Conformal symmetry based relations between Bjorken and Ellis-Jaffe sum rules

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The identity between expressions for the coefficient functions of the Bjorken and Ellis-Jaffe sum rules is derived in the conformal invariant limit of massless $U(1)$ model, which is realised in the perturbative quenched QED (pqQED) approximation and in the conformal invariant limit of $SU(N_c)$ gauge group with fermions. The derivation is based on the comparison of the expressions for the triangle dressed Green functions of singlet Axial vector- Vector-Vector and non-singlet Axial vector-Vector-Vector fermion currents in the limit, when the conformal symmetry is not violated. In the case of pqQED its explicit cross-check at the third order of perturbation theory is discussed. The analytical order α_s^3 approximation for the non-singlet coefficient function, derived in the conformal invariant limit of $SU(N_c)$ group, is reminded. Its possible phenomenological application is outlined.

1 Introduction

The definitions of the massless perturbative expressions for the Bjorken and Ellis-Jaffe sum rules of the polarised lepton-nucleon DIS are well-known and have the following form

$$Bjp(Q^2) = \int_0^1 \big(g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2)\big)dx = \frac{1}{6}g_A C_{NS}(A_s(Q^2)) \quad (1)$$

$$EJ^{lp(n)}(Q^2) = C_{NS}(A_s(Q^2))(\pm\frac{1}{12}a_3 + \frac{1}{36}a_8) + C_{SI}(A_s(Q^2))\frac{1}{9}\Delta\Sigma(Q^2) \quad (2)$$

where $a_3 = \Delta u - \Delta d = g_A$, $a_8 = \Delta u + \Delta d - 2\Delta s$, Δu, Δd and Δs are the polarised parton distributions and the subscript $lp(n)$ indicate the processes of the polarised DIS of charged leptons (l) on protons (p) and neutrons (n). In the $SU(N_c)$ colour gauge theory $A_s = \alpha_s/(4\pi)$. The order $O(A_s^3)$ and $O(A_s^4)$ perturbative expressions for the non-singlet (NS) coefficient function $C_{NS}(A_s)$ were analytically evaluated in [1] and [2] correspondingly, while the analytical expressions for the leading in the number of quarks flavours terms (renormalon contributions) were obtained in [3] (see [4] as well). The singlet (SI) contribution C_{SI} to Eq.(2) contains the coefficient function , calculated in [5] at the $O(A_s^3)$- level in the $\overline{MS}$-scheme, while the SI anomalous dimension term is known analytically from the $O(A_s^2)$ and $O(A_s^3)$ results of [6] and [5] respectively. In all these calculations the $\overline{MS}$-scheme was used. In this scheme the polarised gluon distribution ΔG does not enter into Eq.(2). Our main aim is to prove, that in the *conformal invariant limit* of the perturbative series, obtained in the $SU(N_c)$ quantum field theory model with fermions and in the $U(1)$ model with fermions (i.e. in the perturbative quenched QED (pqQED) approximation) the analytical expressions for C_{NS} and C_{SI}, defined in Eq.(2), are identical in all orders of perturbation theory in the expansion parameters $A_s = \alpha_s/(4\pi)$ or $A = \alpha/(4\pi)$. While proving this identity we follow the studies, given in [7], where the classical Crewther relation [8], derived in the quark-parton era from the three-point Green function of the NS Axial vector-Vector-Vector(AVV) currents, is compared with the three-point Green function of singlet Axial vector-Vector-Vector currents. In the era of continuing understanding of the special features of the relations between NS characteristics of strong interactions, evaluated within perturbative approach in the the $SU(N_c)$ gauge group (see [3], [2], [10]), the detailed studies of the relations, which follow from the three-point Green functions of the NS AVV currents, were considered in [11], [12], [13]. The comment on possible phenomenological applications of the conformal-symmetry motivated expression for the Bjorken polarised sum-rule, which in QCD depends from the scale, fixed within *principle of maximal conformality* [14], [15], is given.

2 Proof of the identity

Theoretical considerations of [8] are based on the property that in the *conformal invariant limit* the dressed expression for the three-point Green functions of NS AVV currents is proportional to the *1-loop* expression of the related three-point diagram [16]. In the momentum space this means, that

$$T^{abc}_{\mu\alpha\beta}(p,q) = i\int <0|TA^a_\mu(y)V^b_\alpha(x)V^c_\beta(0)|0> e^{ipx+iqy}dxdy = d^{abc}\Delta^{(1-loop)}_{\mu\alpha\beta}(p,q) \qquad (3)$$

where $A^a_\mu(y) = \overline{\psi}(y)\gamma_\mu(\lambda^a/2)\gamma_5\psi(y)$ and $V^b_\alpha(x) = \overline{\psi}(x)\gamma_\mu(\lambda^b/2)\psi(x)$ are the NS Axial-vector and Vector currents. In the same limit it is possible to write-down the similar expression for the three-point Green function of SI Axial vector-NS Vector-Vector currents [17]

