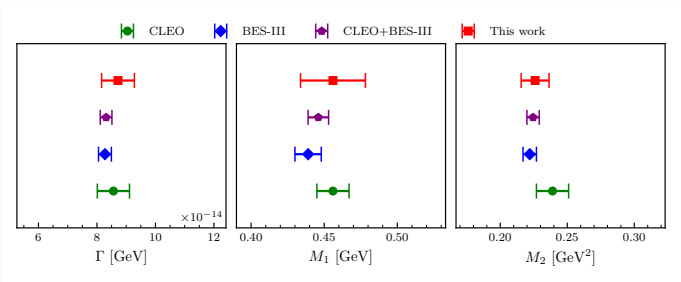
- PDF 2024 values $|V_{cs}| = 0.975(6)$ and $|V_{cd}| = 0.221(4)$)
- The $\bar{c}s$ dominates, $\bar{c}d$ channel is Cabibbo-suppressed

$\bar{f}g$	$\bar{c}s$	$\bar{c}d$
$10^{14} \times \Gamma_{\bar{f}g}$ [GeV]	8.53(55)	12.60(93)
$\Gamma M_{1,\bar{f}g}/\Gamma_{\bar{c}s}$ [GeV]	0.453(24)	0.731(61)
$\Gamma M_{2,\bar{f}g}/\Gamma_{\bar{c}s}$ [GeV^2]	0.223(11)	0.416(43)

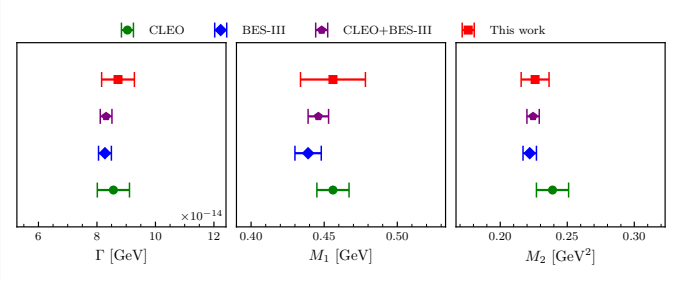
	$\Gamma_{\bar{c}s} \times 10^{14}$
$\bar{X}_{\parallel}^{VV}(q^2)$	4.95(16)
$\bar{X}_{\perp}^{VV}(q^2)$	0.162(87)
$\bar{X}_{\parallel}^{AA}(q^2)$	2.45(16)
$\bar{X}_{\perp}^{AA}(q^2)$	1.30(10)
total	8.86(27)

Kellermann et al. (2025)

Lattice QCD confronts experiments



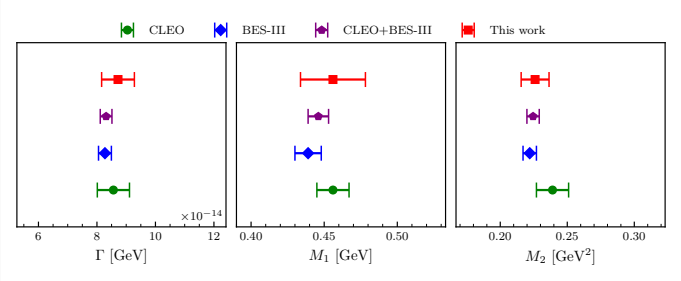
Lattice QCD confronts experiments



$$x = \frac{|V_{cd}|^2}{|V_{cs}|^2}$$

$$\begin{cases} M_1 = \dfrac{\dfrac{\Gamma M_{1,\bar{c}s}}{\Gamma_{\bar{c}s}} + x\,\dfrac{\Gamma M_{1,\bar{c}d}}{\Gamma_{\bar{c}s}}}{1 + x\,\dfrac{\Gamma_{\bar{c}d}}{\Gamma_{\bar{c}s}}} \\ M_2 = \dfrac{\dfrac{\Gamma M_{2,\bar{c}s}}{\Gamma_{\bar{c}s}} + x\,\dfrac{\Gamma M_{2,\bar{c}d}}{\Gamma_{\bar{c}s}}}{1 + x\,\dfrac{\Gamma_{\bar{c}d}}{\Gamma_{\bar{c}s}}} \end{cases}$$

