Next-to-leading and Resummed BFKL Evolution with Saturation Boundary

David Zaslavsky
Collaboration with Emil Avsar, Anna Staśto, Dionysis Triantafyllopoulos

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Outline

Essentials of BFKL evolution at leading and next-to-leading orders and explanation of the resummation scheme used

Outline

Explanation of the traveling wave ansatz and saturation boundary conditions, and how they can be used to study nonlinear equations

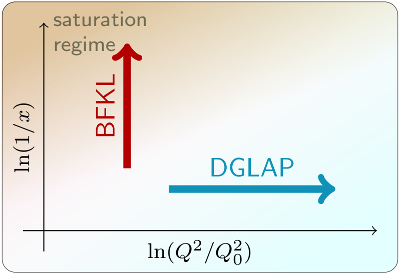
Outline

Presentation and discussion of selected results for the solution of the BFKL equation with saturation boundaries

Outline

Conclusions

BFKL Equation



BFKL evolution: standard phenomelogical tool at small-x (or large rapidity $Y = \ln \frac{1}{x}$)

Mellin transform: $x \to \omega$

Evolution equation:

$$\omega F(\omega, \mathbf{k}) = F_0(\mathbf{k}) + \int \frac{\mathrm{d}^2\mathbf{k}'}{\pi^2} K(\mathbf{k}, \mathbf{k}') F(\omega, \mathbf{k}')$$

Kernel known to NLL order

$$K(\mathbf{k}, \mathbf{k}') = \underbrace{K_0(\mathbf{k}, \mathbf{k}')}_{\mathcal{O}(\bar{\alpha}_s)} + \underbrace{K_1(\mathbf{k}, \mathbf{k}')}_{\mathcal{O}(\bar{\alpha}_s^2)} + \mathcal{O}\left(\bar{\alpha}_s^3\right)$$

NLL BFKL Kernel and Instabilities

(Another) Mellin transform $k^2 \to \gamma$ gives

$$K_0 \to \chi_0(\gamma) = \bar{\alpha}_s \big(2\psi(1) - \psi(\gamma) - \psi(1-\gamma)\big), \quad \psi(\gamma) \equiv \mathrm{d}\ln\Gamma(\gamma)/\mathrm{d}\gamma$$

$$\sim \frac{1}{\gamma} + \frac{1}{1-\gamma}$$

$$K_1 \to \chi_1(\gamma) \sim -\frac{1}{2\gamma^3} - \frac{1}{2(1-\gamma)^3} - \frac{11/12}{\gamma^2} - \frac{11/12 + b}{(1-\gamma)^2} + \mathcal{O}\left(\frac{1}{\gamma}\right)$$

- Running coupling
- Nonsingular terms in DGLAP splitting function
- Kinematic constraints (energy scale dependence)

NLL corrections cause instability of the evolution: negative or oscillating cross sections

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Resummation and Energy Shifts

Resummation fixes instabilities: cancels negative poles and enforces DGLAP limits[†]

$$\chi^\omega(\gamma) = \chi_0^\omega(\gamma) + \chi_{\text{coll}}^\omega(\gamma) + \tilde{\chi}_1^\omega(\gamma)$$

Default choice: symmetric scale: $s_0 = QQ_0$

$$\chi_0^\omega(\gamma) = 2\psi(1) - \psi\left(\gamma + \frac{\omega}{2}\right) - \psi\left(1 - \gamma + \frac{\omega}{2}\right)$$

For saturation physics we need asymmetric scale: $s_0 = Q^2$

$$\chi_0^\omega(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma + \omega)$$

