Resummation of large-x and small-x double logarithms in DIS and semi-inclusive e^+e^- annihilation

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partly with G. Soar, A. Lo Presti, C.H. Kom (UoL), A. Almasy (UoL, now DESY)
[and S. Moch (DESY), J. Vermaseren (NIKHEF)]

- Splitting and coefficient functions and their endpoint behaviour
- Generalized threshold resummation of 1/Mellin-N contributions
- Small-x resummation of $x^{-1}\ln^{\ell} x$ (SIA) and $x^{0}\ln^{\ell} x$ (DIS) terms

Conventions and references

Double-log enhancement: two additional logs L per additional order in α_s

$$Q\Big|_{\alpha_s^n} \sim L^{-\ell_0}(\,\#L^{2n} + \#L^{2n-1} + \#L^{2n-2} + \ldots) + \ldots$$

LL NLL NNLL

LL, NLL, ...: leading logarithms, next-to-leading logarithms, ...

Counting of a resummation, cf. small-x, not of a (stronger) exponentiation, cf. soft gluons: NNLL resummation $\Leftrightarrow$ (re-expanded) NLL exponentiation

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- Non-singlet NNLL (NLL for DY) resummation from physical kernels
  MV, arXiv:0902.2342 (JHEP), 0909.2124 (JHEP)
- Singlet NNLL for fourth-order splitting functions and F_L in DIS
  SMVV, 0912.0369 (NPB), 1008.0952 (Loops & Legs 2010)
- Generalized threshold resummation in inclusive DIS and SIA
  A.V., 1005.1606 (PLB); ASV, 1012.3352 (JHEP); ALPV, 1202.5224 (Radcor 2011), ...
- Small-x resummation of splitting & coefficient functions in SIA and DIS
  A.V., arXiv:1108.2993 (JHEP); KV, ...

Inclusive DIS and SIA in perturbative QCD

Structure function & fragmentation functions F_a: up to $\mathcal{O}(1/Q^{(2)})$ given by

$$F_a^h(x, Q^2) \;=\; \Big[C_{a,i}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2)\Big](x) \;, \quad \otimes: \text{ Mellin convolution}$$

Coefficient functions calculated at renormalization/factorization scale $\mu = Q$

Parton/fragmentation distributions: evolution with splitting functions $P^{S/T}$

$$\frac{d}{d\ln\mu^2}\, f_i^h(\xi, \mu^2) \;=\; \sum_k \Big[P_{ik/ki}^{S,T}(\alpha_s(\mu^2)) \otimes f_k^h(\mu^2)\Big](\xi)$$

Initial conditions uncalculable $\Rightarrow$ fit-analyses of reference processes

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Pert. expansions:

$$P \;=\; \alpha_s\, P^{(0)} + \alpha_s^2\, P^{(1)} + \alpha_s^3\, P^{(2)} + \ldots$$

$$C_a \;=\; \underbrace{\alpha_s^{\,n_a}\Big[\, c_a^{(0)} + \alpha_s\, c_a^{(1)} + \alpha_s^2\, c_a^{(2)}}\; + \alpha_s^3\, c_a^{(3)} + \ldots \Big]$$

NNLO: first quantitative error estimate, . . .

Almost complete for massless DIS / SIA, except for longitudinal fragmentation

Flavour singlet – non-singlet decomposition

Quark-quark splitting functions:

$$P_{\mathrm{q}_i\mathrm{q}_k} = P_{\bar{\mathrm{q}}_i\bar{\mathrm{q}}_k} = \delta_{ik} P^{\mathrm{v}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\mathrm{qq}}$$
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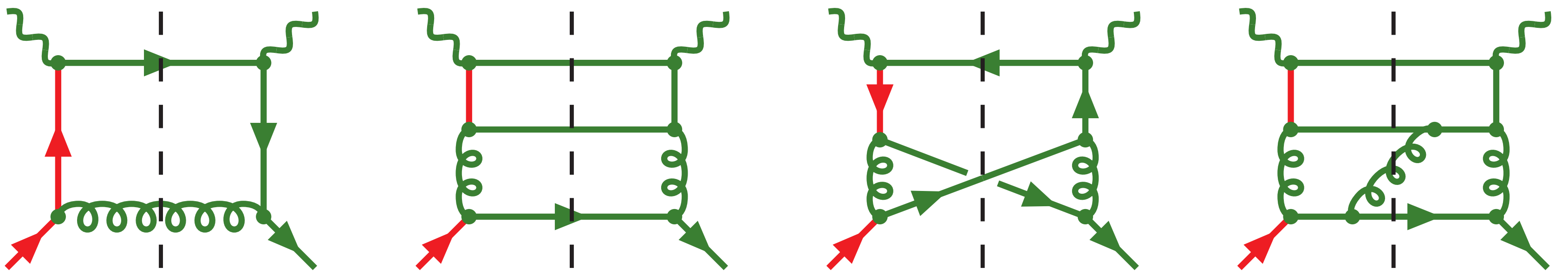

$P^{\mathrm{v}}_{\mathrm{qq}} = \mathcal{O}(\alpha_{\mathrm{s}})$ $\qquad$ $P^{\mathrm{s}}_{\mathrm{qq}},\, P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \,:\, \alpha_{\mathrm{s}}^2$ $\qquad$ $P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} \,:\, \alpha_{\mathrm{s}}^2$ $\qquad$ $P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \neq P^{\mathrm{s}}_{\mathrm{qq}} \,:\, \alpha_{\mathrm{s}}^3$

Three types of difference (non-singlet) combinations: $P^{\pm}_{\mathrm{ns}} = P^{\mathrm{v}}_{\mathrm{qq}} \pm P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}}\,,\, P^{\mathrm{v}}_{\mathrm{ns}}$

Evolution of gluon and flavour-singlet quark distributions g and q_{s}

$$q_{\mathrm{s}} = \sum_{r=1}^{n_f} (q_r + \bar{q}_r)\,, \quad \frac{d}{d\ln\mu^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

with (ps = ‘pure singlet’) $\quad P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + n_f (P^{\mathrm{s}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\bar{\mathrm{q}}\mathrm{q}}) \equiv P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$

Quark coefficient fct’s: analogous decomposition $C_{a,\mathrm{q}\{\bar{\mathrm{q}}\}} = C_{a,\mathrm{ns}} + C_{a,\mathrm{ps}}$

$\overline{\text{MS}}$ splitting functions at large x / large N

Mellin trf. $f(N) = \int_0^1 dx\, (x^{N-1}\{-1\})\, f(x)_{\{+\}}$: M-convolutions $\longrightarrow$ products

$$\frac{\ln^n(1-x)}{(1-x)_+} \overset{\mathrm{M}}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \ldots\,, \quad \ln^n(1-x) \overset{\mathrm{M}}{=} \frac{(-1)^n}{N} \ln^n N + \ldots$$

Diagonal splitting functions: no higher-order enhancement at N^0, N^{-1}

$$P^{(\ell-1)}_{\mathrm{qq/gg}}(N) = A^{(\ell)}_{\mathrm{q/g}} \ln N + B^{(\ell)}_{\mathrm{q/g}} + C^{(\ell)}_{\mathrm{q/g}} \frac{1}{N} \ln N + \ldots\,, \quad A_{\mathrm{g}} = C_A/C_F\, A_{\mathrm{q}}$$

. . . ; Korchemsky (89); MVV(04); Dokshitzer, Marchesini, Salam (05)

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Off-diagonal: double-log behaviour, colour structure with $C_{AF} = C_A - C_F$

$$C_F^{-1} P^{(\ell)}_{\mathrm{gq}} \,/\, n_f^{-1} P^{(\ell)}_{\mathrm{qg}} = \frac{1}{N} \ln^{2\ell} N \ \# C^{l}_{AF}$$
$$+ \frac{1}{N} \ln^{2\ell-1} N\, (\# C_{AF} + \# C_F + \# n_f)\, C^{l-1}_{AF} + \dots$$

Double logs $\ln^n N$, $\ell+1 \le n \le 2\ell$ vanish for $C_F = C_A$ ($\rightarrow$ SUSY case)

$\overline{\text{MS}}$ coefficient functions at large x / large N

'Diagonal' [$\mathcal{O}(1)$] coeff. fct's for $F_{2,3,\phi}$ in DIS, $F_{T,A,\phi}$ in SIA, $F_{\rm DY} = \frac{1}{\sigma_0}\frac{d\sigma_{\rm q\bar{q}}}{dQ^2}$

$$C^{(\ell)}_{2,{\rm q}/\phi,{\rm g}/\ldots} \;=\; \#\;\ln^{2\ell} N + \,\ldots\; + N^{\,-1}(\#\;\ln^{2\ell-1} N + \ldots) \,+\, \ldots$$

N^0 parts: threshold exponentiation   Sterman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log (N^{3}LL) accuracy - mod. $A^{(4)}$
$\Rightarrow$ highest seven (DIS, SIA), six (DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05) (+ SCET papers, from 06), SIA: Blümlein, Ravindran (06); MV, arXiv:0908.2746 (PLB)