Lattice QCD confronts experiments

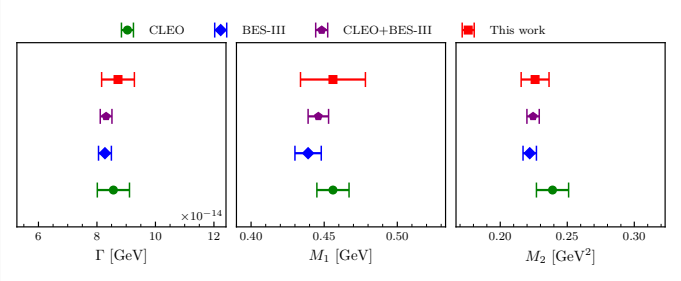


$$x = \frac{|V_{cd}|^2}{|V_{cs}|^2}$$

$$\begin{cases} M_1 = \dfrac{\dfrac{\Gamma M_{1,\bar{c}s}}{\Gamma_{\bar{c}s}} + x\,\dfrac{\Gamma M_{1,\bar{c}d}}{\Gamma_{\bar{c}s}}}{1 + x\,\dfrac{\Gamma_{\bar{c}d}}{\Gamma_{\bar{c}s}}} \\ M_2 = \dfrac{\dfrac{\Gamma M_{2,\bar{c}s}}{\Gamma_{\bar{c}s}} + x\,\dfrac{\Gamma M_{2,\bar{c}d}}{\Gamma_{\bar{c}s}}}{1 + x\,\dfrac{\Gamma_{\bar{c}d}}{\Gamma_{\bar{c}s}}} \end{cases}$$

- The CKM-independent $\Gamma M_{l,\bar{c}s}/\Gamma_{\bar{c}s}$ already agrees with $M_l^{\rm exp}$ $\Longrightarrow$ no chance to get x at better precision than 100%

Lattice QCD confronts experiments



$$x = \frac{|V_{cd}|^2}{|V_{cs}|^2}$$

$$\begin{cases} M_1 = \dfrac{\dfrac{\Gamma M_{1,\bar{c}s}}{\Gamma_{\bar{c}s}} + x\,\dfrac{\Gamma M_{1,\bar{c}d}}{\Gamma_{\bar{c}s}}}{1 + x\,\dfrac{\Gamma_{\bar{c}d}}{\Gamma_{\bar{c}s}}} \\ M_2 = \dfrac{\dfrac{\Gamma M_{2,\bar{c}s}}{\Gamma_{\bar{c}s}} + x\,\dfrac{\Gamma M_{2,\bar{c}d}}{\Gamma_{\bar{c}s}}}{1 + x\,\dfrac{\Gamma_{\bar{c}d}}{\Gamma_{\bar{c}s}}} \end{cases}$$

- The CKM-independent $\Gamma M_{l,\bar{c}s}/\Gamma_{\bar{c}s}$ already agrees with $M_l^{\rm exp} \implies$ no chance to get x at better precision than 100%
- From the decay rate

$$|V_{cs}|^{\rm incl.}_{\rm lqcd} = \sqrt{\frac{\Gamma^{\rm exp} - |V_{cd}|^2_{\rm pdg}\Gamma_{\bar{c}d}}{\Gamma_{\bar{c}s}}} = 0.951(35)(3.5\%)$$

Improvement ...

- Little to do with the methods

Improvement ...

- Little to do with the methods
- Lattice artifacts
  - less conservative analysis

Improvement ...

- Little to do with the methods
- Lattice artifacts
  - less conservative analysis
- Finite volume effects
  - theory/model dependence

Improvement ...