$$T^{ab}_{\mu\alpha\beta}(p,q) = i\int <0|TA_\mu(y)V^a_\alpha(x)V^b_\beta(0)|0> e^{ipx+iqy}dxdy = \delta^{ab}\Delta^{(1-loop)}_{\mu\alpha\beta}(p,q) \qquad (4)$$

where $A_\mu(y) = \overline{\psi}(y)\gamma_\mu\gamma_5\psi(y)$. The SI coefficient function of the Ellis-Jaffe sum rule is defined as the coefficient function of the SI structure in the operator-product expansion of two NS Vector currents,namely

$$i\int TV^a_\alpha V^b_\beta d^4x|_{p^2\to\infty} \approx i\delta^{ab}\epsilon_{\alpha\beta\rho\sigma}\frac{p^\sigma}{P^2}C^{SI}_{EJp}(A_s)\ A_\rho(0) + \ldots \qquad (5)$$

The expression should be compared with the definition of the NS coefficient function , which enters into operator-product of the three-point Green function of Eq.(4) as

$$i\int TV^a_\alpha V^b_\beta e^{ipx} d^4x|_{p^2\to\infty} \approx id^{abc}\epsilon_{\alpha\beta\rho\sigma}\frac{p^\sigma}{P^2}C^{NS}(A_s)\ A^c_\rho(0) + \ldots \qquad (6)$$

Taking now the limit $q^2 \to \infty$ in Eq.(4) and Eq.(3), we get the following Crewther-type identity in the SI channel [17], [7]

$$C_{SI}(A_s) \times C^{SI}_D(A_s) \equiv 1 \quad . \qquad (7)$$

It should be compared with the classical NS Crewther identity, namely with

$$C_{NS}(A_s) \times C^{NS}_D(A_s) \equiv 1 \quad . \qquad (8)$$

It follows from the x-space considerations of [8] (see [9] as well). In the momentum space it was re-derived in [11]. Note, that $C_D^{SI}(A_s)$ and $C_D^{NS}(A_s)$ are the coefficient functions of the massless axial-vector and vector Adler D-functions, defined by taking derivative $Q^2\frac{d}{dQ^2}$ of the mass-independent terms in the correlator of SI axial-vector currents

$$i\int <0|TA_\mu(x)A_\nu(0)|0> e^{iqx}d^4x = \Pi^{SI}_{\mu\nu}(Q^2) = (g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi^{SI}(Q^2) \tag{9}$$

and of the correlator of NS axial-vector currents

$$i\int <0|TA^{(a)}_\mu(x)A^{(b)}_\nu(0)|0> e^{iqx}d^4x = \delta^{ab}\Pi^{NS}_{\mu\nu}(Q^2) = \delta^{ab}(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi^{NS}(Q^2) \tag{10}$$

where $Q^2 = -q^2$ is the Euclidean momentum transfer. The exact chiral invariance of the massless perturbative expressions for the coefficient functions implies, that $C_D^{SI}(A_s) \equiv C_D^{NS}(A_s)$. Keeping this in mind and comparing l.h.s. of Eq.(7) and Eq.(8), we get the following relation

$$C_{NS}(A_s) \equiv C_{SI}(A_s)|_{conformal\ invariant\ limit} \tag{11}$$

which is valid in the conformal-invariant limit of $SU(N_c)$ gauge model and in the pqQED model in all orders of perturbation theory. In the latter case Eq.(11) was proved in [7].

3 Conformal invariant limit of the third order perturbative series

In the pqQED model it is possible to demonstrate explicitly the validity of the identity of Eq.(11) at the level of third order corrections [7]. In these studies the following $O(A^3)$ pqQED expressions were used: order $O(A^3)$ expression for $C_{NS}(A)$, available from [1] and the following dimensional regularisation [18] expression for $C_{SI}(A_s) = \overline{C}_{SI}(A_s)/Z_5^{SI}(A_s)$ [6] where Z_5^{SI} is the finite renormalization constant of the SI Axial-vector current. In order to get the pqQED limit of all functions, contributing to $C_{SI}(A_s)$, in the work [7] $Z_5^{SI}(A)$ was determined from the pqQED limit of $Z_5^{NS}(A_s)$ renormalization constant, analytically evaluated in [1]. Using all these inputs and the validity of identity of Eq.(11) was explicitly demonstrated at the $O(A^3)$ -approximation of pqQED [7]. To fix the $O(A^4)$ pqQED correction to these functions, one can use the pqQED expression of the related analytical result from [2]. This result coincides with the one, obtained in [19] from classical Crewther relation of Eq.(8), supplemented with the pqQED $O(A^4)$ analytical approximation for $C_D^{NS}(A)$, first presented in [20]. The $O(A^4)$ pqQED expression for $C_{NS}(A)$ reads

$$C_{NS}(A) = 1 - 3A + \frac{21}{2}A^2 - \frac{3}{2}A^3 - \left(\frac{4823}{8} + 96\zeta_3\right)A^4 + O(A^5) \quad . \tag{12}$$