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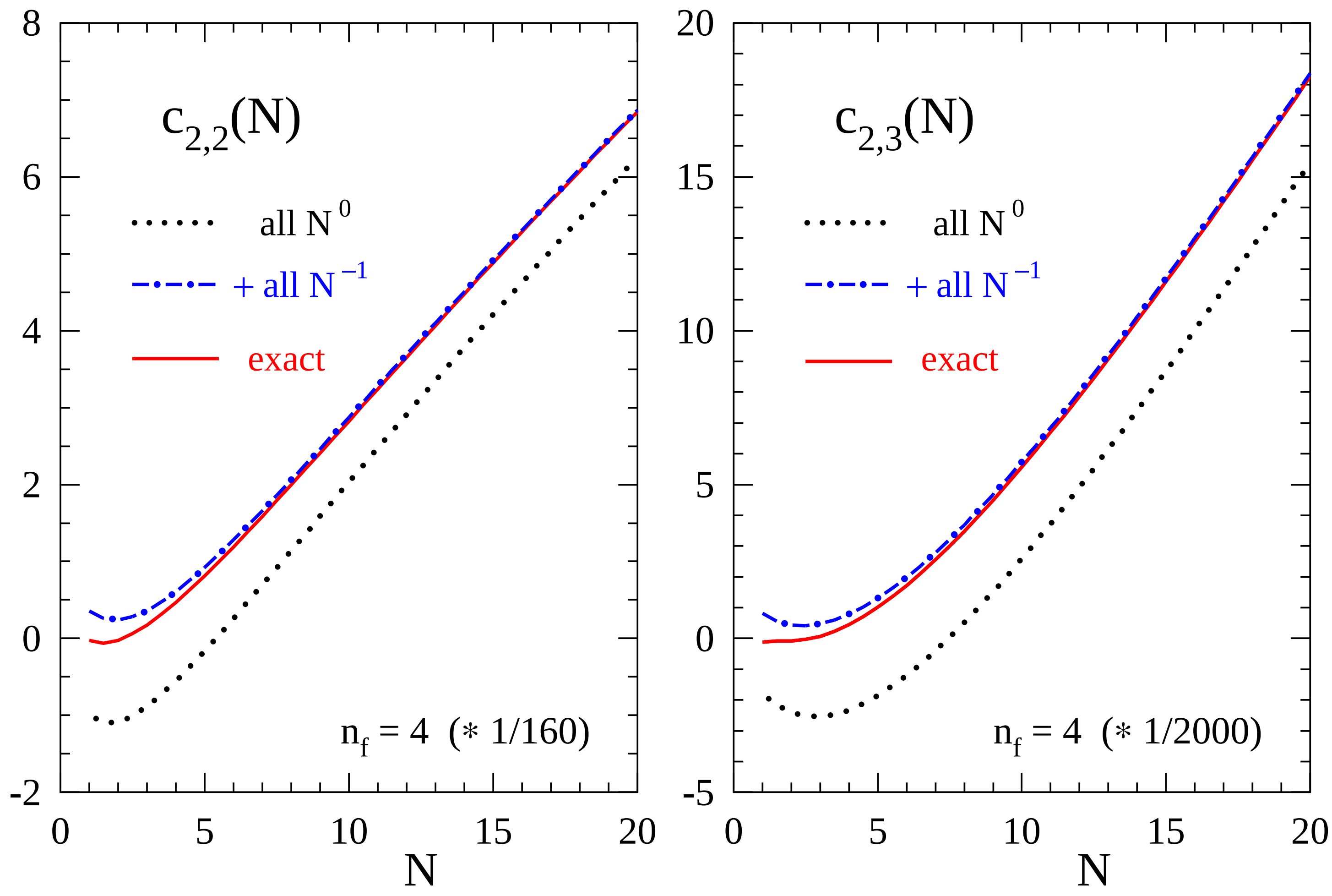
'Off-diagonal' [$\mathcal{O}(\alpha_s)$] quantities: leading N^{-1} double logarithms

$$C^{(\ell)}_{\phi,\mathrm{q}/2,\mathrm{g}/\ldots} \;=\; N^{-1}(\#\ \ln^{2\ell-1} N + \ldots) + \ldots$$

Longitudinal DIS/SIA structure functions [convention: ℓ = order in α_s – 1]

$$C^{(\ell)}_{\mathrm{L,q}} \;=\; N^{-1}(\#\ \ln^{2\ell} N + \ldots) + \ldots\;,\quad C^{(\ell)}_{\mathrm{L,g}} \;=\; N^{-2}(\#\ \ln^{2\ell} N + \ldots) + \ldots$$

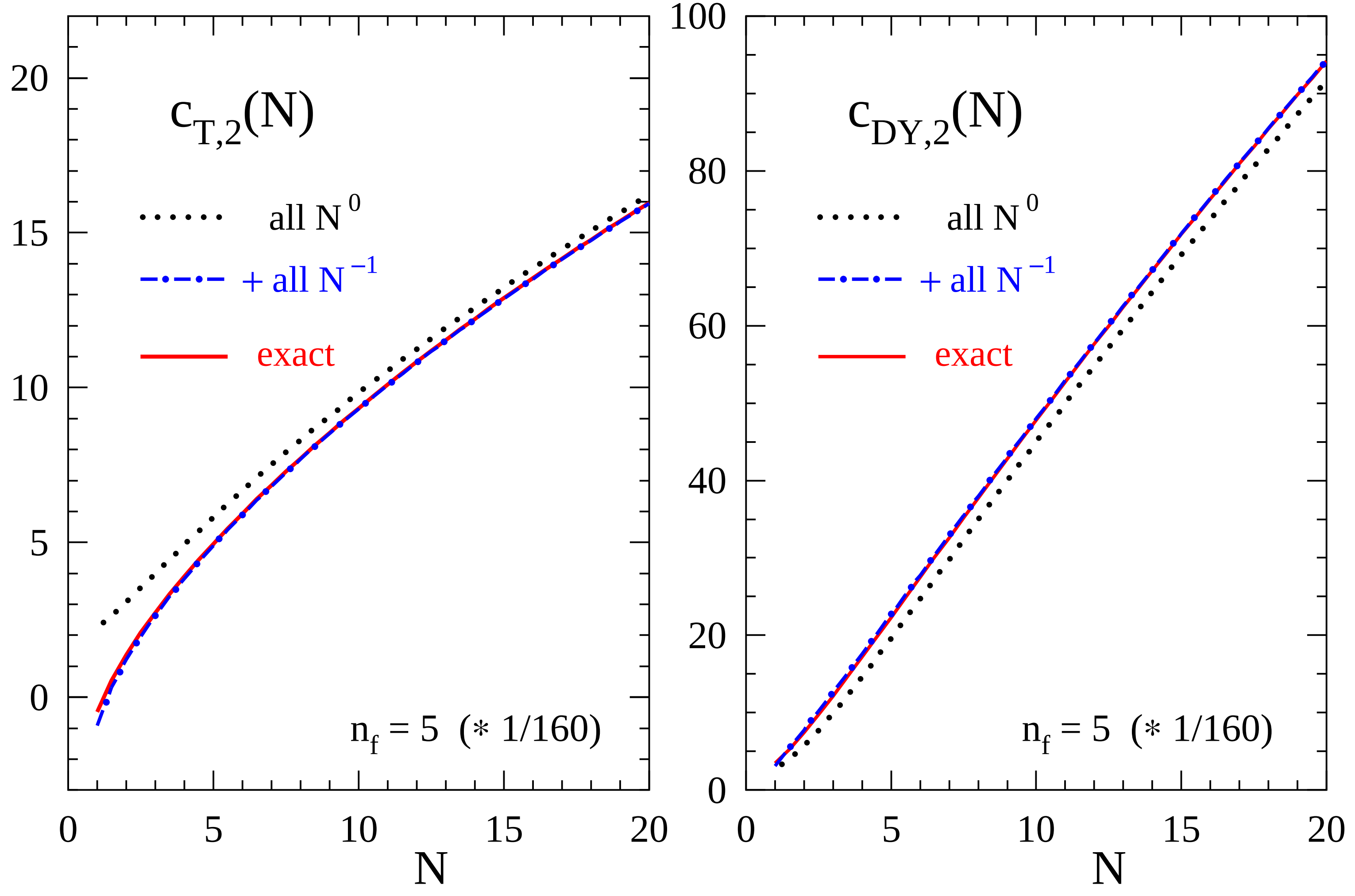
Second- and third-order N-space $C_{2,\mathrm{ns}}$ in DIS



N^{-1} terms relevant over full range shown, $\mathcal{O}(N^{-2})$ sizeable only at $N < 5$

Sum of $N^{-1} \ln^k N$ looks almost constant: half of maximum only at $N \simeq 150$

Second-order non-singlet C_T in SIA and C_{DY}



DIS → SIA → DY : increase of the N^0 terms, N^{-1} corrections less important

$\overline{\text{MS}}$ splitting functions at small $x\,/\,N\!\rightarrow 1$ or 0

Logs in x-space $\Leftrightarrow$ poles in N-space, $\quad x^a \ln^n x \;\stackrel{\text{M}}{=}\; \dfrac{(-1)^n n!}{(N+a)^{n+1}}$

Space-like case, non-singlet: no x^{-1} terms, leading x^0 double logarithms :

LL: Kirschner, Lipatov (83); Blümlein, A.V. (95)

Singlet quantities: dominant x^{-1} terms single-log enhanced

$$P_{\rm ij}^{(\ell)S} \;=\; x^{-1}(\,\#\ln^{\ell-\delta_{\rm iq}} x + \ldots\,) + (\,\#\ln^{2\ell} x + \ldots\,) \,+\, \ldots$$

x^{-1} part: BFKL (77/78); Jaroszewicz (82); Catani, Fiorani, Marchesini (89); Catani, Hautmann (94); . . . , Fadin, Lipatov; Camici, Ciafaloni (98)

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Timelike case: huge x^{-1} double logarithms

