Thermodynamics of the nonlinear O(N) model in 1+1 dimensions

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Model

- many common features with four-dimensional non-Abelian gauge theories
- toy model for QCD
- dimensionless coupling constant, theory is renormalizable
- asymptotical freedom
- dynamically (nonperturbatively) generated mass gap
- conformal invariance
- nonvanishing trace anomaly
- instanton solutions for $N = 3$

Model

- generating functional :

$$Z_{NL} = N \int \mathcal{D}\Phi \delta(\Phi^2 - N/g^2) \exp\left[\int_0^\beta d\tau \int_{-\infty}^{\infty} dx \mathcal{L}_0\right] ,$$

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \Phi^t \partial^\mu \Phi$$

where $\Phi^t = (\sigma, \pi_1, \ldots, \pi_{N-1})$ and *g* is the coupling constant

- nonlinear constraint : $\Phi^2 = N/g^2$.

the thermodynamics is constraint on an *N – 1* dimensional sphere

Model

- using

$$\delta(\Phi^2 - N/g^2) = \lim_{\varepsilon \to 0^+} N \int \mathcal{D}\alpha\, e^{\left\{- \int_0^\beta d\tau \int_{-\infty}^{\infty} dx \left[\frac{i}{2}\alpha(\Phi^2 - N/g^2) + \frac{\varepsilon}{2}\alpha^2\right]\right\}}$$

one can write

$$Z_{NL} = \lim_{\varepsilon \to 0^+} N \int \mathcal{D}\alpha \mathcal{D}\Phi \exp\left[-\int_0^\beta d\tau \int_{-\infty}^{\infty} dx \mathcal{L}\right] ,$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^t \partial^\mu \Phi - U(\Phi, \alpha) \ , \quad U(\Phi, \alpha) = \frac{i}{2}\alpha(\Phi^2 - N/g^2) + \frac{\varepsilon}{2}\alpha^2$$

where *α* is an auxiliary field

CJT effective potential

$$V_{eff} = U(\phi, \alpha_0) + \frac{1}{2} \sum_{i=\sigma,\vec{\pi},\alpha} \int_k [\ln G_i^{-1}(k) + D_i^{-1}(k;\phi,\alpha_0) G_i(k) - 1] + V_2$$

- tree-level potential:

$$U = \frac{i}{2}(\alpha_0 + \alpha)(\sigma^2 + \pi_i^2 + 2\sigma\phi + \phi^2 - N/g^2) + \frac{N\varepsilon}{8}(\alpha_0 + \alpha)^2$$

σ and *α* attain nonvanishing expectation values:

$$\sigma \to \phi + \sigma \qquad \alpha \to \alpha_0 + \alpha$$

this produces a bilinear mixing term: $-i\alpha\sigma\phi,$

which is eliminated by a shift of the auxiliary field

$$\alpha \to \alpha - 4\frac{i\phi\sigma}{N\varepsilon}$$

CJT effective potential

- shifted Lagrangian :

$$\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi_i\partial^\mu\pi_i - \frac{\sigma^2}{2}\left(i\alpha_0 + \phi^2/\varepsilon\right) - \frac{\pi_i^2}{2}\left(i\alpha_0\right) - \frac{\alpha^2}{2}\varepsilon - \frac{i}{2}\alpha(\sigma^2+\pi_i^2) - \frac{\phi\sigma}{2\varepsilon}(\sigma^2+\pi_i^2) - \frac{i}{2}\alpha_0(\phi^2 - N/g^2) - \frac{\varepsilon}{2}\alpha_0^2$$

- inverse tree-level propagators and the tree-level masses :

$$D_i^{-1}(k;\phi,\alpha_0) = -k^2 + m_i^2 \; ; \quad i = \sigma \; , \; \vec{\pi}$$

$$m_\sigma^2 = i\alpha_0 + \frac{\phi^2}{\varepsilon} \; , \quad m_\pi^2 = i\alpha_0 \; ,$$

$$D_\alpha^{-1}(k;\phi,\alpha_0) = m_\alpha^2 = \varepsilon$$

One-loop CJT effective potential

- the 2 PI contribution vanishes: $V_2 = 0$

$$V_{eff}(\phi, \alpha_0, G_\sigma, G_\pi, G_\alpha) = \frac{i}{2}\alpha_0(\phi^2 - N/g^2) + \frac{\varepsilon}{2}\alpha_0^2$$

$$+\frac{1}{2}\sum_{i=\sigma,\vec{\pi},\alpha}\int_k [\ln G_i^{-1}(k) + D_i^{-1}(k;\phi,\alpha_0)G_i(k) - 1]$$