† Scheme B from hep-ph/0307188

Traveling Waves and Saturation

Diffusive equations of the form

$$\partial_t u(t,x) = \mathbb{D}_x \cdot u(t,x) + f\big(u(t,x)\big)$$

exhibit traveling wave solutions

$$u(t,x) = u(x - vt)$$

Example: BFKL with nonlinear term (diffusive approximation of BK)

$$\underbrace{\frac{\partial F}{\partial Y} = \left[A\left(\frac{\partial}{\partial \ln k^2}\right)^2 + B\left(\frac{\partial}{\partial \ln k^2}\right) + C\right]F}_{\text{BFKL}} \; \underbrace{-\bar{\alpha}_s F^2}_{\text{nonlinear term}}$$

exhibits geometric scaling (note: F is related to σ)

$$\sigma(Y, Q^2) = \sigma\left(\frac{Q^2}{Q_s^2(Y)}\right)$$

Correspondence:

$$t \leftrightarrow Y \qquad\qquad x \leftrightarrow \ln k^2$$

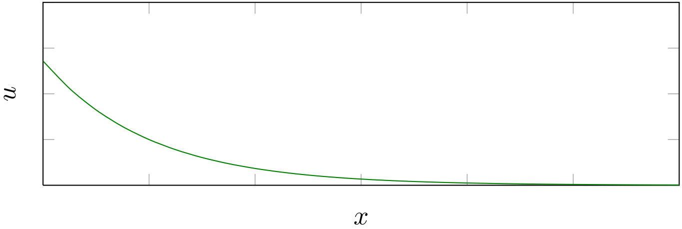
Traveling Wave Diffusion

For a general diffusive equation

$$\partial_t u(x,t) = \mathbb{D}_x \cdot u(x,t) + f\big(u(x,t)\big)$$

where $f(u)$ has an unstable fixed point at $u = 0$

Traveling wave solution:



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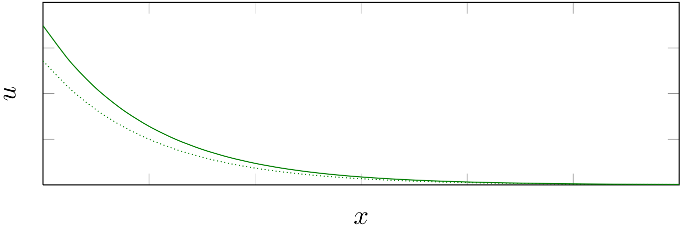

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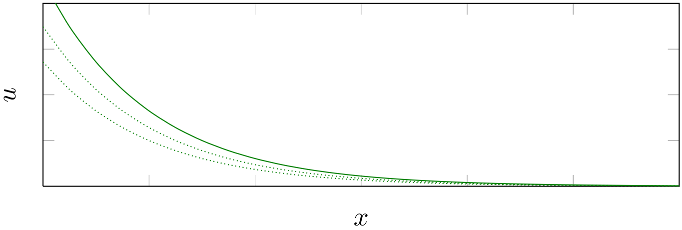

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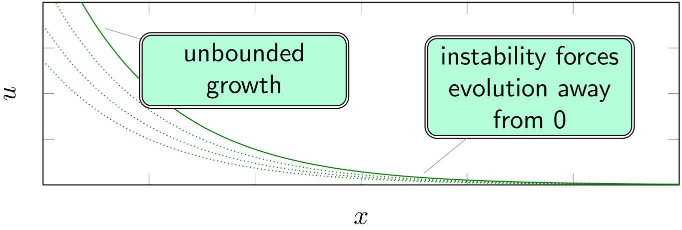

Suppression of Growth

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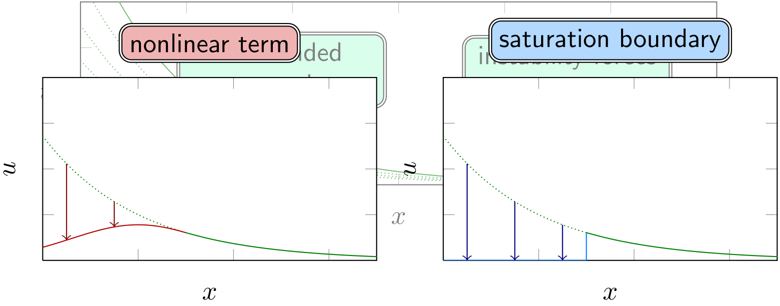