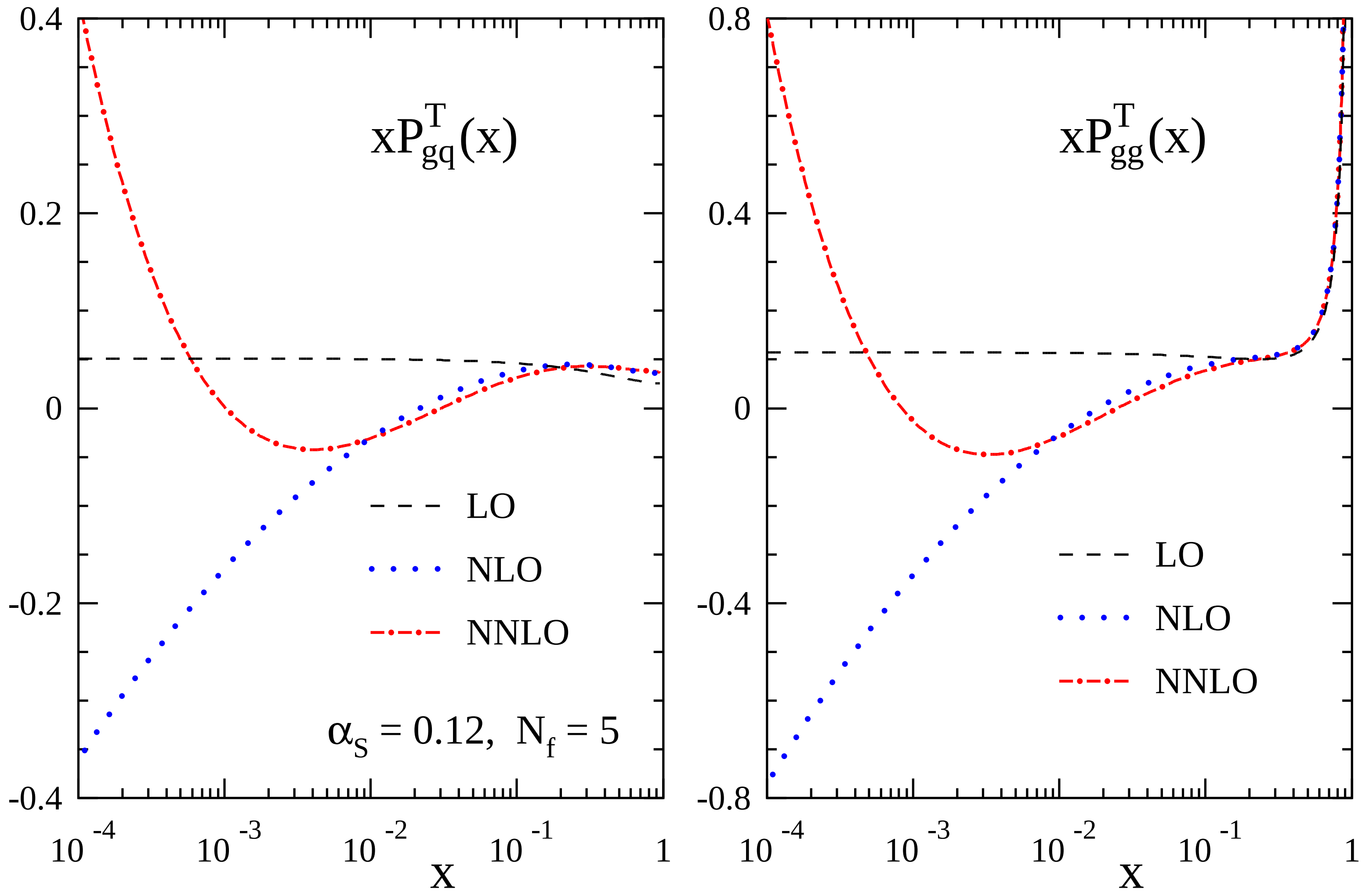
$$P_{\rm ij}^{(\ell)T} \;=\; x^{-1}(\#\;\ln^{2\ell-\delta_{\rm iq}} x + \ldots\,) + (\#\;\ln^{2\ell} x + \ldots\,) \;+\; \ldots$$

LL: Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82). NLL: Mueller (83)

– but latter not in $\overline{\text{MS}}$, see Albino, Bolzoni, Kniehl, Kotikov (2011)

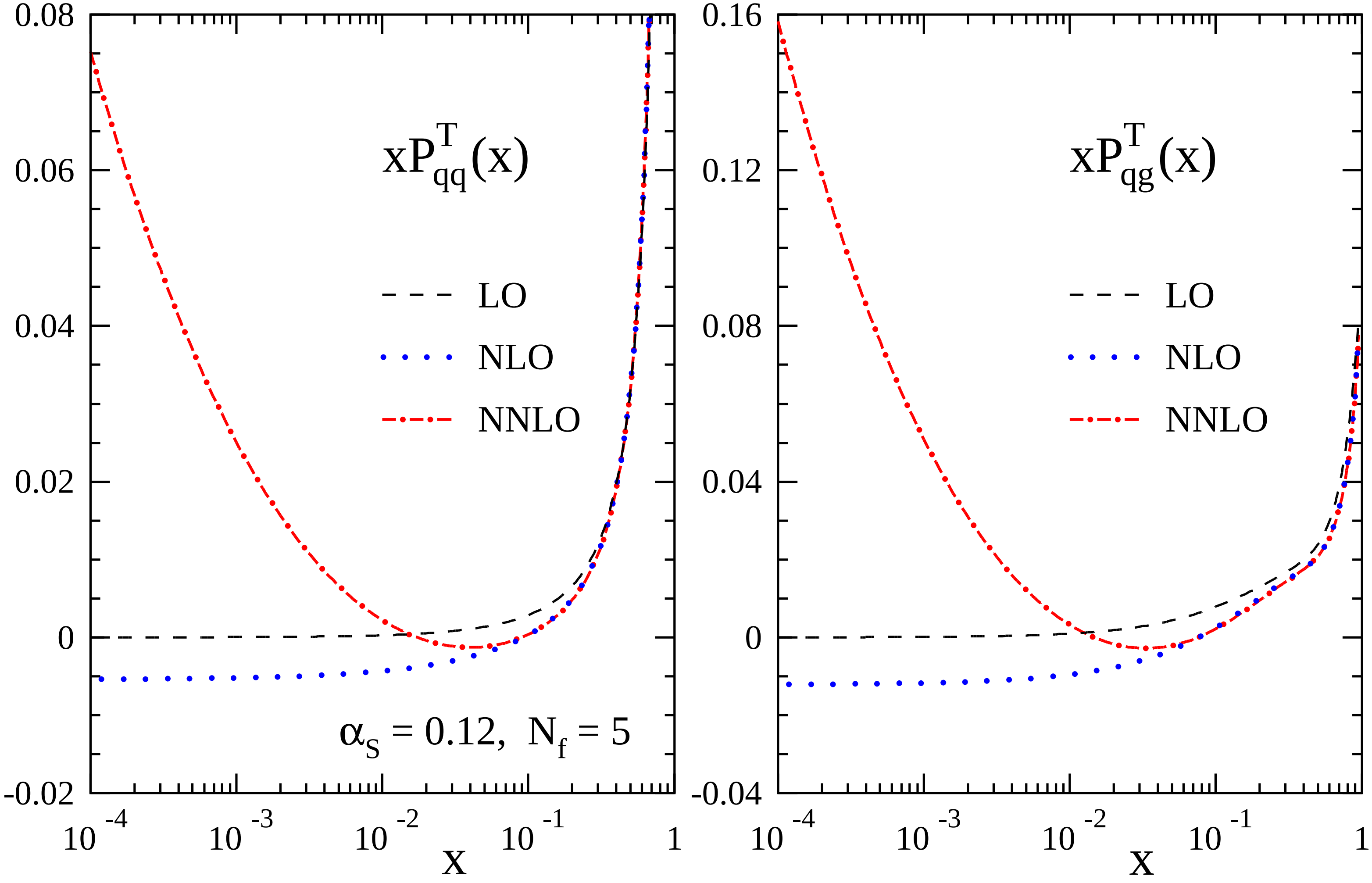
Behaviour of gauge-boson exchange coefficient functions analogous

NNLO approximations for $P_{gi}^{T}(x, \alpha_s)$



NLO/NNLO: terms up to $x^{-1}\ln^2 x$ / $x^{-1}\ln^4 x$. Unstable at $x \lesssim 0.005$

NNLO approximations for $P^T_{\mathrm{qi}}(x, \alpha_s)$



NLO: no $x^{-1} \ln x$ terms. NNLO: up to $x^{-1} \ln^3 x$. Unstable at $x \lesssim 0.02$

Threshold logarithms before factorization (I)

Unfactorized partonic structure functions in $D = 4-2\varepsilon$ dimensions

$$T_{a,j} = \widetilde{C}_{a,i}\, Z_{ij}\ , \quad -\gamma \equiv P = \frac{dZ}{d\ln Q^2}\, Z^{-1}\ , \quad \frac{da_s}{d\ln Q^2} = -\varepsilon a_s + \beta_{D=4}$$

a_s^n: $\varepsilon^{-n} \ldots \varepsilon^{-2}$: lower-order terms, ε^{-1}: n-loop splitting functions + ...,
ε^0: n-loop coefficient fct's + ..., ε^k, $0 < k < l$: required for order $n+l$

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N^0 and N^{-1} transition functions Z to next-to-leading log (NLL) accuracy

$$Z\Big|_{a_s^n} = \frac{1}{\varepsilon^n}\frac{\gamma_0^{n-1}}{n!}\left[\gamma_0 - \frac{\beta_0}{2}n(n-1)\right] + \sum_{\ell=1}^{n-1}\frac{1}{\varepsilon^{n-\ell}}\sum_{k=1}^{n-\ell-1}\gamma_0^{n-\ell-k-1}\gamma_\ell\gamma_0^k\,\frac{(\ell+k)!}{n!\,\ell!}$$
$$-\frac{\beta_0}{2}\sum_{\ell=1}^{n-2}\frac{1}{\varepsilon^{n-\ell}}\sum_{k=1}^{n-\ell-2}\gamma_0^{n-\ell-k-2}\gamma_\ell\gamma_0^k\,\frac{(\ell+k)!}{n!\,\ell!}\left(n(n-1)-\ell(\ell+k+1)\right)$$

$+$ NNLL contributions (explicit expressions) $+ \ \ldots$

$\varepsilon^{-n+\ell}$ off-diagonal entries: contributions up to $N^{-1}\ln^{n+\ell-1}N$

Diagonal cases: γ_0 only for N^0 part, second term with $\ell = 1$ for N^{-1} NLL

Threshold logarithms before factorization (II)

D-dimensional coefficient functions $\widetilde{C}_a$: finite for $\varepsilon \to 0$

$$\widetilde{C}_{a,i} = 1_{(\text{diagonal cases})} + \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} a_{\mathrm{s}}^n \, \varepsilon^\ell \, c_{a,i}^{(n,\ell)}$$

$c_{a,i}^{(n,\ell)}$: ℓ additional factors $\ln N$ relative to $c_{a,i}^{(n,0)} \equiv c_{a,i}^{(n)}$ discussed above

Full N$^{\mathrm{m}}$LO calc. of $T_{a,j}$: highest $m+1$ powers of ε^{-1} to all orders in α_{s}