- stationary conditions :

$$\frac{\delta V_{eff}}{\delta\phi} = 0\ , \quad \frac{\delta V_{eff}}{\delta\alpha_0} = 0\ , \quad \frac{\delta V_{eff}}{\delta G_i(k)} = 0\ ; \quad i = \sigma\ ,\ \vec{\pi}\ ,\ \alpha$$

One-loop approximation

- equations for the two condensates :

$$h = i\alpha_0\phi + \frac{4\phi}{N\varepsilon}\int_k G_\sigma(k) \ ,$$

$$i\alpha_0 = \frac{2}{N\varepsilon}\left(\phi^2 - \frac{N}{g^2} + \int_k G_\sigma(k) + (N-1)\int_k G_\pi(k)\right)$$

- equations for the full propagators :

$$G_i^{-1} \ = \ -k^2 + M_i^2 \ ; \quad i = \sigma \ , \ \vec{\pi}$$

$$M_\sigma^2 \ = \ i\alpha_0 + \frac{\phi^2}{\varepsilon} \ , \ \ M_\pi^2 = i\alpha_0 \ ,$$

$$G_\alpha^{-1} \ = \ \varepsilon$$

One-loop approximation

- eliminating the auxiliary field, in the limit *ε → 0* the equations read

$$0 = \phi M_\pi^2 \ ,$$
$$M_\sigma^2 = M_\pi^2 + \frac{4\phi^2}{N\varepsilon} \ ,$$
$$\phi^2 = \frac{N}{g^2} - (N-1)\int_k G_\pi(k)]$$

- in two dimensions there is only one meaningful solution :

$$\phi = 0 \ ,$$
$$M_\sigma^2 = M_\pi^2 = M^2,$$
$$\frac{N}{g^2} = N\int_k G(k)$$

- *Mermin-Wagner-Coleman Theorem:*

no spontaneous breakding of a continuous symmetry in *1+1 d*

One-loop approximation

- thermodynamical pressure : $P = -V_{eff}^{min}$
- energy density : $\rho = T\frac{dP}{dT} - P$
- trace anomaly : $\theta = \rho - P$

Renormalization

- renormalized coupling constant

$$\frac{1}{g_{ren}^2} = \frac{2}{N}\frac{dP_{reg}^{T=0}}{dm^2}\Big|_{m^2=\mu^2}$$

$$= \frac{1}{g^2} - \frac{1}{4\pi}\ln\frac{\Lambda^2}{\mu^2} = \frac{1}{g^2}\left[1 - \frac{g^2}{4\pi}\ln\frac{\Lambda^2}{\mu^2}\right]$$

- renormalized gap equation

$$\frac{1}{g_{ren}^2} = \int_0^\infty \frac{dk}{\pi}\frac{1}{\omega_k}\frac{1}{\exp\{\omega_k/T\}-1} + \frac{1}{4\pi}\ln\frac{\mu^2}{M^2}$$

- renormalized pressure

$$P^{ren} = N\frac{M^2}{2g_{ren}^2} + N\int_0^\Lambda \frac{dk}{\pi}\frac{k^2}{\omega_k}\frac{1}{\exp\{\omega_k/T\}-1}$$

$$-N\frac{M^2}{8\pi}\left(1+\ln\frac{\mu^2}{M^2}\right) + N\frac{m^2}{8\pi}$$

Renormalization

- asymptotic freedom :

$$
\begin{aligned}
\mu \frac{dg_{ren}^2}{d\mu} &= \mu \frac{d}{d\mu}\left[g^2 \left(1 - \frac{g^2}{4\pi} \ln \frac{\Lambda^2}{\mu^2} \right)^{-1} \right] \\
&= -\mu g^2 \left[1 - \frac{g^2}{4\pi} \ln \frac{\Lambda^2}{\mu^2} \right]^{-2} \left(-\frac{g^2}{4\pi} \right) \left(-\frac{2}{\mu} \right) \\
&= -\frac{1}{2\pi} \left[g^2 \left(1 - \frac{g^2}{4\pi} \ln \frac{\Lambda^2}{\mu^2} \right)^{-1} \right]^2 \\
&= -\frac{g_{ren}^4}{2\pi} < 0
\end{aligned}
$$

