

# TMD Theory Overview

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Transverse momentum dependent (TMD) distribution and fragmentation functions are described as Fourier transforms of matrix elements containing non-local combinations of quark and gluon fields. The  $x$  and  $p_T$  dependent TMD functions appear in the parametrization of light-front correlators including a transverse (space-like) non-locality. The TMD functions relevant at leading order include spin-spin densities as well as momentum-spin densities and they are able to describe single-spin and azimuthal asymmetries, such as Sivers and Collins effects in SIDIS. Their moments involve higher-twist operators evaluated at zero-momentum (gluonic poles). They appear in observables with process-specific gluonic pole factors such as the sign in SIDIS versus Drell-Yan, which can be traced back to having TMD's with non-trivial process-dependent past- or future-pointing gauge links.

To incorporate transverse momentum dependent (TMD) distribution functions (PDF) and fragmentation functions (FF), in short referred to as TMD's, the starting point are forward matrix elements of parton fields, such as the quark-quark correlator

$$\Phi_{ij}(p|p) = \int \frac{d^4\xi}{(2\pi)^4} e^{ip\cdot\xi} \langle P|\bar{\psi}_j(0)\psi_i(\xi)|P\rangle, \quad (1)$$

where a summation over color indices is understood. For a single incoming fermion one would have  $\Phi \propto (\not{p} + m)$ . The quark-quark-gluon correlator is defined

$$\Phi_{Aij}^\mu(p-p_1, p_1|p) = \int \frac{d^4\xi d^4\eta}{(2\pi)^8} e^{i(p-p_1)\cdot\xi} e^{ip_1\cdot\eta} \langle P|\bar{\psi}_j(0)A^\mu(\eta)\psi_i(\xi)|P\rangle. \quad (2)$$

The basic idea is to isolate these hadronic (soft) parts in a full diagrammatic approach and parametrize them in terms of PDFs. This requires high energies in which case the momenta of different hadrons obey  $P\cdot P' \propto Q^2$ , where  $s \sim Q^2$  is the hard scale in the process. In that case one can for each hadron correlator employ light-like vectors  $P$  and  $n$  such that  $P\cdot n = 1$  (for instance  $n = P'/P\cdot P'$ ) and make a Sudakov expansion of the parton momenta,

$$p = xP + p_T + (p\cdot P - xM^2)n, \quad (3)$$

with  $x = p^+ = p\cdot n$ . In any contraction with vectors outside the correlator, the component  $xP$  contributes at order  $Q$ , the transverse component at order  $M$  and the remaining component contributes at order  $M^2/Q$ . This allows consecutive integration of the components to obtain from the fully un-integrated result in Eq. 1 the TMD light-front (LF) correlator

$$\Phi_{ij}(x, p_T; n) = \int \frac{d\xi\cdot P d^2\xi_T}{(2\pi)^3} e^{ip\cdot\xi} \langle P|\bar{\psi}_j(0)\psi_i(\xi)|P\rangle \Big|_{\xi\cdot n=0}, \quad (4)$$

the collinear light-cone (LC) correlator

$$\Phi_{ij}(x) = \int \frac{d\xi \cdot P}{2\pi} e^{i p \cdot \xi} \langle P | \bar{\psi}_j(0) \psi_i(\xi) | P \rangle \Big|_{\xi \cdot n = \xi_T = 0 \text{ or } \xi^2 = 0}, \quad (5)$$

or the local matrix element

$$\Phi_{ij} = \langle P | \bar{\psi}_j(0) \psi_i(\xi) | P \rangle \Big|_{\xi=0}. \quad (6)$$

The importance of integrating at least the light-cone (minus) component  $p^- = p \cdot P$  is that the expression is at equal time, i.e. time-ordering is not relevant anymore for TMD or collinear PDFs [1]. For local matrix elements one can calculate the anomalous dimensions, which show up as the Mellin moments of the splitting functions that govern the scaling behavior of the collinear correlator  $\Phi(x)$ . We note that the collinear correlator is not simply an integrated TMD. The dependence on upper limit  $\Phi(x; Q^2) = \int^Q d^2 p_T \Phi(x, p_T)$  is found from the anomalous dimensions (splitting functions). One has a  $\alpha_s/p_T^2$  behavior of TMD's that is calculable using collinear TMD's and which matches to the intrinsic non-perturbative  $p_T$ -behavior [2]. We note that in operator product expansion language, the collinear correlators involve operators of definite twist, while TMD correlators involve operators of various twist.