Extension to all powers of ε: all-order resummation of highest $m+1$ logs

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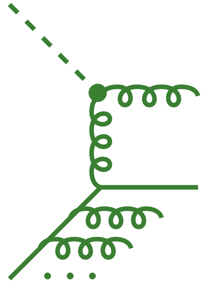
Example: Leading-log (LL) $1/N$ terms of $T_{\phi,\rm q}^{(n)}$ and $T_{2,\rm g}^{(n)}$, with $L \equiv \ln N$

$$\frac{1}{C_F} T_{\phi,\rm q}^{(n)} = \frac{1}{n_f} T_{2,\rm g}^{(n)} = \frac{L^{n-1}}{N \varepsilon^n} \sum_{k=0}^{\infty} (\varepsilon L)^k \mathcal{L}_{n,k} \left(C_F^{n-1} + C_F^{n-2} C_A + \ldots + C_A^{n-1} \right)$$

to all orders in ε (calc. + D-dim. structure), with same coefficients $\mathcal{L}_{n,k}$

$\Rightarrow$ all-order relation for one colour structure of either amplitude sufficient

All-order off-diagonal leading-log amplitudes



$$\left. T^{(n)}_{\phi,\mathrm{q}} \right|_{C_F \text{ only}} \overset{\mathrm{LL}}{=} \frac{1}{n} T^{(1)}_{\phi,\mathrm{q}} \underbrace{T^{(n-1)}_{2,\mathrm{q}}}_{\frac{1}{(n-1)!}(T^{(1)}_{2,\mathrm{q}})^{n-1}} \overset{\mathrm{LL}}{=} \frac{1}{n!} T^{(1)}_{\phi,\mathrm{q}} \, (T^{(1)}_{2,\mathrm{q}})^{n-1}$$

Three-loop diagram calculation + $P^{(3)}_{\mathrm{gq}} \overset{\mathrm{LL}}{=} 0$ + general mass factorization:

first four powers in ε known at any order. Rest $\longrightarrow$ higher-order predictions

$$\left. T_{\phi,\mathrm{q}} \right|_{C_F \text{ only}} \overset{\mathrm{LL}}{=} T^{(1)}_{\phi,\mathrm{q}} \, \frac{\exp(a_{\mathrm{s}} T^{(1)}_{2,\mathrm{q}}) - 1}{T^{(1)}_{2,\mathrm{q}}}$$

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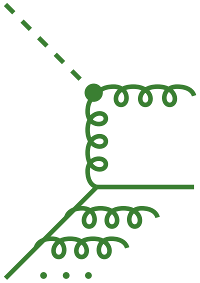

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Exact D-dimensional leading-log expressions for the one-loop amplitudes

$$T^{(1)}_{\phi,\mathrm{q}} \overset{\mathrm{LL}}{=} -2C_F\, \frac{1}{\varepsilon}(1-x)^{-\varepsilon} \overset{\mathrm{M}}{=} -\frac{2C_F}{N}\, \frac{1}{\varepsilon}\exp(\varepsilon \ln N)$$

$$T^{(1)}_{2,\mathrm{q}} \overset{\mathrm{LL}}{=} -4C_F\, \frac{1}{\varepsilon}(1-x)^{-1-\varepsilon} + \text{virtual} \overset{\mathrm{M}}{=} 4C_F\, \frac{1}{\varepsilon^2}(\exp(\varepsilon \ln N) - 1)$$

$\Rightarrow$ leading-log expression for $T_{\phi,\mathrm{q}}$ and $T_{2,\mathrm{g}}$ completely determined

Leading-log splitting and coefficient functions

Expansions and iterative mass factorization to ‘any’ order [done in FORM]

$\Rightarrow$ All-order expressions for LL off-diagonal splitting and coefficient fct’s

$$P_{\rm qg}^{\rm LL}(N,\alpha_{\rm s}) \;=\; \frac{n_f}{N}\,\frac{\alpha_{\rm s}}{2\pi}\,\sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,\tilde{a}_{\rm s}^{\;n}\;, \qquad \tilde{a}_{\rm s} = \frac{\alpha_{\rm s}}{\pi}\,(C_A - C_F)\,\ln^2 N$$

Bernoulli numbers B_n: zero for odd $n \geq 3$ $\;\Rightarrow\;$ $P_{\rm gq}^{(3)}(N) \stackrel{\rm LL}{=} 0$ not accidental

$$B_0 = 1\,,\; B_1 = -\frac{1}{2}\,,\; B_2 = \frac{1}{6}\,,\; B_4 = -\frac{1}{30}\,,\; B_6 = \frac{1}{42}\,,\;\ldots\,,\; B_{12} = -\frac{691}{2730}\,,\;\ldots$$

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$$C_{2,\rm g}^{\rm LL} \;=\; \frac{1}{2N\ln N}\,\frac{n_f}{C_A - C_F}\,\left\{\exp(2C_F\,a_{\rm s}\ln^2 N)\,\mathcal{B}_0(\tilde{a}_{\rm s}) - \exp(2C_A\,a_{\rm s}\ln^2 N)\right\}$$

$\exp(\ldots)$: LL soft-gluon exponentials   Parisi; Curci, Greco; Amati et al. (80)

$$\mathcal{B}_0(x) \;=\; \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,x^n$$

$P_{\rm gq}^{\rm LL}$, $C_{\phi,\rm q}^{\rm LL}$: same functions but with $C_F \leftrightarrow C_A$ (also in $\tilde{a}_{\rm s}$), then $n_f \rightarrow C_F$

First properties of the new $\mathcal{B}$-functions

Relation between even-n Bernoulli numbers and the Riemann ζ-function

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\,\zeta_{2n}\left(\frac{x}{2\pi}\right)^{2n}$$

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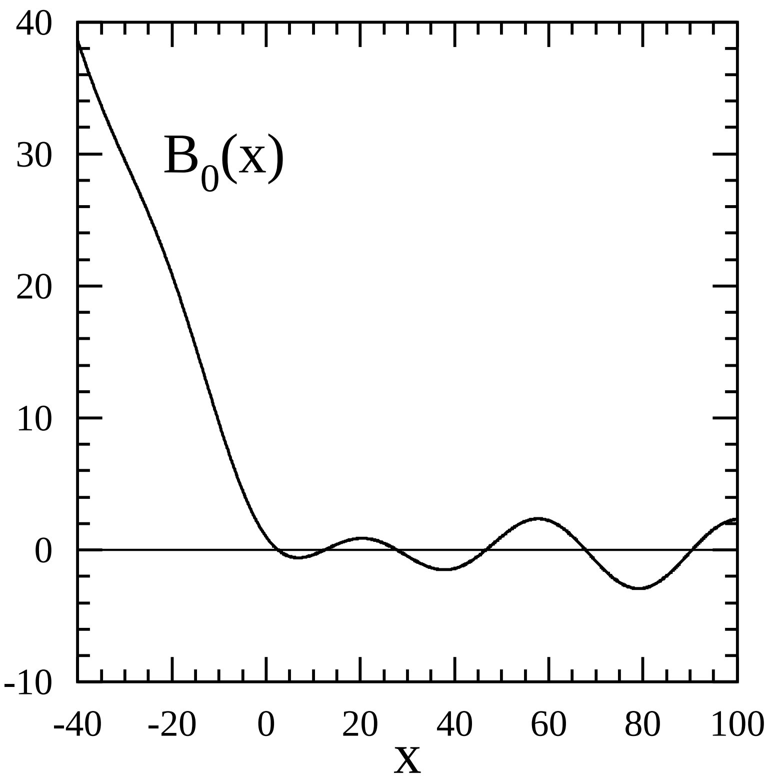

Further $\mathcal{B}$-functions for later use

$$\mathcal{B}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!}\,x^n$$

$$\mathcal{B}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!}\,x^n$$

Relations to $\mathcal{B}_0(x)$

$$\frac{d^k}{dx^k}(x^k\mathcal{B}_k) = \mathcal{B}_0\,, \quad x^k\frac{d^k}{dx^k}\mathcal{B}_0 = \mathcal{B}_{-k}$$

D-dim. structure of unfactorized observables

Maximal phase space for deep-inelastic scattering/semi-incl. annihilation

$\mathbf{NLO}: \; 2 \to 2 \,/\, 1 \to 1+2 \qquad (1-x)^{-1-\varepsilon} \; x^{\cdots} \int_0^1$ **one other variable**

$\mathbf{N^2LO}: \; 2 \to 3 \,/\, 1 \to 1+3 \qquad (1-x)^{-1-2\varepsilon} \; x^{\cdots} \int_0^1$ **four other variables**

$\mathbf{N^3LO}: \; 2 \to 4 \,/\, 1 \to 1+4 \qquad (1-x)^{-1-3\varepsilon} \; x^{\cdots} \int_0^1$ **seven other variables**

$\dots$

N^{2}LO: Matsuura, van Neerven (88), Rijken, vN (95), N$^{n\geq 3}$LO, indirectly: MV[V] (05)

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$\mathbf{N^2LO}: \; 2 \to 3 \;/\; 1 \to 1+3$ $\qquad (1-x)^{-1-2\varepsilon} \; x^{\cdots} \int_0^1$ four other variables

$\mathbf{N^3LO}: \; 2 \to 4 \;/\; 1 \to 1+4$ $\qquad (1-x)^{-1-3\varepsilon} \; x^{\cdots} \int_0^1$ seven other variables

$\ldots$

$\mathrm{N^2LO}$: Matsuura, van Neerven (88), Rijken, vN (95), $\mathrm{N}^{n\geq 3}\mathrm{LO}$, indirectly: MV[V] (05)