Two-loop approximation

$$
\begin{aligned}
\mathcal{L}_\sigma &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi_i\partial^\mu\pi_i - \frac{\sigma^2}{2}\left(i\alpha_0 + \phi^2/\varepsilon\right) \\
&\quad - \frac{\pi_i^2}{2}\left(i\alpha_0\right) - \frac{\alpha^2}{2}\varepsilon - \frac{i}{2}\alpha(\sigma^2+\pi_i^2) - \frac{\phi\sigma}{2\varepsilon}(\sigma^2+\pi_i^2) \\
&\quad - \frac{i}{2}\alpha_0(\phi^2 - N/g^2) - \frac{\varepsilon}{2}\alpha_0^2
\end{aligned}
$$

- using that in *1+1 d* $\phi = 0$ only $\frac{i}{2}\alpha(\sigma^2+\pi_i^2)$ contributes to V_2 :

$$V_2 = \frac{N}{4}\int_k\int_p G(k)G_\alpha(p)G(k+p)$$

- effective potential to two-loop order :

$$
\begin{aligned}
V_{eff}(\alpha_0, G, G_\alpha) &= -\frac{i}{2}\alpha_0 N/g^2 + \frac{\varepsilon}{2}\alpha_0^2 \\
&\quad + \frac{1}{2}\sum_i\int_k \left[\ln G_i^{-1}(k) + D_i^{-1}(k;\alpha_0)G_i(k) - 1\right] + V_2
\end{aligned}
$$

Two-loop approximation

- equation for the condensate :

$$i\alpha_0 = \frac{1}{2\varepsilon}\left(-\frac{1}{g^2} + \int_k G(k)\right)$$

- equations for the full propagtors :

$$\begin{aligned} G^{-1}(k) &= D^{-1}(k) + \Sigma(k) \ , \\ G_\alpha^{-1}(k) &= D_\alpha^{-1}(k) + \Sigma_\alpha(k) \end{aligned}$$

- self energies :

$$\begin{aligned} \Sigma(k) &= \frac{2}{N}\frac{\delta V_2}{\delta G(k)} = \int_p G_\alpha(p) G(p+k) \ , \\ \Sigma_\alpha(k) &= 2\frac{\delta V_2}{\delta G_\alpha(k)} = \frac{N}{2}\int_p G(p) G(p+k) \end{aligned}$$

Lattice

- discretised action : $S = \beta \sum_{\langle i,j \rangle} (1 - \vec{s}_i \cdot \vec{s}_j)$