- Little to do with the methods
- Lattice artifacts
  - less conservative analysis
- Finite volume effects
  - theory/model dependence
- $\sigma \mapsto 0$
  - subtraction of ground state

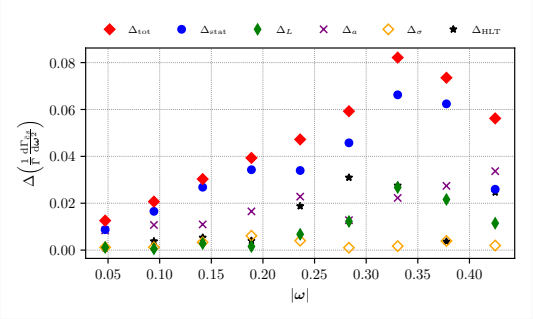
but what is the main source or error?

Improvement ...

- Little to do with the methods
- Lattice artifacts
  - less conservative analysis
- Finite volume effects
  - theory/model dependence
- $\sigma \mapsto 0$
  - subtraction of ground state

but what is the main source or error?

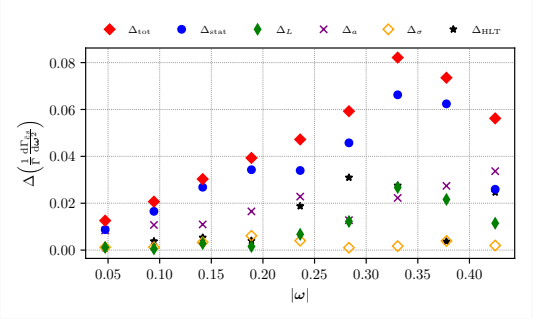
Statistical errors which can be easily improved!



Improvement ...

- Little to do with the methods
- Lattice artifacts
  - less conservative analysis
- Finite volume effects
  - theory/model dependence
- $\sigma \mapsto 0$
  - subtraction of ground state

but what is the main source or error?



Statistical errors which can be easily improved!

- To extend the calculation to B-mesons the main difficulty is $m_b \mapsto m_b^{\rm phys}$
- (work in progress) achievable through HQET, ratio methods or simulating the physical b, stay tuned!

Take-home messages

Inclusive semileptonic decays of heavy mesons are feasible nowadays thanks to **new reliable methods**

recent solid studies of $D_s \mapsto X\ell\bar{\nu}$ highlight that the **precision** achievable by the lattice is already **relevant for phenomenological comparisons**

Take-home messages

Inclusive semileptonic decays of heavy mesons are feasible nowadays thanks to **new reliable methods**

recent solid studies of $D_s \mapsto X\ell\bar{\nu}$ highlight that the **precision** achievable by the lattice is already **relevant for phenomenological comparisons**

this open to the real perspective of computing the inclusive $B_s \mapsto X_c\ell\bar{\nu}$ soon and shed light on the long-standing $|V_{cb}|$ puzzle

Take-home messages

Inclusive semileptonic decays of heavy mesons are feasible nowadays thanks to **new reliable methods**

recent solid studies of $D_s \mapsto X\ell\bar{\nu}$ highlight that the **precision** achievable by the lattice is already **relevant for phenomenological comparisons**

this open to the real perspective of computing the inclusive $B_s \mapsto X_c\ell\bar{\nu}$ soon and shed light on the long-standing $|V_{cb}|$ puzzle

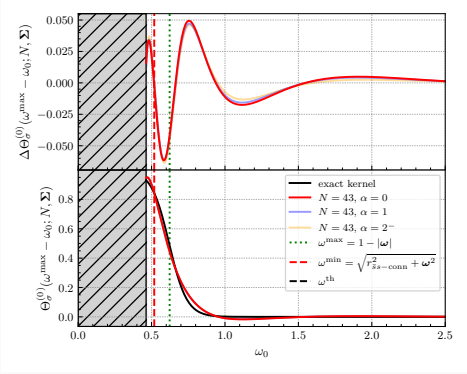
... but the era of inclusive processes on the lattice has already begun!

R-ratio (2022), τ decay (2023), τ decay (2024), ...