It should coincide with the pqQED limit of still unknown $O(A^4)$ coefficient of $C_{SI}(A_s)$ contribution into the Ellis-Jaffe sum rule. In the case of $SU(N_c)$ gauge group with fermions the similar expressions for the coinciding SI and NS coefficient functions are known at present at the $O(A_s^3)$ level. They can be obtained from the $O(A_s^3)$ expression for the coefficient function $C_{NS}(A_s)$

$$C_{NS}(A_s) = 1 - 3C_FA_s + \left(\frac{21}{2}C_F^2 - C_FC_A\right)A_s^2 - \left[\frac{3}{2}C_F^3 + 65C_F^2C_A + \left(\frac{523}{12} - 216\zeta_3\right)C_FC_A^2\right]A_s^3 \tag{13}$$

derived in the conformal invariant limit of $SU(N_c)$ in [10] using the β-expansion approach of [21]. Here C_F and C_A are the Casimir operators of the $SU(N_c)$ group. This result should coincide with the similar approximation of $C_{SI}(A_s)$-contribution into Eq.(2). Considering now the ratios of the corresponding approximations for the Ellis-Jaffe and Bjorken sum rule, namely

$$\frac{EJ^{lp(n)}(Q^2)}{Bjp(Q^2)} = \pm\frac{1}{2} + \frac{a_8}{6\ a_3} + \frac{2\Delta\Sigma}{3a_3} \tag{14}$$

where $a_8 = 3a_3 - 4D$, $a3$, a_8 and $\Delta\Sigma$ are defined through the polarised parton distributions below Eqs.(2) and D is the hyperon decay constant, we recover the massless quark-parton relations. Indeed, these ratios can be re-written as

$$\frac{EJ^{lp}(Q^2)}{Bjp(Q^2)} = 1 + \frac{2(\Delta\Sigma - D)}{3\ a_3} \quad ; \quad \frac{EJ^{ln}(Q^2)}{Bjp(Q^2)} = +\frac{2}{3}\frac{(\Delta\Sigma - D)}{a_3} \quad .$$

They lead to the quark-parton model definition of the Bjorken sum rule through and Ellis-Jaffe sum rule

$$Bjp \equiv EJ^{lp} - EJ^{ln} \quad . \tag{15}$$

Thus, one can see that our considerations are self-consistent.

4 Conformal symmetry and the Bjorken sum rule

To conclude the discussions presented above we note, we outline the ideas how the obtained in [10] $O(A_s^3)$ approximation for the coefficient function $C_{NS}(A_s)$ of the Bjorken sum rule, given in Eq.(13), can be used in the phenomenological studies. As mentioned above, this result was obtained with the help of formulated in [21] β-expansion approach, which prescribes to consider the following representations of the $O(A_s^2)$ and $O(A_s^3)$ coefficients

$$c_2 = \beta_0 c_2[1] + c_2[0] \quad , \quad c_3 = \beta_0^2 c_3[2] + \beta_1 c_3[0,1] + \beta_0 c_3[1] + c_3[0] \quad , \ldots \tag{16}$$

of the coefficient $C_{NS}(A_s)$ function of the Bjorken sum rule which is defined as

$$C_{NS}(A_s) = 1 + \sum_{l\geq 0} c_l A_s^{l+1}(Q^2) \quad . \tag{17}$$

In Eq.(16) β_i are the perturbative coefficients of the $SU(N_c)$ gauge group β-function, which can be written down as

$$\mu^2\frac{\partial A_s}{\partial \mu^2} = -\sum_{i\geq 0}\beta_i A_s^{i+1} \quad . \tag{18}$$

The conformal-invariant contribution $c_l[0]$ to the coefficients in Eq.(16) are obtained in the conformal-invariant limit of the $SU(N_c)$ group, which corresponds to the imaginable theory, which has all perurbative coefficients of the β-function identically equal to zero. They were obtained in [10] and are presented in the expression of Eq.(13). To relate this pure theoretical expression to the real world, one should fix absorb into the scale Q^2_{PMC} of the expansion parameter A_s the terms, proportional to the coefficients of the $SU(N_c)$ β-function. This can achieved using the concrete realisation of the principle of maximal conformality (PMC), introduced and applied in the works [14], [15]. This PMC approach should be considered as the analog of the formulated in [22] generalization of the original BLM procedure [23]. The resulting expression coincide with Eq.(13), where all coefficients of the β-function in Eq.(16) should be absorbed into the scale of the expansion parameter $A_s(Q^2_{PMC})$ The details of the derivation of the related expressions the corresponding scales and possible phenomenological applications will be considered elsewhere.

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