Purely real contributions to unfactorized structure functions

$$T_{a,j}^{(n)\mathrm{R}} = (1-x)^{-1-n\varepsilon} \sum_{\xi=0} (1-x)^{\xi} \, \frac{1}{\varepsilon^{2n-1}} \left\{ R_{a,j,\xi}^{(n)\mathrm{LL}} + \varepsilon R_{a,j,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Mixed contributions ($2 \to r+1$ with $n-r$ loops in DIS)

$$T_{a,j}^{(n)\mathrm{M}} = \sum_{l=r}^{n} (1-x)^{-1-l\varepsilon} \sum_{\xi=0} (1-x)^{\xi} \, \frac{1}{\varepsilon^{2n-1}} \left\{ M_{a,j,l,\xi}^{(n)\mathrm{LL}} + \varepsilon M_{a,j,l,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Purely virtual part (diagonal cases, $\eta = 0$ present): $\gamma^* qq$, Hgg form factors

$$T_{a,j}^{(n)\mathrm{V}} = \delta(1-x) \, \frac{1}{\varepsilon^{2n}} \left\{ V_{a,j}^{(n)\mathrm{LL}} + \varepsilon V_{a,j}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Resulting resummation of large-x double logs

KLN cancellation between purely real, mixed and purely virtual contributions

$$T_{a,j}^{(n)} = T_{a,j}^{(n)\mathrm{R}} + T_{a,j}^{(n)\mathrm{M}} \left(+ T_{a,j}^{(n)\mathrm{V}} \right) = \frac{1}{\varepsilon^n} \left\{ T_{a,j}^{(n)0} + \varepsilon T_{a,j}^{(n)1} + \ldots \right\}$$

$\Rightarrow$ **Up to $n-1$ relations between the coeff's of $(1-x)^{-1-l\varepsilon}$, $l = 1, \ldots, n$**

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Log expansion: N^{k}LL higher-order coefficients completely fixed, if first $k+1$ powers of ε known to all orders – provided by N^{k}LO calculation, see above

Present situation: (a) N^{3}LO for non-singlet $F_{a \neq L}$ in DIS – recall DMS (05)
(b) N^{2}LO for SIA, non-singlet F_L in DIS, and singlet DIS

$\Rightarrow$ resummation of the (a) four and (b) three highest $N^{-1} \ln^k N$ terms to all orders in α_s: consistent with, and extending, our previous results

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Soft-gluon exponentiation of the $(1-x)^{-1}/N^0$ diagonal coefficient functions:

$(1-x)^{-1-\varepsilon}, \ldots, (1-x)^{-1-(n-1)\varepsilon}$ at order n: products of lower-order quantities

$\Rightarrow$ N^{n}LO [+$A^{(n+1)}$] $\rightarrow$ N^{n}LL exponentiation; $2n[+1]$ highest logs predicted

NS results, off-diagonal splitting fct's and $C_{L,\mathrm{g}}$

NS: physical-kernel results confirmed and extended by fourth log for $c_{a,\mathrm{ns}}^{(n\geq 4)}$

also: Grunberg (2010)

Off-diagonal splitting functions

$$\tilde{a}_{\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{\pi}\,(C_A - C_F)\,\ln^2 \widetilde{N}$$

$$N P_{\mathrm{qg}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;=\; 2 a_{\mathrm{s}} n_f \,\mathcal{B}_0(\tilde{a}_{\mathrm{s}}) + a_{\mathrm{s}}^2 \ln\widetilde{N}\, n_f \left\{ (6C_F - \beta_0)\left(\frac{2}{\tilde{a}_{\mathrm{s}}}\mathcal{B}_{-1}(\tilde{a}_{\mathrm{s}}) + \mathcal{B}_1(\tilde{a}_{\mathrm{s}})\right) + \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-2}(\tilde{a}_{\mathrm{s}}) \right\}$$

$$N P_{\mathrm{gq}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;=\; 2 a_{\mathrm{s}} C_F \,\mathcal{B}_0(-\tilde{a}_{\mathrm{s}}) \;+\; a_{\mathrm{s}}^2 \ln\widetilde{N}\, C_F \left\{ (12 C_F - 6\beta_0)\,\frac{1}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-1}(-\tilde{a}_{\mathrm{s}}) - \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-2}(-\tilde{a}_{\mathrm{s}}) + \,(14 C_F - 8 C_A - \beta_0)\,\mathcal{B}_1(-\tilde{a}_{\mathrm{s}}) \right\}$$

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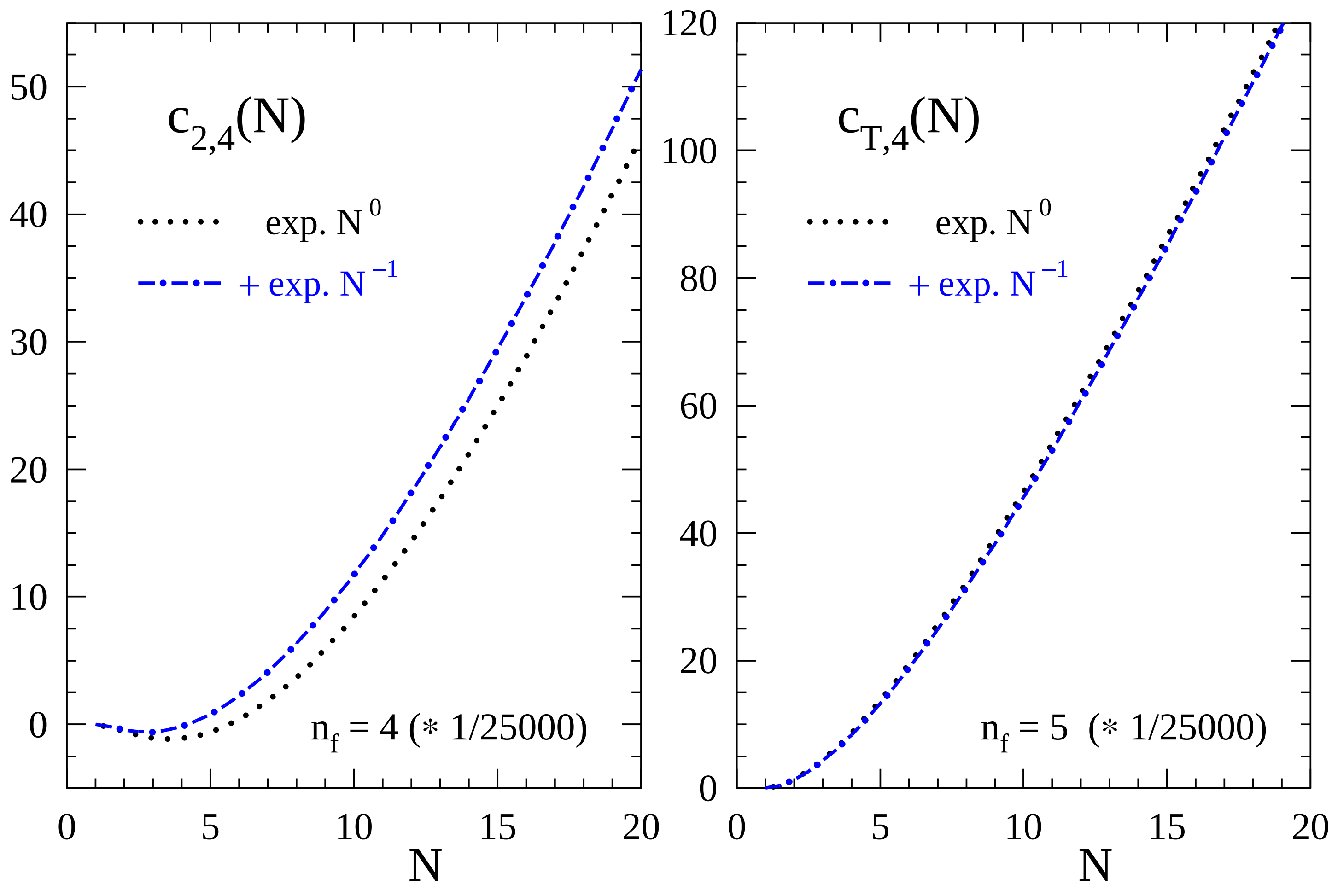
Gluon contribution to F_L – 'non-singlet' $C_F = 0$ part done before MV (09)

$$N^2 C_{L,\mathrm{g}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;=\; 8 a_{\mathrm{s}} n_f \exp(2 C_A a_{\mathrm{s}} \ln^2 \widetilde{N}) \;+\; 4 a_{\mathrm{s}} C_F N C_{2,\mathrm{g}}^{\mathrm{LL}}(N,\alpha_{\mathrm{s}}) + 16 a_{\mathrm{s}}^2 \ln\widetilde{N}\, n_f \left\{ 4C_A - C_F + \frac{1}{3} a_{\mathrm{s}} \ln^2\widetilde{N}\, C_A \beta_0 \right\} \exp(2 C_A a_{\mathrm{s}} \ln^2 \widetilde{N})$$

NNLL terms known to 'any' order, but no closed expressions (except $C_{L,\mathrm{g}}$)