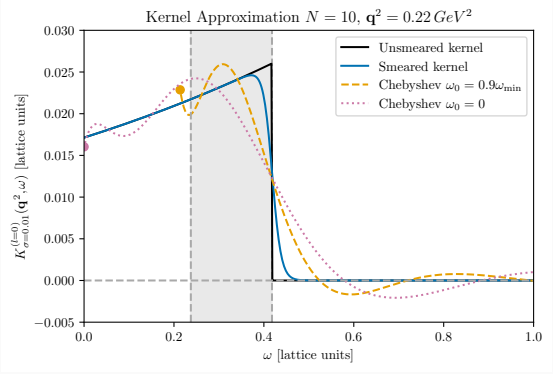
Thank you for the attention!

Backup slides

Kernel approximations



- HLT



- Chebyshev (additional factor $e^{\omega_0 a}$)

left: De Santis et al. (2025) - right: Kellermann et al. (2025)

Decomposition of the hadronic tensor

$$m_{D_s}^3 H^{\mu\nu}(p,\omega) = g^{\mu\nu} m_{D_s}^2 \; h^{(1)} + p^\mu p^\nu \; h^{(2)} + (p-\omega)^\mu (p-\omega)^\nu \; h^{(3)} + \left[p^\mu (p-\omega)^\nu + (p-\omega)^\mu p^\nu\right] h^{(4)} + i\epsilon^{\mu\nu\alpha\beta} p_\alpha (p-\omega)_\beta \; h^{(5)}$$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\boldsymbol{\omega}^2 \mathrm{d}\omega_0 de_\ell} = \frac{m_{D_s}^5 G_F^2 S_{\mathrm{EW}}}{32\pi^4} \Bigg(-2\left\{(1-\omega_0)^2 - \boldsymbol{\omega}^2\right\} h^{(1)} + \left\{\boldsymbol{\omega}^2 - (1-\omega_0-2e_\ell)^2\right\} h^{(2)} + 2\left[(1-\omega_0)^2 - \boldsymbol{\omega}^2\right][2e_\ell - (1-\omega_0)] \; h^{(5)} \Bigg)$$

$$h^{(1)} = \mathcal{Y}^{(1)}$$

$$h^{(2)} = \frac{(1-\omega_0)^2 - \boldsymbol{\omega}^2}{\boldsymbol{\omega}^2}\mathcal{Y}^{(1)} + \mathcal{Y}^{(2)} + \frac{(1-\omega_0)^2}{\boldsymbol{\omega}^2}\mathcal{Y}^{(3)} - \frac{2(1-\omega_0)}{|\boldsymbol{\omega}|}\mathcal{Y}^{(4)}$$

$$h^{(3)} = \frac{\mathcal{Y}^{(1)} + \mathcal{Y}^{(3)}}{\boldsymbol{\omega}^2}$$

$$h^{(4)} = -\frac{1-\omega_0}{\boldsymbol{\omega}^2}(\mathcal{Y}^{(1)} + \mathcal{Y}^{(3)}) + \frac{\mathcal{Y}^{(4)}}{|\boldsymbol{\omega}|}$$

$$h^{(5)} = \frac{\mathcal{Y}^{(5)}}{|\boldsymbol{\omega}|}$$

$$\mathcal{Y}^{(1)} = -\frac{m_{D_s}}{2}\sum_{r=1}^{2}\sum_{i,j=1}^{3} \hat{n}_r^i \hat{n}_r^j H^{ij}$$

$$\mathcal{Y}^{(2)} = m_{D_s} H^{00}$$

$$\mathcal{Y}^{(3)} = m_{D_s}\sum_{i,j=1}^{3} \hat{\omega}^i \hat{\omega}^j H^{ij}$$

$$\mathcal{Y}^{(4)} = -m_{D_s}\sum_{i=1}^{3} \hat{\omega}^i H^{0i}$$

$$\mathcal{Y}^{(5)} = -\frac{i m_{D_s}}{2}\sum_{i,j,k=1}^{3} \varepsilon^{ijk}\hat{\omega}^k H^{ij}$$

