The singular behavior of one-loop massive QCD amplitudes with one external soft gluon

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based on work with M. Czakon and A. Mitov:

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Introduction

This calculation was done as part of the NNLO top–quark calculation of M. Czakon et al.

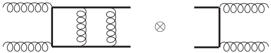
Why NNLO with massive quarks:

- Background process: Discovering BSM particles in $t\bar{t}$–channel:
  High–energy jets, leptons and missing transverse energy are typical signals for SUSY as well as for $t\bar{t}$–decays
- Precise measurement of the top–quark mass:
  consistency check of the SM Higgs boson mass
- Precise measurement of the top–quark cross section:
  improvement of the gluon pdf at large x.

$\rightarrow$ More on top–quarks:
The Heavy Quark Session of yesterday afternoon

However: the one–loop soft–gluon current is universal.

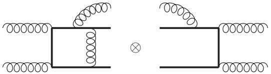
NNLO - contributions



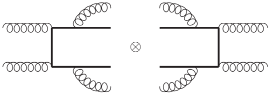
$qq \to QQ$: numerically: Czakon '07

$gg \to QQ$: to come soon: Czakon, Bärnreuther

Kniehl, Körner, Merebashvili, Rogal '05 - 08; Anastasiou Mert Aybat '08



$\Leftarrow$ We are here,
in the limit of a soft external gluon

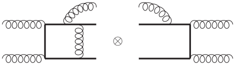


Czakon '10-11

See also A. Mitov's talk in the HQ session of yesterday afternoon!

NNLO: Virtual - Real

One needs to integrate over the real external gluon



→ this generates a divergence when the gluon is soft and/or collinear to the external legs.

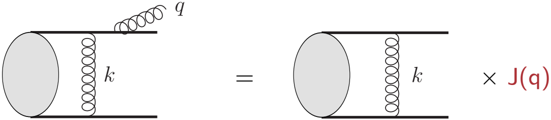
→ The general idea:
Construct a subtraction term in the soft/collinear limit for the squared amplitude, ST:

$$|M|^2 = \underbrace{|M|^2 - ST}_{finite} + ST$$

The subtraction term is singular in phase space integration, but a simple(r) analytically integrable function.

How to get the subtraction terms for massive one–loop amplitudes?

- *Collinear limits:* luckily at this point, we are in the massive case where emissions off massive lines are finite
- *Soft limit* of one–loop massive amplitudes:
  External gluon becomes soft: $q \to \lambda q, \lambda \to 0$;



The emission of a soft gluon does not effect momenta or spin of the radiating hard parton, BUT colour (see later)!
$\to$ no complete factorisation, but color correlations: $J(q)$ process–independent

Consider an (n+1)–point amplitude with one external soft gluon of momentum q.
We want to "factorize" the singularity and derive a form:

$$M_a(n+1;q) \quad = \quad J_a(q)M(n) + O(\lambda) \quad \text{for} \qquad q \to \lambda q, \lambda \to 0$$

$J_a(q)$ is the process–independent soft–gluon current.

All terms have an expansion in the strong coupling constant:

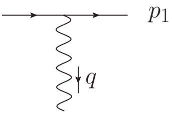
$$\begin{aligned} J_a(q) &= g_S \mu^\epsilon \left(J_a^{(0)}(q) + J_a^{(1)}(q) + \ldots \right), \\ M(n) &= M^{(0)}(n) + M^{(1)}(n) + \ldots \\ M_a(n+1;q) &= M_a^{(0)}(n+1;q) + M_a^{(1)}(n+1;q) + \ldots, \end{aligned}$$

where

$$J_a^{(n)}(q) \equiv \varepsilon^\mu(q) J_a^{\mu(n)}(q)$$

The calculation is done in eikonal approximation.

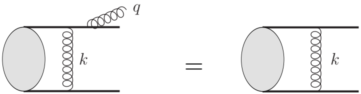
Eikonal approximation:



$$\simeq \frac{p_1^\mu}{p_1 \cdot q} T^a$$

Tree–level massive and massless: $J_a^{\mu(0)}(q) = \sum_{i=1}^{n} T_i^a \frac{p_i^\mu}{p_i \cdot q} \equiv \sum_{i=1}^{n} T_i^a e_i^\mu \,,$

$$M_a^{(0)}(n+1; q) \simeq g_S \mu^\epsilon J_a^{(0)}(q) M^{(0)}(n)$$

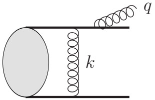


$\times$ J(q)

$J_a^{(1)}(q)$ controlls the soft limit of one-loop amplitudes.