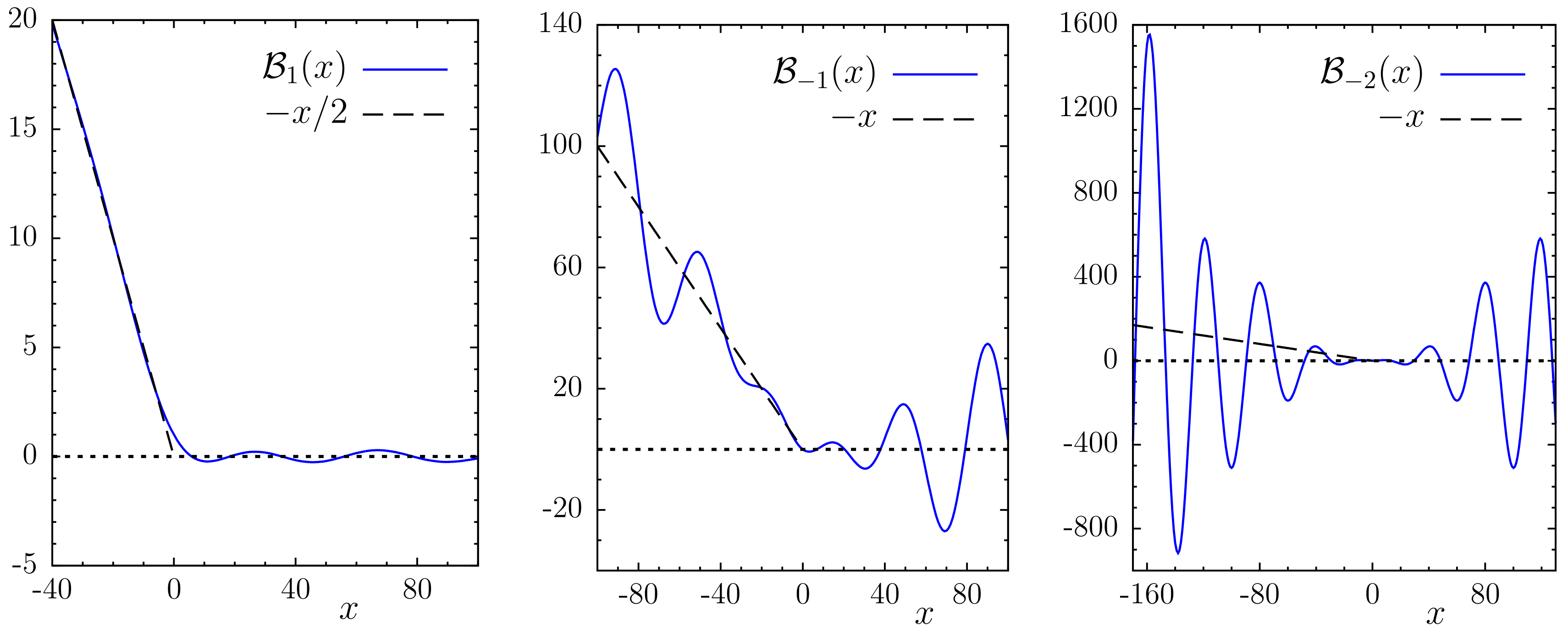
Fourth-order C_2 (DIS) and C_T (SIA) at large N



Exp. N^0: 7 of 8 logs, exp. N^{-1}: 4 of 7 logs $\Rightarrow$ large-x higher-twist analyses

N^{-1} contributions again relevant for F_2, but small for F_T at least at $N > 5$

NLL: $\mathcal{B}$-functions with index unequal zero



$x > 0$: all functions $\mathcal{B}_k(x)$ oscillate about $y = 0$

$x < 0$: oscillations about $y = -\frac{x}{(k+1)!}$ for $k \geq 0$ and $y = -x$ for $k < 0$

Amplitudes increase very rapidly with decreasing k

Oscillation of $\mathcal{B}_0$ continuous (much more irregularly) to very large x

D. Broadhurst, private communication

Resummed gluon coefficient function for F_2

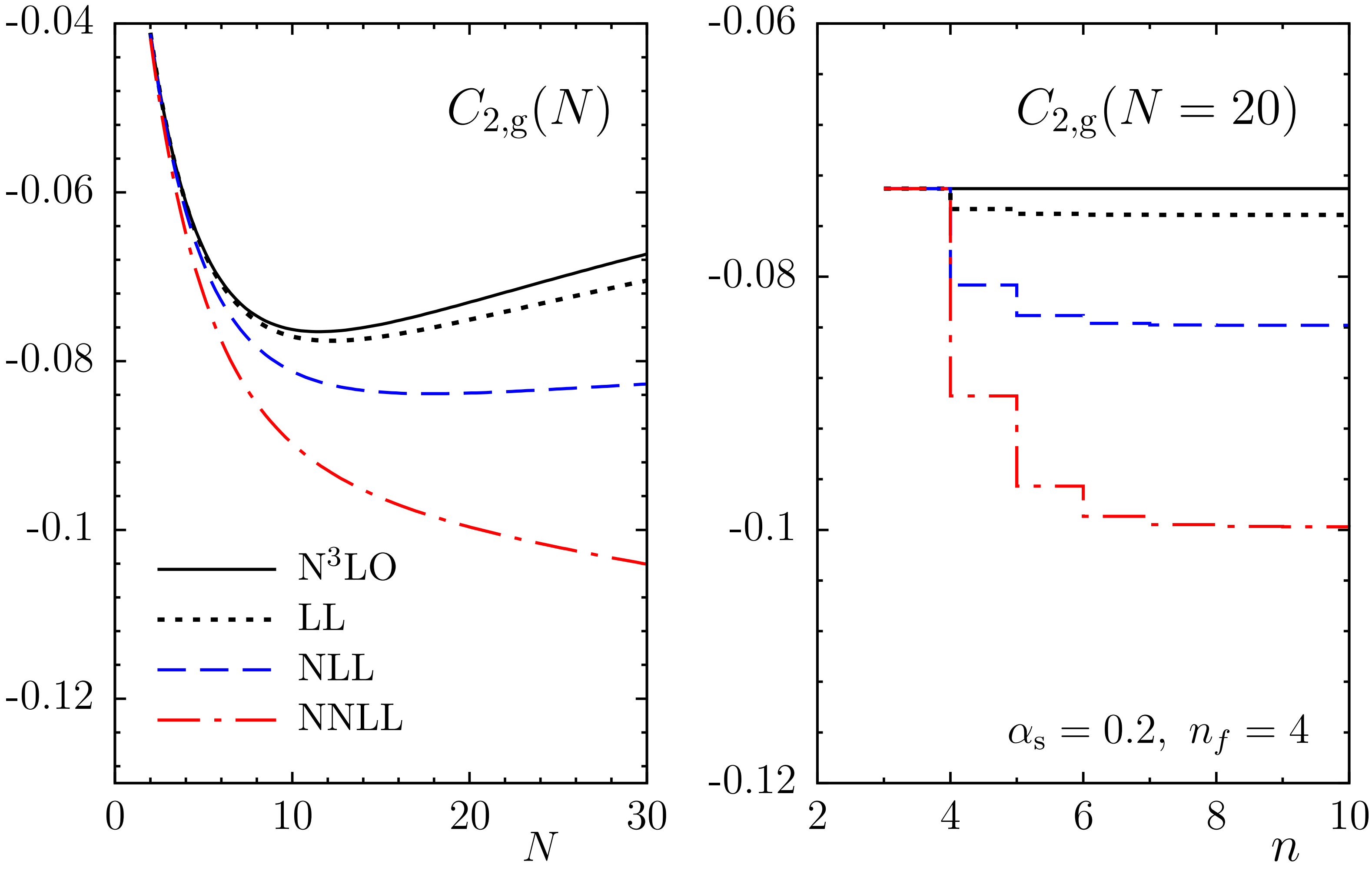
$$
\begin{aligned}
N C_{2,\mathrm{g}}(N,\alpha_{\mathrm{s}}) \;=&\\
& \frac{1}{2\ln\widetilde{N}}\,\frac{n_f}{C_A - C_F}\left[\exp(2a_{\mathrm{s}}C_F\ln^2\widetilde{N})\mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - \exp(2a_{\mathrm{s}}C_A\ln^2\widetilde{N})\right]\\
& -\frac{1}{8\ln^2\widetilde{N}}\,\frac{n_f(3C_F-\beta_0)}{(C_A - C_F)^2}\left[\exp(2a_{\mathrm{s}}C_F\ln^2\widetilde{N})\mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - \exp(2a_{\mathrm{s}}C_A\ln^2\widetilde{N})\right]\\
& -\frac{a_{\mathrm{s}}}{4}\,\frac{n_f}{C_A - C_F}\,\exp\left(2a_{\mathrm{s}}C_A\ln^2\widetilde{N}\right)\left(8C_A + 4C_F - \beta_0\right)\\
& -\frac{a_{\mathrm{s}}}{4}\,\frac{n_f}{C_A - C_F}\,\exp\left(2a_{\mathrm{s}}C_F\ln^2\widetilde{N}\right)\Big[-6C_F\mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - (6C_F-\beta_0)\mathcal{B}_1(\tilde{a}_{\mathrm{s}})\\
& \qquad -(12C_F-4\beta_0)\,\frac{1}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-1}(\tilde{a}_{\mathrm{s}}) - \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-2}(\tilde{a}_{\mathrm{s}})\Big]\\
& -\frac{a_{\mathrm{s}}^2}{3}\,\beta_0\ln^2\widetilde{N}\,\frac{n_f}{C_A - C_F}\left[C_A\exp(2a_{\mathrm{s}}C_A\ln^2\widetilde{N}) - C_F\exp(2a_{\mathrm{s}}C_F\ln^2\widetilde{N})\mathcal{B}_0(\tilde{a}_{\mathrm{s}})\right]
\end{aligned}
$$

$+$ **known NNLL contributions (tables)** $+\;\ldots$

$C_{\phi,\mathrm{q}}$ analogous. Analytic forms identified via the physical kernel for (F_2, F_ϕ)

Resummed timelike splitting and coefficient functions: same structure

Numerical illustration of $C_{2,\mathrm{g}}$



NNLL terms dominate $\Rightarrow$ impact of high orders presumably underestimated

About 35% correction at $N = 20$, 4^{th}-order coefficient $\approx$ Padé estimate

Small-x resummation via unfactorized SIA

Phase-space integrations: $x^{a\varepsilon}$ terms analogous to $(1-x)^{b\varepsilon}$ large-x factors

2nd order: **Matsuura, van Neerven (88), Rijken, vN (95)**

Decomposition of the D-dim. partonic fragmentation functions for $a = T, \phi$

$$\widehat{F}_{a,\mathrm{g}}^{(n)} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\varepsilon} \left\{ A_{a,\mathrm{g}}^{(\ell,n)} + \varepsilon B_{a,\mathrm{g}}^{(\ell,n)} + \varepsilon^2 C_{a,\mathrm{g}}^{(\ell,n)} + \ldots \right\}$$

Leading log: terms of the form $x^{-1} \ln^{n+\delta-1} x$ at all orders $\varepsilon^{-n+\delta}$ with $\delta = 0,\ 1,\ 2,\ \ldots$, and $\widehat{F}_{a,\mathrm{g}}^{(n)}$ is decomposed into n contributions of the form

$$\varepsilon^{-2n+1}\, x^{-1-k\,\varepsilon} \;=\; \varepsilon^{-2n+1}\, x^{-1} \left[1 \;-\; k\,\varepsilon \ln x \;+\; \frac{1}{2}(k\,\varepsilon)^2 \ln^2 x \;+\; \ldots \right], \qquad k = 2,\ 4,\ \ldots,\ 2n$$