Lepton-energy moments

$$\frac{\mathrm{d}M_1^{(p)}(\sigma)}{\mathrm{d}\boldsymbol{\omega}^2} = \bar{M}_1 |\boldsymbol{\omega}|^{4-p} \times \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \theta_\sigma(\omega_0^{\max} - \omega_0)(\omega_0^{\max} - \omega_0)^p Z_1^{(p)}(\omega_0, \boldsymbol{\omega})^2$$

$$\bar{M}_1 = \frac{1}{\Gamma} \frac{m_{D_s}^6 G_F^2 S_{\mathrm{EW}}}{96\pi^4}$$

$$Z_1^{(0)} = \mathcal{Y}^{(2)} + \mathcal{Y}^{(3)} - 2\mathcal{Y}^{(4)}$$
$$Z_1^{(1)} = -4\mathcal{Y}^{(1)} + \mathcal{Y}^{(2)} + 3\mathcal{Y}^{(3)} - 4\mathcal{Y}^{(4)} + 2\mathcal{Y}^{(5)}$$
$$Z_1^{(2)} = -6\mathcal{Y}^{(1)} + 3\mathcal{Y}^{(3)} - 2\mathcal{Y}^{(4)} + \mathcal{Y}^{(5)}$$
$$Z_1^{(3)} = -2\mathcal{Y}^{(1)} + \mathcal{Y}^{(3)}$$

$$\frac{\mathrm{d}M_2^{(p)}(\sigma)}{\mathrm{d}\boldsymbol{\omega}^2} = \bar{M}_2 |\boldsymbol{\omega}|^{5-p} \times \int_{\omega_0^{\min}}^{\infty} \mathrm{d}\omega_0 \theta_\sigma(\omega_0^{\max} - \omega_0)(\omega_0^{\max} - \omega_0)^p Z_2^{(p)}(\omega_0, \boldsymbol{\omega})^2$$

$$\bar{M}_2 = \frac{1}{\Gamma} \frac{m_{D_s}^7 G_F^2 S_{\mathrm{EW}}}{960\pi^4}$$

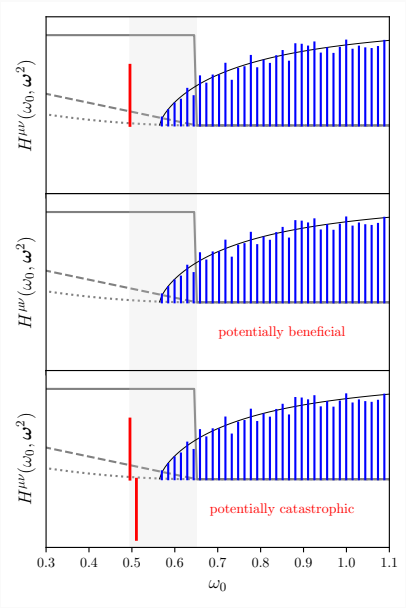
$$Z_2^{(0)} = 6(\mathcal{Y}^{(2)} + \mathcal{Y}^{(3)} - 2\mathcal{Y}^{(4)})$$
$$Z_2^{(1)} = 2(-14\mathcal{Y}^{(1)} + 5\mathcal{Y}^{(2)} + 11\mathcal{Y}^{(3)} - 16\mathcal{Y}^{(4)} + 10\mathcal{Y}^{(5)})$$
$$Z_2^{(2)} = -54\mathcal{Y}^{(1)} + 5\mathcal{Y}^{(2)} + 31\mathcal{Y}^{(3)} - 30\mathcal{Y}^{(4)} + 30\mathcal{Y}^{(5)}$$
$$Z_2^{(3)} = -10(4\mathcal{Y}^{(1)} - 2\mathcal{Y}^{(3)} + \mathcal{Y}^{(4)} - \mathcal{Y}^{(5)})$$
$$Z_2^{(4)} = 5(-2\mathcal{Y}^{(1)} + \mathcal{Y}^{(3)})$$