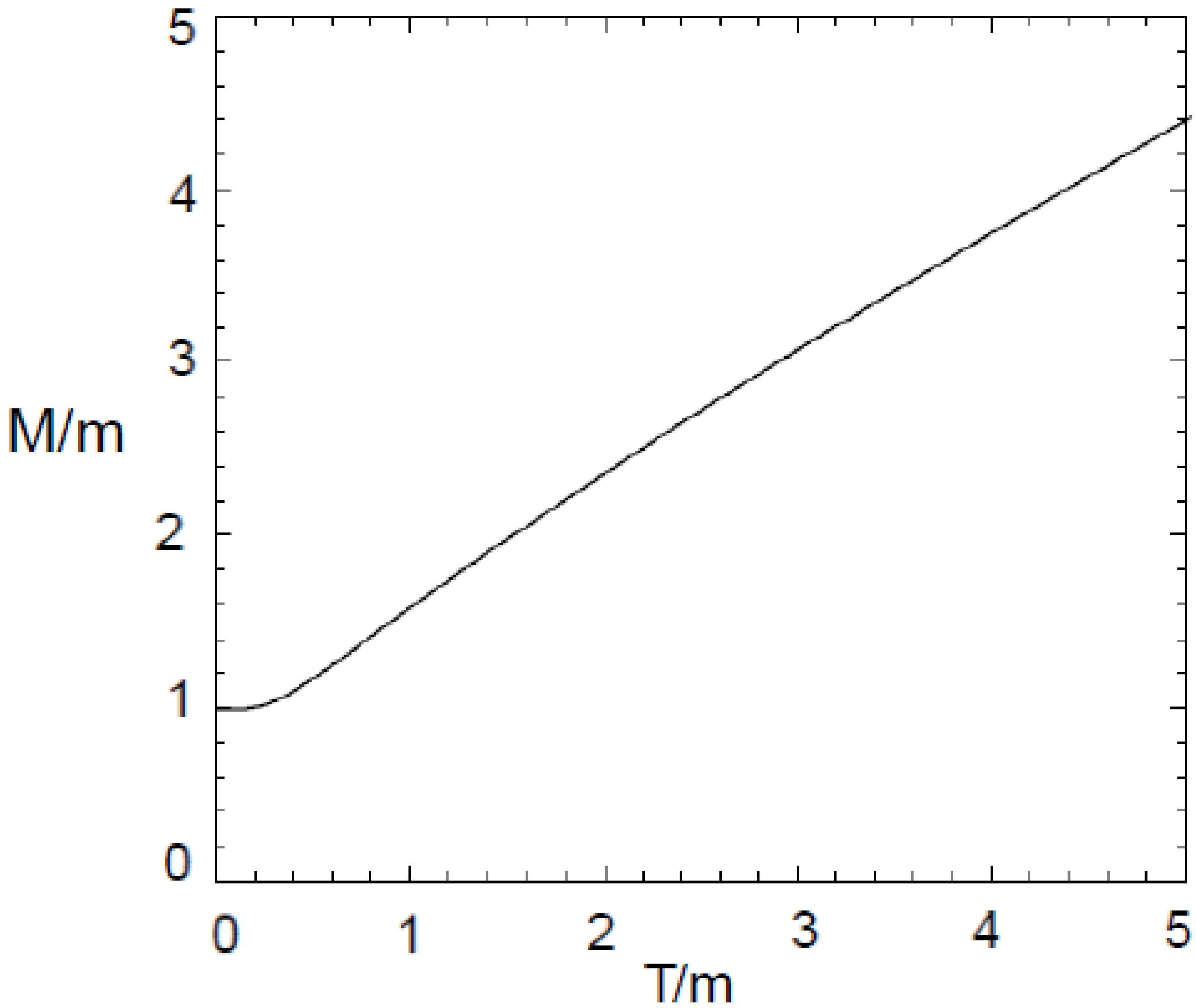
where $\beta = N/g^2$

- finite temperature: $\frac{1}{T} = a(\beta) N_t$

- scale setting: spin-spin correlation at *T = 0* $\langle \vec{s}(0) \cdot \vec{s}(t) \rangle \sim e^{-amt}$
- each β $\rightarrow$ a discretisation
- each $N_t \rightarrow$ a temperature
- continuum expression: $p_*(T) = T \frac{\partial \ln Z}{\partial V}$, $Z = \int_{S^2} \{d\vec{s}_i\} \, e^{\beta \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j}$

- integral method: $\frac{p(T)}{T^2} = N_t^2 \int_0^\beta \left(\langle \ell_x + \ell_t \rangle_{\beta', N_t} - 2 \langle \ell \rangle_{\beta', \infty} \right) d\beta'$
- where $\ell_e = \vec{s}_i \cdot \vec{s}_{i+\hat{e}}$