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$n-1$ KLN-type cancellations – $\widehat{F}_{a,\mathrm{g}}^{(n)}$ starts at order $1/\varepsilon^n$ – plus 3 constraints from the NNLO results $\Rightarrow$ $n+2$ linear equations for n coefficients $A_{a,\mathrm{g}}^{(\ell,n)}$

Thus: $\mathrm{N^nLO}$ known $\Rightarrow$ highest $n+1$ ($\mathrm{N^nLL}$) double logs fixed at all orders

'All-order' mass factorization: NNLL timelike splitting & coefficient functions

Resumed splitting and coefficient functions

$$\frac{C_A}{C_F}\, P^{T}_{\rm gq}(N,\alpha_{\rm s}) \stackrel{\rm LL}{=} P^{T}_{\rm gg}(N,\alpha_{\rm s}) \stackrel{\rm LL}{=} \frac{1}{4}(N-1)\left\{(1-4\xi)^{1/2}-1\right\}, \quad \xi = -\frac{8C_A a_{\rm s}}{(N-1)^2}$$

Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82)

NNL contributions to the $\overline{\rm MS}$ splitting functions: only partially in closed form

$$\left[P^{T}_{\rm gg}\right]^{\rm NLL}_{C_F=0} = \left\{(1-4\xi)^{-1/2}+1\right\} a_{\rm s}\left(\frac{11}{6}C_A + \frac{1}{3}n_f\right)$$

$$\left[\frac{C_A}{C_F}\, P^{T}_{\rm gq}\right]^{\rm NLL}_{C_F=0} = \left[P^{T}_{\rm gg}\right]^{\rm NLL}_{C_F=0} + \left\{(1-4\xi)^{1/2}-1\right\}\frac{1}{24}(N-1)^2(1+n_f/C_A)$$

Resummed splitting and coefficient functions

$$\frac{C_A}{C_F}\, P^T_{\rm gq}(N,\alpha_s) \;\stackrel{\rm LL}{=}\; P^T_{\rm gg}(N,\alpha_s) \;\stackrel{\rm LL}{=}\; \tfrac{1}{4}(N-1)\left\{(1-4\xi)^{1/2}-1\right\}, \quad \xi = -\,\frac{8\,C_A\,a_s}{(N-1)^2}$$

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LL coefficient functions for F_T and F_ϕ [also: Albino, Bolzoni, Kniehl, Kotikov (11)]

$$C^{\rm LL}_{T,\,{\rm g}} \;=\; \frac{C_F}{C_A}\left(C^{T,\,{\rm LL}}_{\phi,{\rm g}} - 1\right) \;=\; \frac{C_F}{C_A}\left\{(1-4\xi)^{-1/4}-1\right\} \quad \text{in } \overline{\rm MS}$$

'Everything else', including all of $P^T_{\rm qq}$, $P^T_{\rm qg}$, the quark coefficient fct's, $C_{L,{\rm i}}$:

Tables of coefficients to order α_s^{16} – numerically sufficient for $x \gtrsim 10^{-4}$ – e.g.

$$P^{(n)T}_{\rm gg,\,NLL}(N) \;=\; -\,\frac{(-8)^n\,C_A^{n-1}}{3(N-1)^{2n}}\left[(11C_A^2 + 2\,C_A n_f)\,B^{(n)}_{\rm gg,1} \;-\; 2\,C_F n_f\,B^{(n)}_{\rm gg,2}\right]$$

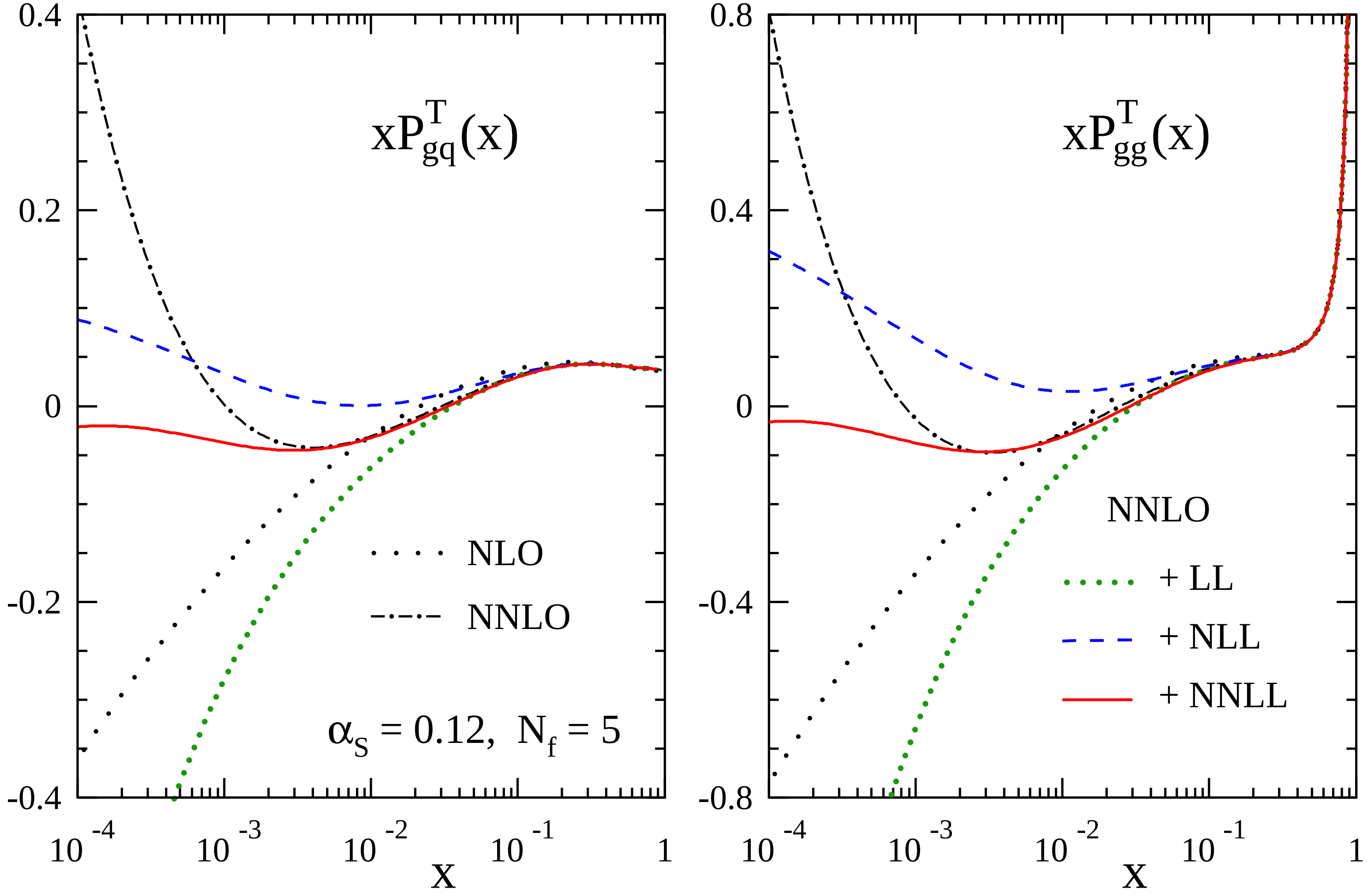
Normalized LL, NLL splitting-fct. coefficients

n	$A_{\rm gi}^{(n)}$	$B_{\rm gg,1}^{(n)}$	$B_{\rm gg,2}^{(n)}$	$B_{\rm gq,1}^{(n)}$	$B_{\rm gq,2}^{(n)}$	$B_{\rm gq,3}^{(n)}$	$A_{\rm qi}^{(n)}$
0	1	1	–	9	–	–	–
1	1	1	2	9	–	–	–
2	2	3	5	29	1	1	1
3	5	10	$\frac{49}{3}$	100	5	$\frac{19}{3}$	$\frac{11}{3}$
4	14	35	$\frac{347}{6}$	357	21	$\frac{179}{6}$	$\frac{73}{6}$
5	42	126	$\frac{6353}{30}$	1302	84	$\frac{3833}{30}$	$\frac{1207}{30}$
6	132	462	$\frac{11839}{15}$	4818	330	$\frac{7879}{15}$	$\frac{2021}{15}$
7	429	1716	$\frac{624557}{210}$	18018	1287	$\frac{444377}{210}$	$\frac{96163}{210}$
8	1430	6435	$\frac{316175}{28}$	67925	5005	$\frac{236095}{28}$	$\frac{44185}{28}$
9	4862	24310	$\frac{54324719}{1260}$	257686	19448	$\frac{42072479}{1260}$	$\frac{6936481}{1260}$

All integer series known, $B_{\rm gg,2}^{(n)} - B_{\rm gq,3}^{(n)} = 2A_{\rm gi}^{(n)}, \; A_{\rm qi}^{(n)} + B_{\rm gg,2}^{(n)} = 2B_{\rm gg,1}^{(n)}$