Subtraction of the Ground State (GS)

$$H^{\mu\nu}(\omega_0, \boldsymbol{\omega}^2) = \delta(\omega_0 - \omega_0^{\min})\, \rho_{\rm GS}^{\mu\nu}(\omega_0^{\min}, \boldsymbol{\omega}^2) + H_{\rm excit.}^{\mu\nu}(\omega_0 > \omega_0^{\min}, \boldsymbol{\omega}^2)$$

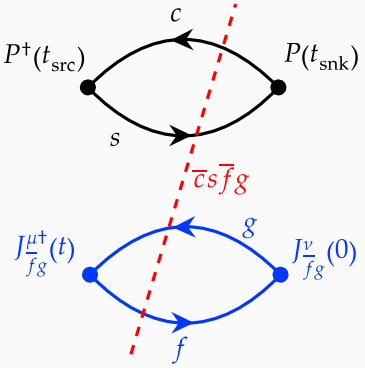
$$\rho_{\rm GS}^{\mu\nu} = \frac{\pi}{2m_{D_s}^3 \omega_0^{\min}} \langle D_s | J^{\mu\dagger}(0) | {\rm GS} \rangle \langle {\rm GS} | J^{\nu} | D_s \rangle$$

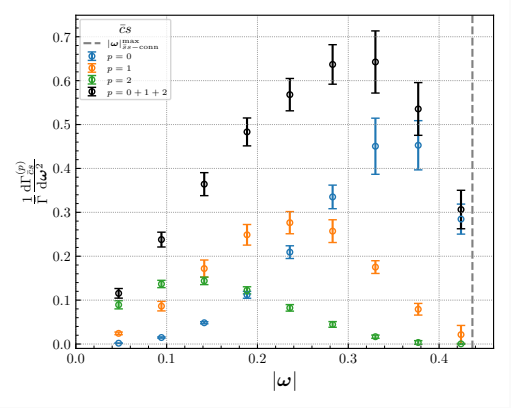
- The subtraction of states is a delicate business
- If not done properly it can lead to uncontrolled effects

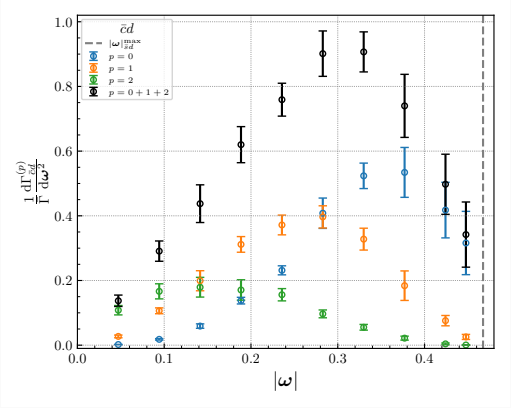


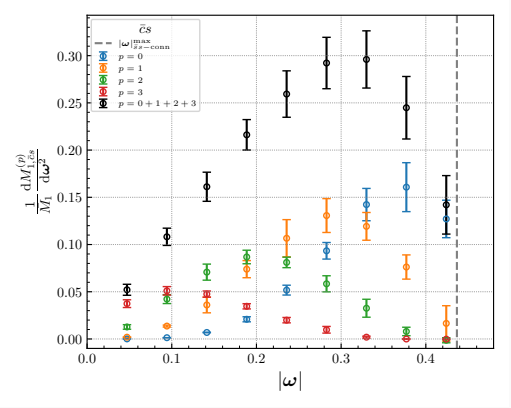
Quark-disconnected current-current contribution

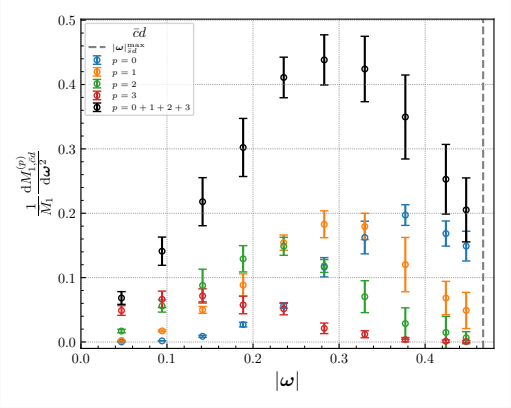
- Present in each channel
- There are no known states $\bar{c}s\bar{f}g$ such that $\omega_0 m_{D_s} \leq m_{D_s} \implies$ it does not contribute to decay rate
- Supported by numerical evidence