Results



- similar *T*-dependence to the gluon mass in the deconfined phase

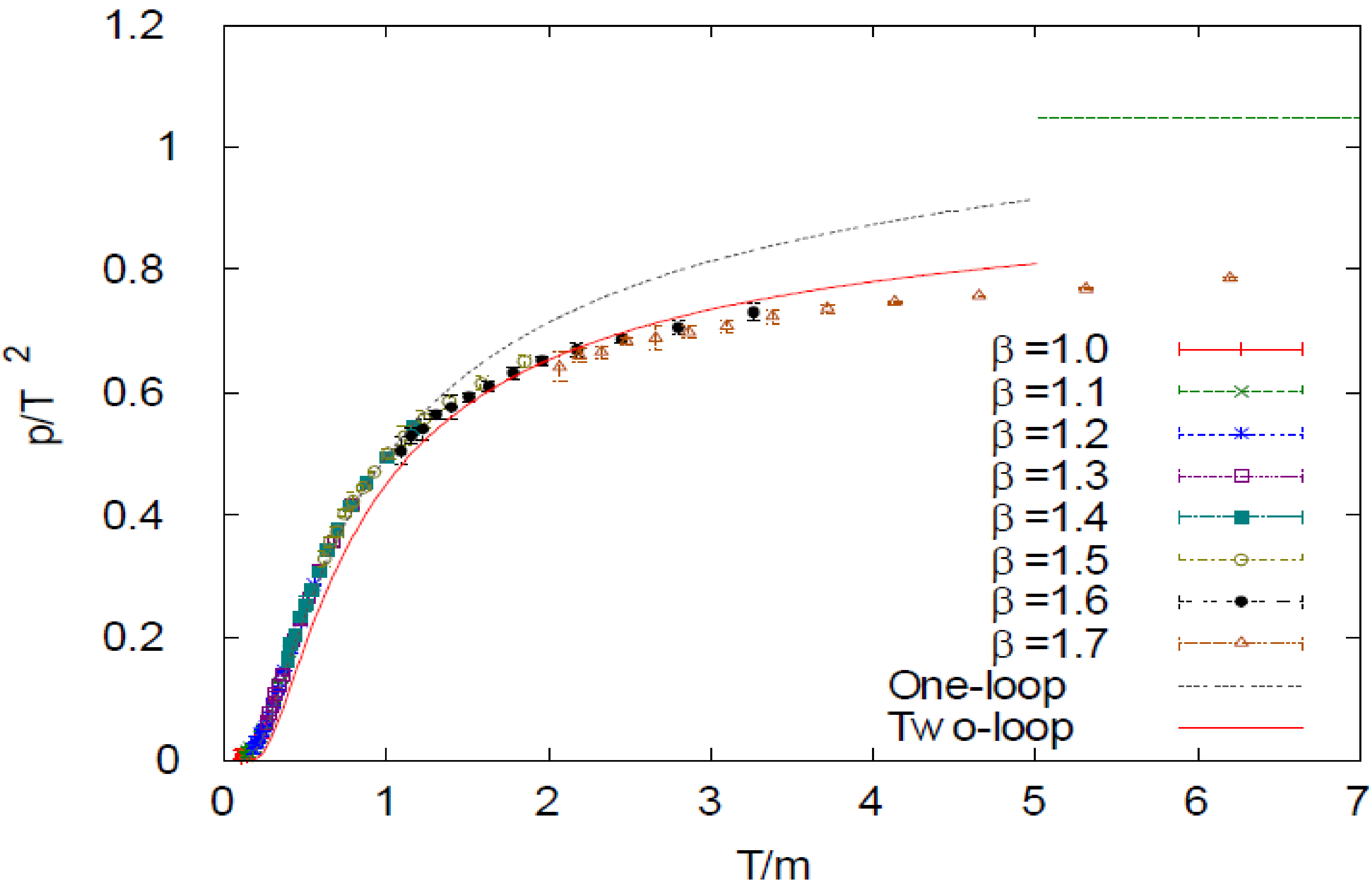
- dimensional transmutation :