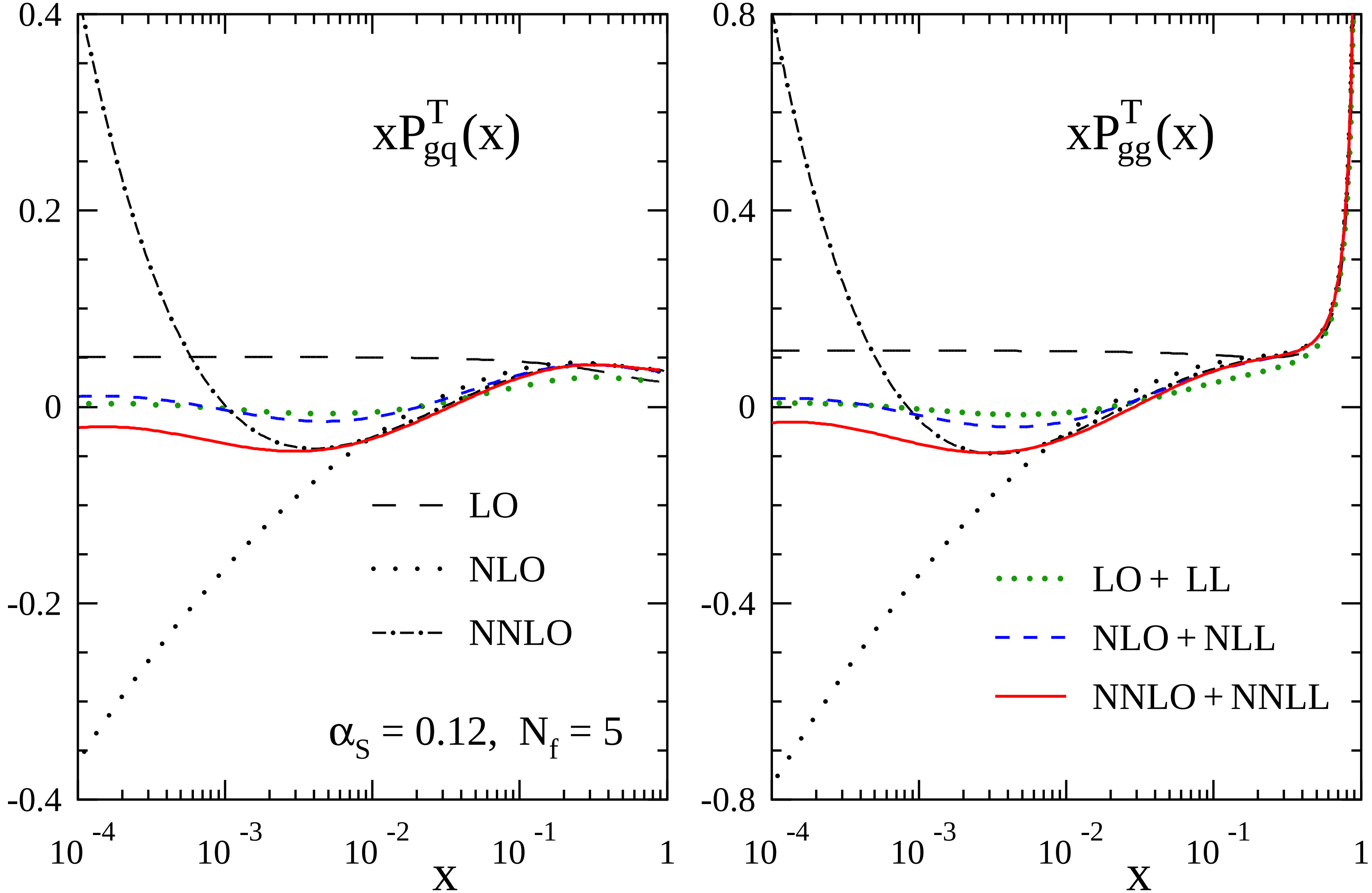
Solution of one non-integer series: analytic structure of all NLL contributions

Small-x gluon-parton splitting functions (I)



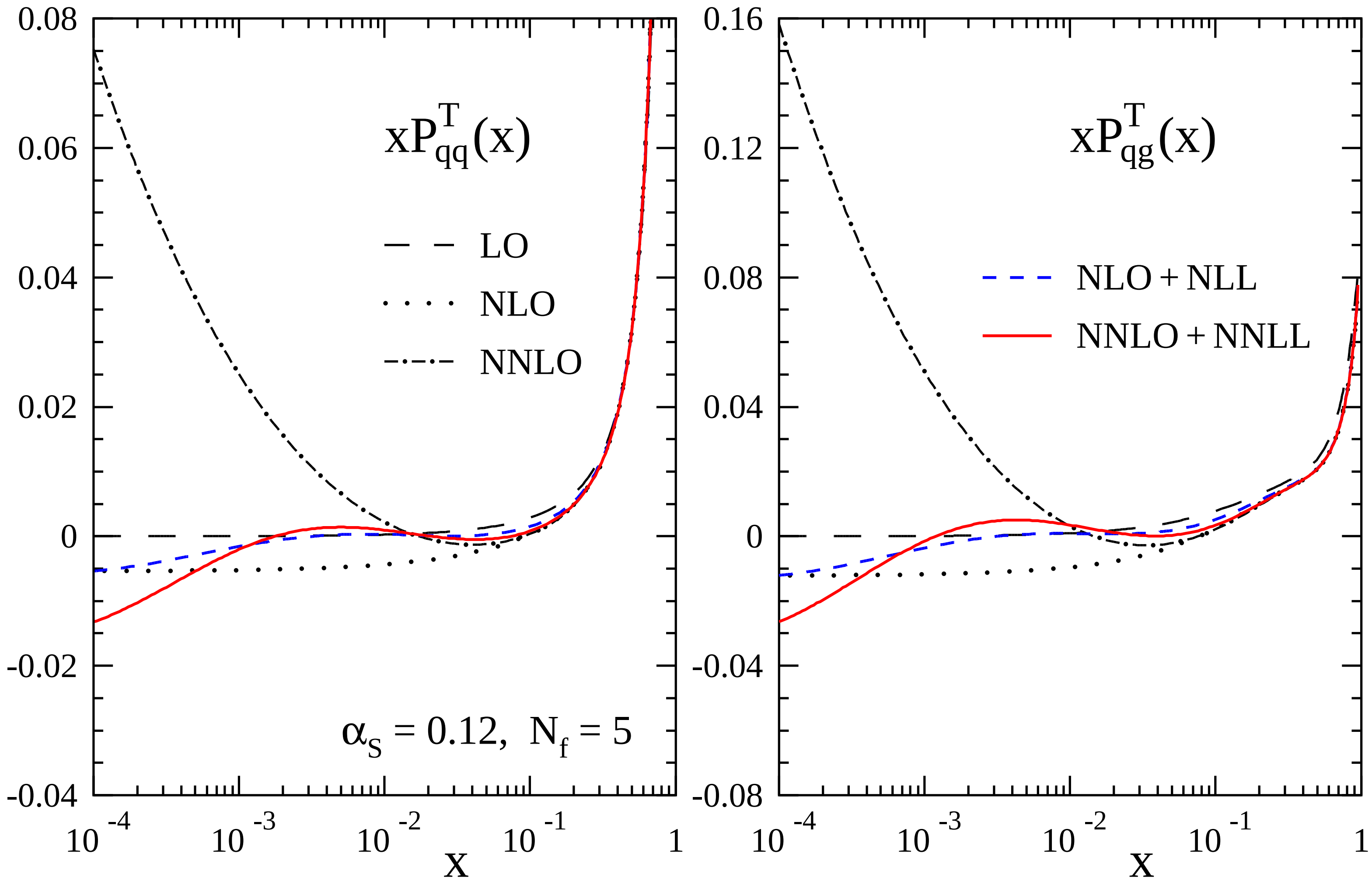
LL insufficient, near-perfect cancellation of NNLO rise by NNLL resummation

Small-x gluon-parton splitting functions (II)



Approximation sequence LO+LL, NLO+NLL, NNLO+NNLL rather stable to very small x

Small-x quark-parton splitting functions



Also consistent with $xP^T_{ji} \approx 0$ at $x < 10^{-2}$ (N^3LL corr's known and positive)

Results for $\ln^{\ell} x$ contributions in DIS

Analogous to SIA: highest three $x^0 \ln^{\ell} x$ double logarithms (to order α_s^{16}) derived for non-singlet$^+$ and flavour-singlet splitting & coefficient functions

Splitting functions $P_{\mathrm{NS}+}^{(n\geq 1)}$: all-order expressions for coefficients to NNLL

LL: $2^{n+1} C(n)\, C_F^{n+1}$ — as in Blümlein, A.V. (95)

NLL: $2^n C(n)\,(n+1)\, C_F^n\,(C_F - \frac{1}{2}\beta_0)$

NNLL:
$$2^n C(n-1) \Big\{ [n(n+1) - 4 - \zeta_2(48n - 44)]\, C_F^{n+1} + \left[\frac{10}{3} n + 48\zeta_2(n-1)\right] C_F^n C_A + n\left(n - \frac{14}{3}\right) C_F^n \beta_0 - 15\zeta_2\,(n-1)\, C_F^{n-1} C_A^2 + \frac{1}{4}\, n(n-1)\, C_F^{n-1} \beta_0^2 \Big\}$$

in terms of the Catalan numbers $C(n) = (2n)!\,[\,n!\,(n+1)!\,]^{-1}$

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$$2^n C(n-1) \Big\{ [n(n+1) - 4 - \zeta_2(48n - 44)]\, C_F^{n+1} + \Big[\frac{10}{3} n + 48\zeta_2(n-1)\Big] C_F^n C_A + n\Big(n - \frac{14}{3}\Big) C_F^n \beta_0 - 15\zeta_2\,(n-1)\, C_F^{n-1} C_A^2 + \frac{1}{4}\, n(n-1)\, C_F^{n-1} \beta_0^2 \Big\}$$

in terms of the Catalan numbers $C(n) = (2n)!\, [\, n!\, (n+1)!\,]^{-1}$

Other quantities not (yet?) in closed all-n form beyond leading logarithms

NNLL expressions alone insufficient for stable results – details another time
Combine with future fixed Mellin-N fourth-order calculations, . . .

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- Complementary: Grunberg; Laenen, Gardi, Magnea, Stavenga, White
- Applications, now: assess relevance of NS $1/N$ terms, large-x DIS fits
- Near$/$mid future: combine with other results, esp. fixed-N calculations
  (close to) feasible now: 4-loop sum rules Baikov, Chetyrkin, Kühn (10)
- Extension to Drell-Yan, Higgs production needs more insights

Small-x summary and outlook

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  Required for using NNLO results in SIA below $x \approx 10^{-2} \ldots 10^{-3}$

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- Unlike large-x case: no direct generalization to all (higher) a in $x^a \ln^\ell x$
  But works for higher even a in SIA – DIS case not checked yet
- Does not work for the odd-N quantities F_3 and g_1 in DIS, F_A in SIA
  E.g., leading logs with group factor $d_{abc}d^{abc}$ at third order in F_3 and F_A
  cf. Dokshitzer, Marchesini (2007)

All large-x and many, but not all, small-x double logarithms in SIA and DIS appear to be 'inherited' from lower-order results.