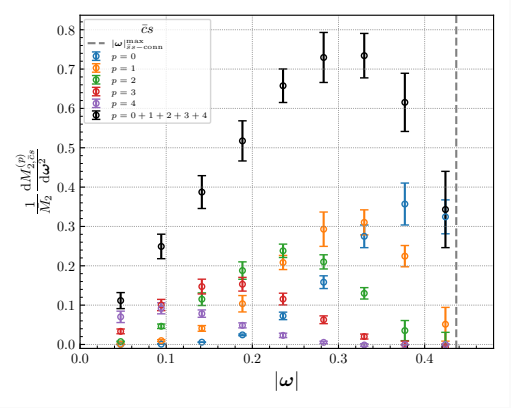


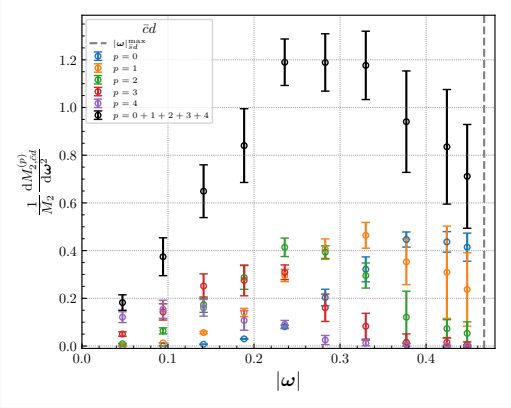

$\bar{c}s$
$|\omega|^{\max}_{\bar{s}s-\mathrm{conn}}$
$p=0$
$p=1$
$p=2$
$p=0+1+2$
$\frac{1}{\Gamma}\frac{\mathrm{d}\Gamma^{(p)}_{\bar{c}s}}{\mathrm{d}\omega^2}$
$|\omega|$


$\bar{c}d$
$|\omega|^{\max}_{\bar{s}d}$
$p=0$
$p=1$
$p=2$
$p=0+1+2$
$\frac{1}{\Gamma}\frac{\mathrm{d}\Gamma^{(p)}_{\bar{c}d}}{\mathrm{d}\omega^2}$
$|\omega|$

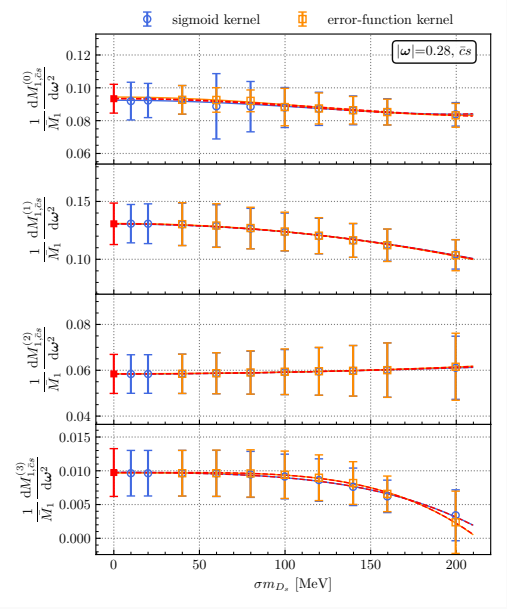

$\bar{c}s$
$|\boldsymbol{\omega}|^{\max}_{s\bar{s}-\mathrm{conn}}$
$p=0$
$p=1$
$p=2$
$p=3$
$p=0+1+2+3$
$\frac{1}{M_1}\frac{\mathrm{d}M_{1,\bar{c}s}^{(p)}}{\mathrm{d}\boldsymbol{\omega}^2}$
$|\boldsymbol{\omega}|$