One–loop current $J^{(1)}$ is known in the massless case: Bern, Del Duca, Kilgore, Schmidt '98-99, Catani, Grazzini '00

We follow the approach for the massless case by Catani, Grazzini:



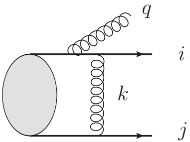
$$M_a^{(1)}(n+1,q) \simeq \underbrace{J_a^{(0)}(q)M^{(1)}(n)}_{factorizable} + \underbrace{J_a^{(1)}(q)M^{(0)}(n)}_{non-factorizable}$$

$$\mathrm{J}_a^{(1)}(q)\mathrm{M}^{(0)}(n) = \left(M_{\mathrm{soft}}^{(1)}(n+1,q) - g_s\mu^{\epsilon}\epsilon^{\mu}(q)J_{a,\mu}^{(0)}(q)M_{\mathrm{soft}}^{(1)}(n)\right)$$

Hence, we need to consider all diagrams contributing to:

All possible insertions of soft gluons k (loop) and q (external) on the hard external lines $-$ factorising terms $J_a^{(0)}(q) M_{\text{soft}}^{(1)}(n)$

After some considerations of the structure of the diagrams, including their color structure and the gauge invariance already of certain subsets of diagrams, one ends up with the calculation of the (non–abelian part) of the following diagrams:



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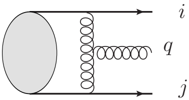

with $T_j^b T_i^b T_i^a \to i f_{abc} T_i^b T_i^c$

The one–loop soft current is non–abelian.

The one–loop UV un–renormalized soft–gluon current

$$J_a^{\mu(1)}(q) = i f_{abc} \sum_{i \neq j=1}^{n} T_i^b T_j^c \left(e_i^\mu - e_j^\mu \right) g_{ij}^{(1)}(\epsilon, q, p_i, p_j)\,,$$

$$\begin{aligned} g_{ij}^{(1)} = a_S^b \mu^{2\epsilon} & \frac{p_i \cdot p_j}{m_i^2 (p_j \cdot q)^2 - 2(p_i \cdot p_j)(p_i \cdot q)(p_j \cdot q) + m_j^2 (p_i \cdot q)^2} \\ & \times \Bigg\{ (p_i \cdot q)(p_j \cdot q) \left[(p_j \cdot q) M_1 + (p_i \cdot q) \hat{M}_1 \right] \\ & + \frac{(p_j \cdot q)}{2} \left[(p_i \cdot p_j)(p_i \cdot q) - m_i^2 (p_j \cdot q) \right] M_2 + \frac{(p_i \cdot q)}{2} \left[(p_i \cdot p_j)(p_j \cdot q) - m_j^2 (p_i \cdot q) \right] \hat{M}_2 \\ & + \left[(p_i \cdot p_j)(p_i \cdot q)(p_j \cdot q) - m_i^2 (p_j \cdot q)^2 - m_j^2 (p_i \cdot q)^2 \right] \frac{(p_i \cdot q)(p_j \cdot q)}{p_i \cdot p_j} M_3 \Bigg\}\,. \end{aligned}$$

The bare coupling $a_S^b = \alpha_s^b S_\epsilon / (2\pi)$ with $S_\epsilon = (4\pi)^\epsilon \exp(-\epsilon \gamma_E)$,

$\hat{M}_k \equiv M_k(p_i \leftrightarrow p_j)$, $k = 1, 2, 3$, $\hat{M}_3 = M_3 \;\Rightarrow\; g_{ij}^{(1)} = g_{ji}^{(1)}$.