  mass gap at *T = 0*

$$m^2 = \mu^2 \exp\left(-\frac{4\pi}{g_{ren}^2}\right)$$

- at large *T* : $M \sim \dfrac{T}{\log T}$

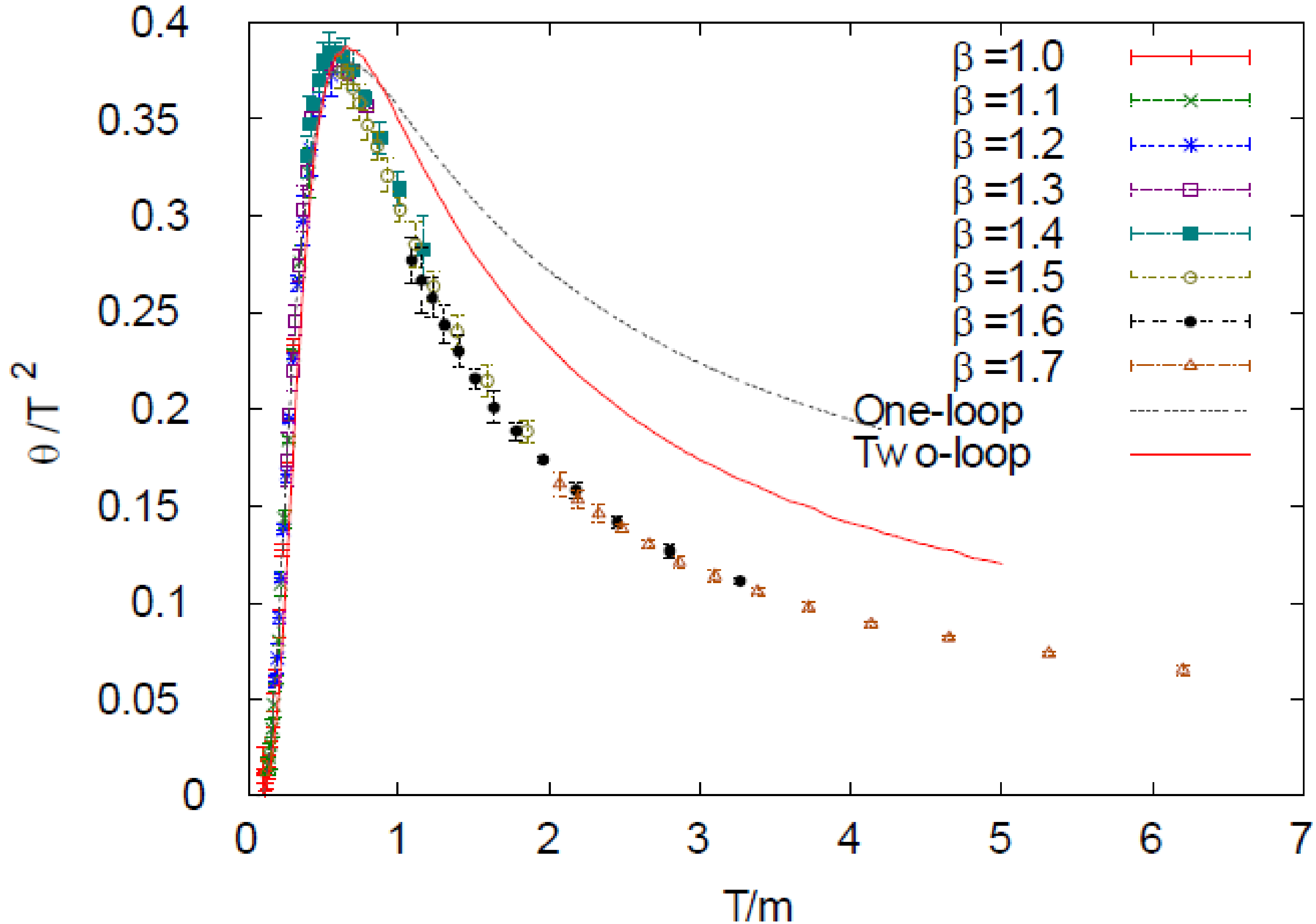
Results



- at high *T* : SB limit of *N - 1* non-interacting bosons:

$$P_{SB}/N = (N-1)\pi T^2/6N$$

Results



- similar structure to the QCD trace anomaly

Summary

- study of the thermodynamics of the nonlinear *O(N)* model in *1+1 d*
- CJT formalism to one- and two-loop order
- fully nonperturbative lattice calculations at finite *T*
- high *T* limit better described by two-loop order
- better correspondence between the CJT formalism and the lattice simulations at low *T*
- improvements : try other theoretical approaches,
higher order terms

Thank you
for your attention