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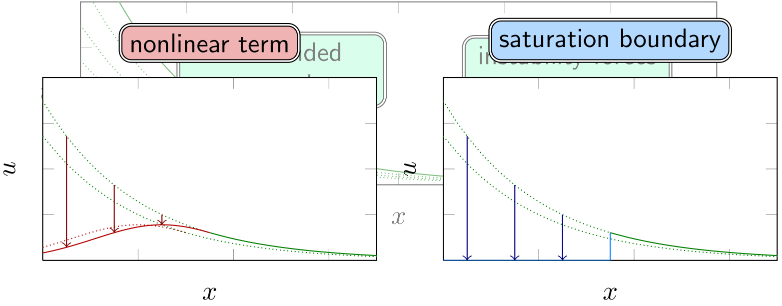

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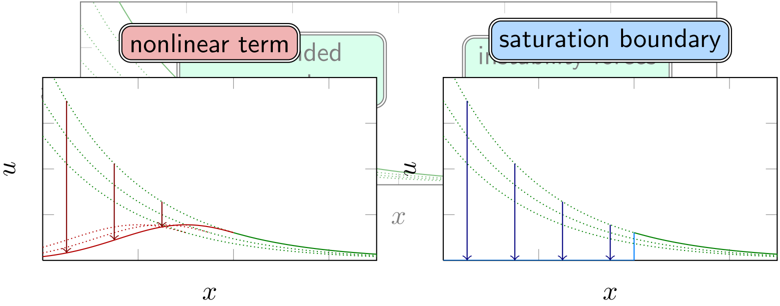

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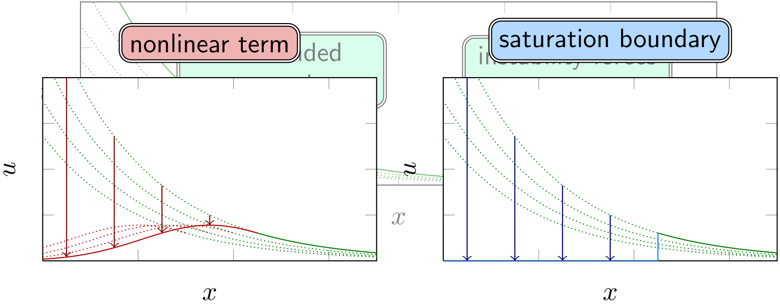
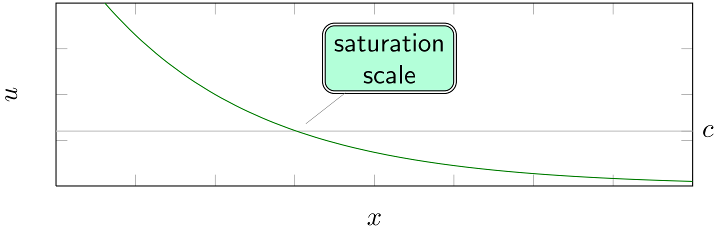


Absorptive Boundary

Two boundary implementations:

Frozen sets $F =$ const. below threshold momentum

Absorptive sets $F = 0$ below threshold momentum



Energy dependence of Q_s is mostly independent of boundary choice

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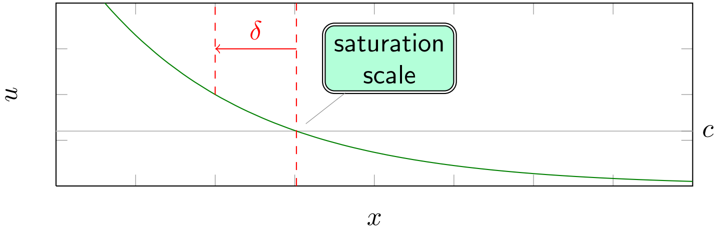