We consider three kinematical regions (q always in the final state):
p_i outgoing, $p_i^2 = m_i^2 > 0$ and

1. *Case 1*: $p_j^2 = 0$, p_j incoming
2. *Case 2*: $p_j^2 = 0$, p_j outgoing
3. *Case 3*: $p_j^2 = m_j^2 > 0$, p_j outgoing

The three integrals are increasingly complicated:

$$
\begin{aligned}
M_1 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][-p_j \cdot k]} \\
M_2 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]} \\
M_3 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]} ,
\end{aligned}
$$

$$M_1 = \Phi \, \frac{\pi^{-2+\epsilon}}{16} \Gamma(-\epsilon)\Gamma(2\epsilon) \frac{m_j^{2\epsilon}}{\left[-(p_j \cdot q) - i\delta\right]^{1+2\epsilon}} \, .$$

$$\begin{aligned} M_2 = & \; \Phi \, \frac{\pi^{-2+\epsilon}}{4} \Gamma(-\epsilon)\Gamma(2\epsilon) \left[-(p_i \cdot q) - i\delta\right]^{-2\epsilon} \left[-2(p_i \cdot p_j) - i\delta\right]^{-1+\epsilon} \\ & \times \left\{ \frac{\Gamma(1+\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \, v^{-1+2\epsilon} \, \beta_j^{-\epsilon} \right. \\ & \left. \qquad - \frac{2\,\beta_i^{\epsilon}}{1+v} \, {}_2F_1\left(1, 1-\epsilon, 1+\epsilon; \frac{1-v}{1+v}\right) \right\} . \end{aligned}$$

with $\beta_k \equiv m_k^2/(-2(p_i \cdot p_j) - i\delta)$, $k = i, j$, and the relative velocity $v = \sqrt{1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}}$.

M_3: For p_j incoming calculable in terms of Appell function F_1:

$$M_3^{(SL)} = -\,\Phi\, \frac{\pi^{-2+\epsilon}}{16} \frac{\Gamma(-\epsilon)\Gamma(2\epsilon)}{(p_i\cdot q)(p_j\cdot q)} \left(\frac{p_i\cdot p_j}{(p_i\cdot q)(p_j\cdot q)}\right)^{\epsilon}$$
$$\times \left\{ (-1-i\delta)^{-2\epsilon}\, \alpha_j^{\epsilon} F_1\left(-2\epsilon,-\epsilon,-\epsilon,1-2\epsilon;\frac{1}{x_1^s},\frac{1}{x_2^s}\right)\right.$$
$$\left. +\ \alpha_i^{\epsilon} F_1\left(-2\epsilon,-\epsilon,-\epsilon,1-2\epsilon;\frac{1}{1-x_2^s},\frac{1}{1-x_1^s}\right)\right\}.$$

Much more difficult for the here needed "time–like" case of p_j outgoing: explicit expansion up to order ϵ^1 available (electronic form).
Involves multiple polylogs

$$F_c(x_1,x_2) = \int_0^1 dt\, \frac{\ln(1-t)\ln(1-t\frac{x_2}{x_1})}{\frac{1}{x_2}-t}$$

The one–loop soft–gluon current for *Case 1*: $p_i^2 = m_i^2 > 0,\ p_j^2 = 0,\ p_j$ incoming

$$g_{ij}^{(1)}(C1) \quad = \quad R_{ij}^{[C1]} + i\pi I_{ij}^{[C1]} \equiv a_S^b \left(\frac{2(p_i \cdot p_j)\mu^2}{2(p_i \cdot q) 2(p_j \cdot q)} \right)^{\epsilon} \sum_{n=-2}^{2} \epsilon^n \left(R_{ij}^{(n)[C1]} + i\pi I_{ij}^{(n)[C1]} \right) ,$$

$$I_{ij}^{(-2)[C1]} = 0 \,, \qquad I_{ij}^{(-1)[C1]} = -\frac{1}{2} \,, \qquad R_S\, I_{ij}^{(0)[C1]} = 2 m_i^2 (p_j \cdot q) \ln \left(\frac{\alpha_i}{2} \right) ,$$

$$R_S\, I_{ij}^{(1)[C1]} = 4 \left[(p_i \cdot p_j)(p_i \cdot q) - m_i^2 (p_j \cdot q) \right] \mathrm{Li}_2 \left(1 - \frac{\alpha_i}{2} \right) + m_i^2 (p_j \cdot q) \ln^2 \left(\frac{\alpha_i}{2} \right) + \pi^2 \frac{-2(p_i \cdot p_j)(p_i \cdot q) + m_i^2 (p_j \cdot q)}{2}$$