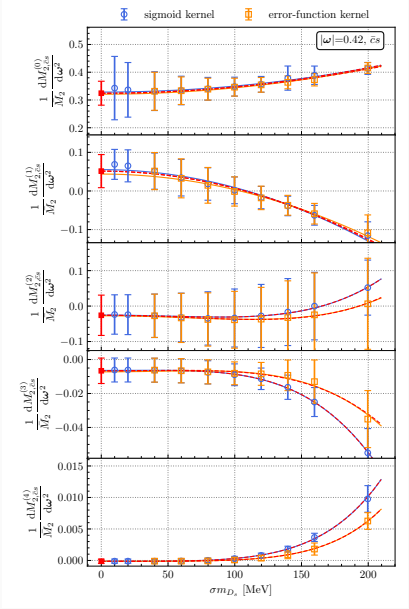

$\bar{c}d$
$|\boldsymbol{\omega}|^{\max}_{\bar{c}d}$
$p=0$
$p=1$
$p=2$
$p=3$
$p=0+1+2+3$
$\frac{1}{M_1}\frac{\mathrm{d}M_{1,\bar{c}d}^{(p)}}{\mathrm{d}\boldsymbol{\omega}^2}$
$|\boldsymbol{\omega}|$


$\bar{c}s$
$|\boldsymbol{\omega}|^{\max}_{s-\mathrm{conn}}$
$p=0$
$p=1$
$p=2$
$p=3$
$p=4$
$p=0+1+2+3+4$
$\frac{1}{M_2}\frac{\mathrm{d}M_{2,\bar{c}s}^{(p)}}{\mathrm{d}\boldsymbol{\omega}^2}$
$|\boldsymbol{\omega}|$

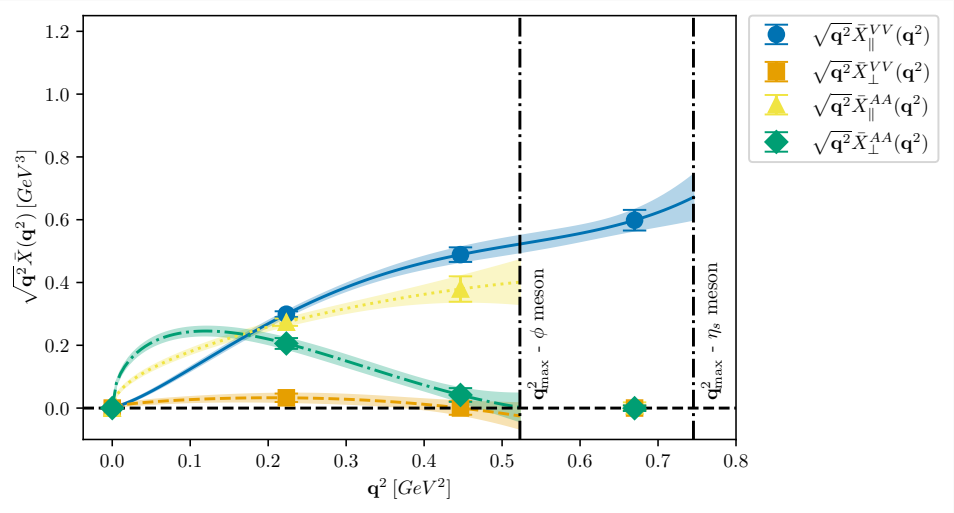

$\bar{c}d$
$|\boldsymbol{\omega}|^{\max}_{\bar{c}d}$
$p=0$
$p=1$
$p=2$
$p=3$
$p=4$
$p=0+1+2+3+4$
$\frac{1}{M_2}\frac{\mathrm{d}M_{2,\bar{c}d}^{(p)}}{\mathrm{d}\boldsymbol{\omega}^2}$
$|\boldsymbol{\omega}|$

$\sigma \mapsto 0$ for lepton-energy moments





Decomposition of the decay rate into $\Gamma_{||}^{VV}$, $\Gamma_{\perp}^{VV}$, $\Gamma_{||}^{AA}$, $\Gamma_{\perp}^{AA}$



Kellermann et al. (2025)