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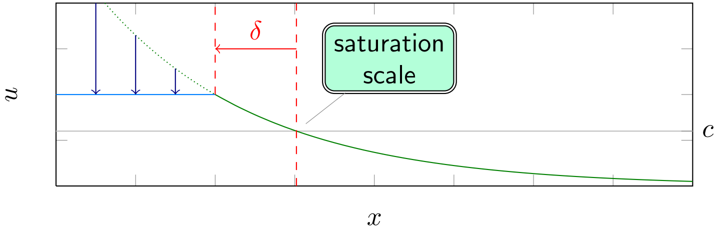

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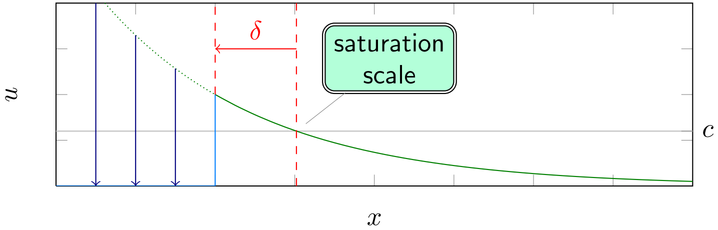

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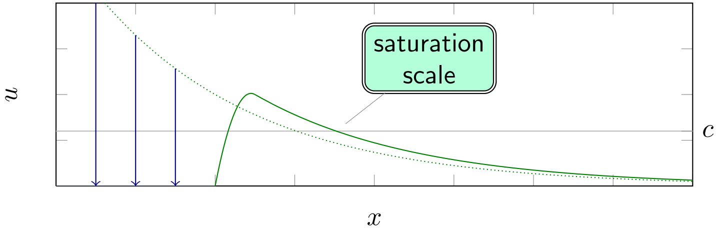

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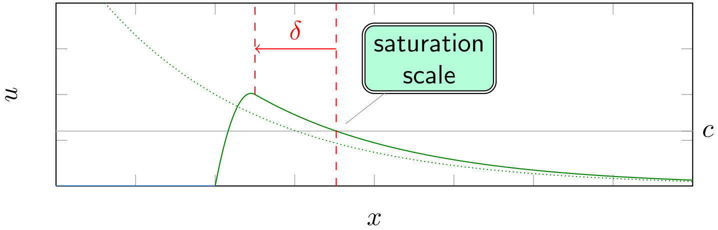

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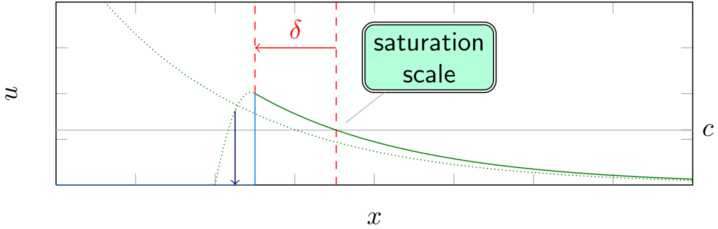

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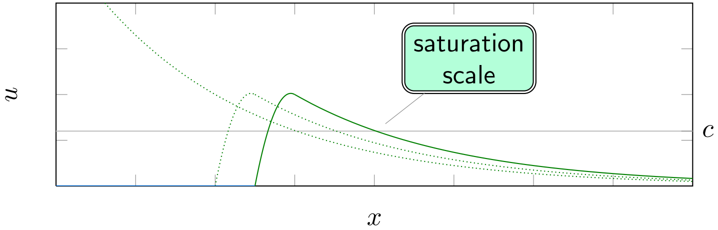

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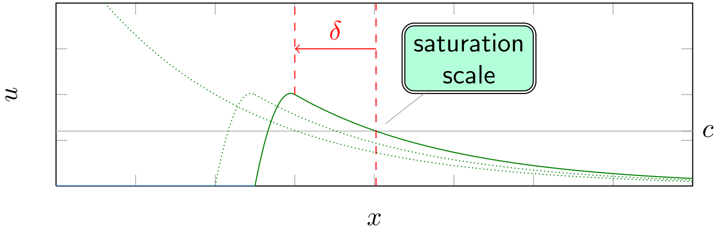