$$R_S\, I_{ij}^{(2)[C1]}$$

$$= 4 \left[(p_i \cdot p_j)(p_i \cdot q) - m_i^2 (p_j \cdot q) \right] \left[\mathrm{Li}_3 \left(1 - \frac{\alpha_i}{2} \right) + \mathrm{Li}_3 \left(\frac{\alpha_i}{2} \right) \right] - \zeta_3 \frac{40 (p_i \cdot p_j)(p_i \cdot q) - 26 m_i^2 (p_j \cdot q)}{3}$$
$$+ 2 \left[(p_i \cdot p_j)(p_i \cdot q) - m_i^2 (p_j \cdot q) \right] \ln \left(1 - \frac{\alpha_i}{2} \right) \ln^2 \left(\frac{\alpha_i}{2} \right) + \frac{m_i^2 (p_j \cdot q)}{3} \ln^3 \left(\frac{\alpha_i}{2} \right)$$
$$+ \ln \left(\frac{\alpha_i}{2} \right) \left(\pi^2 \frac{-4 (p_i \cdot p_j)(p_i \cdot q) + m_i^2 (p_j \cdot q)}{6} + 4 \left[(p_i \cdot p_j)(p_i \cdot q) - m_i^2 (p_j \cdot q) \right] \mathrm{Li}_2 \left(1 - \frac{\alpha_i}{2} \right) \right)$$

$$R_S = 4 \left[m_i^2 (p_j \cdot q) - 2 (p_i \cdot p_j)(p_i \cdot q) \right] , \ \alpha_i = \frac{m_i^2 (p_j \cdot q)}{(p_i \cdot q)(p_i \cdot p_j)} , \ \alpha_j = \frac{m_j^2 (p_i \cdot q)}{(p_j \cdot q)(p_i \cdot p_j)} .$$

Checks:

- Some independent calculations of the integrals via Feynman Parameter or Mellin–Barnes
- The pole terms of the one-loop soft current agree with what is expected based on the structure of the singularities of massive gauge theory amplitudes
- *Case 2* as limit of *Case 3* for $m_j \to 0$
- Small mass limit $m_i = 0$ and $m_j = 0$ agrees with Catani, Grazzini
- We understood the analytic continuation from spacelike to timelike and used that for a check:
  Case 2 can be obtained from *Case 1* (for both $p_j^2 = 0$) by replacement $p_j \to -p_j$, which leaves $g_{ij}^1(\textit{Case 1})$ unchanged, hence these cases are identical!
- Several numerical checks

The square of a Born amplitude in the soft–gluon limit:

$$\langle M_a^{(0)}(n+1;q)|M_a^{(0)}(n+1;q)\rangle =$$
$$-4\pi\alpha_S\mu^{2\epsilon}\left\{\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle + \sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(0)}(n)\rangle\right\} + \mathcal{O}(\lambda)\,.$$

with $e_{ij}\equiv e_i\cdot e_j$ and $\mathcal{C}_i\equiv T_i\cdot T_i$.

The interference term between the Born and one-loop amplitude:

$$\langle M_a^{(0)}(n+1;q)|M_a^{(1)}(n+1;q)\rangle + c.c. = -4\pi\alpha_S\mu^{2\epsilon}\Bigg\{$$
$$2C_A\sum_{i\neq j=1}^{n}\left(e_{ij}-e_{ii}\right)R_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle$$
$$-4\pi\sum_{i\neq j\neq k=1}^{n} e_{ik}I_{ij}\langle M^{(0)}(n)|f^{abc}T_i^aT_j^bT_k^c|M^{(0)}(n)\rangle$$
$$+\left(\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(1)}(n)\rangle + c.c.\right) + \left(\sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(1)}(n)\rangle + c.c.\right)\Bigg\} + \mathcal{O}(\lambda)\,.$$

Conclusions

- We calculated the singluar limit of massive QCD one–loop amplitudes with one external soft gluon
- The one–loop soft–gluon current is process independent and universal
- This was the last missing theoretical input for the NNLO top–quark production calculation... outlook? $\rightarrow$ *see A. Mitov's talk*

THANKS !