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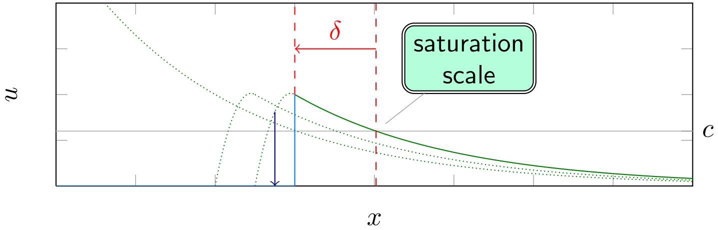

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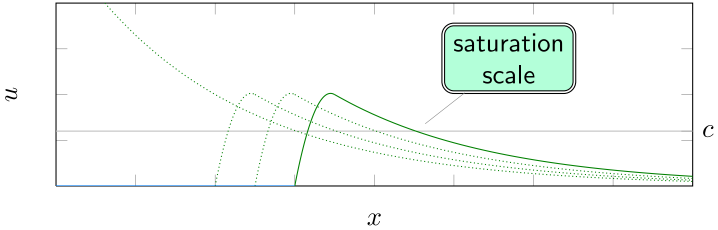

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Putting It All Together

Two effects that complicate parton dynamics at small x:

Higher Order BFKL Series for $\chi(\gamma)$ is not well behaved

- Solution: **resummation**

Saturation Nonlinear effects become important

- Solution: **absorptive boundary**

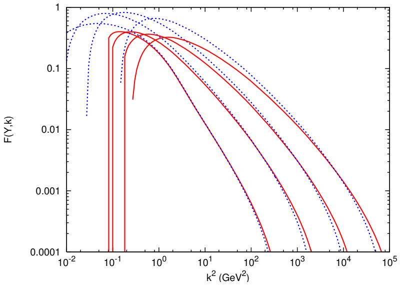
Goal: understand the interaction between these two effects

Method

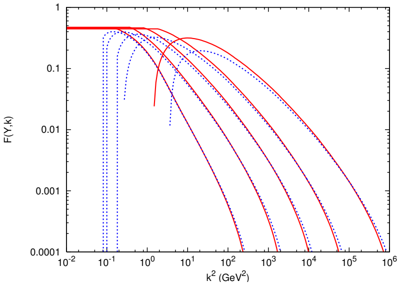
We have numerically solved the full momentum-space BFKL equation using LL, NLL, and resummed kernels with the saturation boundary

Fixed Coupling Traveling Wave Solutions

Results for F at NLL with fixed coupling:



Dotted blue: no boundary
Solid red: absorptive boundary
$Y = 2, 6, 10, 14$

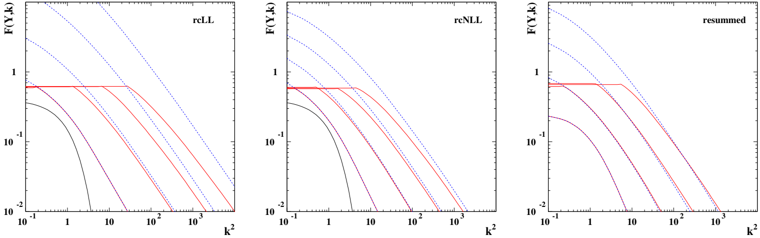


Dotted blue: absorptive boundary
Solid red: frozen boundary
$Y = 2, 6, 10, 14, 20$

The NLL solution goes negative at low momenta without the boundary, and at high rapidities even with the frozen boundary

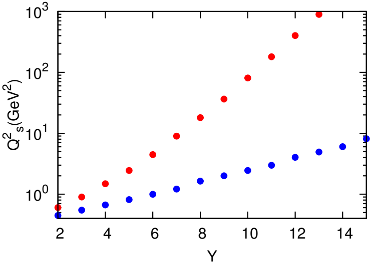
Running Coupling Traveling Wave Solutions

Results for F with LL, NLL, and resummed kernels with running coupling:

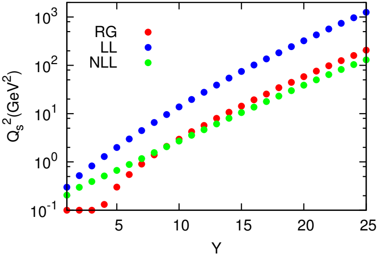


Initially the boundary slows the wave propagation progressively less as higher-order terms are included.

Saturation Scale



Red (top): LL • Blue (bottom): NLL



Blue (top): LL • Green (middle): NLL • Red (bottom): resummed

Fixed coupling Saturation scale at LL shows exponential growth; NLL growth is suppressed

Running coupling Asymptotic $\exp(\sqrt{Y+Y_0})$ growth in all cases, but still large differences between LL and NLL

Normalization of Q_s is arbitrary!

note differing Y limits on plots

Conclusions

Saturation does not fix instabilities — resummation and NLL terms beyond running coupling are necessary

Our method does reproduce asymptotic growth of Q_s which is similar for all cases (rcLL, rcNLL, resummed)

Preasymptotic terms delay the growth of Q_s in the resummed evolution — these may be important for future phenomenological studies

Full BFKL Equation

Cross section factorizes into impact factors and gluon Green's function G

$$\sigma^{AB}(s, Q_A, Q_B) = \int \frac{\mathrm{d}\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega} \int \frac{\mathrm{d}^2\mathbf{k}_1}{k_1^2}\frac{\mathrm{d}^2\mathbf{k}^2}{k_2^2}\times$$
$$\Phi_A(Q_A, \mathbf{k}_1)G(\omega; \mathbf{k}_1, \mathbf{k}_2)\Phi_B(Q_B, \mathbf{k}_2)$$

F defined as

$$F(\omega, \mathbf{k}; Q_B) = \int \frac{\mathrm{d}^2\mathbf{k}_2}{k_2^2} G(\omega; \mathbf{k}, \mathbf{k}_2)\Phi_B(Q_B, \mathbf{k}_2)$$

BFKL equation is

$$\omega G(\omega; \mathbf{k}, \mathbf{k}_0) = \delta^2(\mathbf{k} - \mathbf{k}_0) + \int \frac{\mathrm{d}^2\mathbf{k}'}{\pi^2} K(\mathbf{k}, \mathbf{k}')G(\omega; \mathbf{k}', \mathbf{k}_0)$$
$$\omega F(\omega, \mathbf{k}) = F_0(\mathbf{k}) + \int \frac{\mathrm{d}^2\mathbf{k}'}{\pi^2} K(\mathbf{k}, \mathbf{k}')F(\omega, \mathbf{k}')$$

Resummed Kernel

Resummation scheme B from `hep-ph/0307188`

$$\chi^\omega(\gamma) = \chi_0^\omega(\gamma) + \chi_{\mathsf{coll}}^\omega(\gamma) + \tilde{\chi}_1^\omega(\gamma)$$

Includes shifts and collinear resummation

$$\chi_0^\omega(\gamma) = 2\psi(1) - \psi\left(\gamma + \frac{\omega}{2}\right) - \psi\left(1 - \gamma + \frac{\omega}{2}\right)$$

$$\chi_{\mathsf{coll}}^\omega(\gamma) = \frac{\omega A_1(\omega)}{\gamma + \frac{\omega}{2}} + \frac{\omega A_1(\omega)}{1 - \gamma + \frac{\omega}{2}}$$

$$\tilde{\chi}_1^\omega(\gamma) = \chi_1(\gamma) + \frac{1}{2}\chi_0(\gamma)\frac{\pi^2}{\sin^2(\pi\gamma)} - \chi_0(\gamma)\frac{A_1(0)}{\gamma(1-\gamma)} - \left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right)C(0) + \left(\frac{1}{\gamma + \frac{\omega}{2}} + \frac{1}{1 - \gamma + \frac{\omega}{2}}\right)C(\omega)[1 + \omega A(\omega)]$$

Dip in Splitting Function

As displayed in hep-ph/0307188, the small-z resummed splitting function with 1-loop DGLAP and BFKL for comparison