In order to determine the importance of a particular correlator in a hard process, one can do a dimensional analysis to find out when they contribute in an expansion in the inverse hard scale. Dominant are the ones with lowest canonical dimension obtained by maximizing contractions with  $n$ , for instance for quark or gluon fields the minimal canonical dimensions  $\dim[\bar{\psi}(0) \not{n} \psi(\xi)] = \dim[F^{n\alpha}(0) F^{n\beta}(\xi)] = 2$ , while an example for a multi-parton combination gives  $\dim[\bar{\psi}(0) \not{n} A_T^\alpha(\eta) \psi(\xi)] = 3$ . Equivalently, one can maximize the number of  $P$ 's in the parametrization of  $\Phi_{ij}$ . Of course one immediately sees that any number of collinear  $n \cdot A(\eta) = A^n(\eta)$  fields doesn't matter. Furthermore one must take care of color gauge invariance, for instance when dealing with the gluon fields and one must include derivatives in color gauge invariant combinations. With dimension zero there is  $iD^n = i\partial^n + gA^n$  and with dimension one there is  $iD_T^\alpha = i\partial_T^\alpha + gA_T^\alpha$ . The color gauge-invariant expressions for quark and gluon distribution functions actually include gauge-link operators,

$$U_{[0,\xi]} = \mathcal{P} \exp \left( -i \int_0^\xi d\zeta_\mu A^\mu(\zeta) \right) \quad (7)$$

connecting the non-local fields,

$$\Phi_{qij}^{[U]}(x, p_T; n) = \int \frac{d\xi \cdot P d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \langle P | \bar{\psi}_j(0) U_{[0,\xi]} \psi_i(\xi) | P \rangle \Big|_{LF}, \quad (8)$$

$$\Phi_g^{[U,U']\mu\nu}(x, p_T) = \int \frac{d(\xi \cdot P) d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \text{Tr} \langle P, S | F^{n\mu}(0) U_{[0,\xi]} F^{n\nu}(\xi) U'_{[\xi,0]} | P, S \rangle \Big|_{LF}. \quad (9)$$

For transverse separations, the gauge links involve gauge links running along the minus direction to  $\pm\infty$  (dimensionally preferred), which are closed with one or more transverse pieces at lightcone infinity. The two simplest possibilities are  $U^{[\pm]} = U_{[0,\pm\infty]}^n U_{[0_T,\xi_T]}^T U_{[\pm\infty,\xi]}^n$ , leading to gauge-link dependent quark TMDs  $\Phi_q^{[\pm]}(x, p_T)$ . For gluons, the correlator involves color gauge-invariant traces of field-operators  $F^{n\alpha}$ , which are written in the color-triplet representation, requiring the inclusion of *two* gauge-links  $U_{[0,\xi]}$  and  $U'_{[\xi,0]}$ . Again the simplest possibilities are

the past- and future-pointing gauge links  $U^{[\pm]}$ , giving even in the simplest case four gluon TMDs  $\Phi_g^{[\pm, \pm]}(x, p_T)$ .

Using the dimensional analysis to collect the leading contributions in an expansion in the inverse hard scale, one will need the above quark and gluon TMDs for the description of azimuthal dependence. Taking the Drell-Yan process as an example, one can look at the cross section depending on the (small!) transverse momentum  $q_T$  of the produced lepton pair,

$$\sigma(x_1, x_2, q_T) = \int d^2 p_{1T} d^2 p_{2T} \delta^2(p_{1T} + p_{2T} - q_T) \Phi_1^{[-]}(x_1, p_{1T}) \overline{\Phi}_2^{[-\dagger]}(x_2, p_{2T}) \hat{\sigma}(x_1, x_2, Q), \quad (10)$$

which involves a convolution of TMDs. What is more important, it is the color flow in the process, in this case neutralized in initial state, that determines the path in the gauge link in the TMDs, in this case past-pointing ones. In contrast in semi-inclusive deep inelastic scattering one finds that the relevant TMD is  $\Phi^{[+]}$  with a future-pointing gauge link. In a general process one can find more complex gauge links including besides Wilson line elements also Wilson loops. In particular when the transverse momentum of more than one hadron is involved, such as e.g. in the DY case above, it may be impossible to have just a single TMD for a given hadron because color gets entangled [3, 4].

The correlators including a gauge link can be parametrized in terms of TMD PDFs depending on  $x$  and  $p_T^2$ . For quarks, these include not only the functions that survive upon  $p_T$  integration,  $f_1^q(x) = q(x)$ ,  $g_1^q(x) = \Delta q(x)$  and  $h_1^q(x) = \delta q(x)$ , which are the well-known collinear spin-spin densities (involving quark and nucleon spin) but also momentum-spin densities such as the Sivers function  $f_{1T}^{\perp q}(x, p_T^2)$  (unpolarized quarks in transversely polarized nucleon) and spin-spin-momentum densities such as  $g_{1T}(x, p_T^2)$  (longitudinally polarized quarks in a transversely polarized nucleon).

In many cases, it is convenient to construct moments of TMDs in the same way as one considers moments of collinear functions. For  $\Phi(x)$  in Eq. 5 one constructs moments

$$\begin{aligned} x^N \Phi(x) &= \int \frac{d\xi \cdot P}{2\pi} e^{i p \cdot \xi} \langle P | \bar{\psi}(0) (i\partial^n)^N U_{[0, \xi]}^n \psi(\xi) | P \rangle \Big|_{LC} \\ &= \int \frac{d\xi \cdot P}{2\pi} e^{i p \cdot \xi} \langle P | \bar{\psi}(0) U_{[0, \xi]}^n (iD^n)^N \psi(\xi) | P \rangle \Big|_{LC}. \end{aligned} \quad (11)$$

Integrating over  $x$  one finds the connection of the Mellin moments of PDFs with local matrix elements with specific anomalous dimensions, which via an inverse Mellin transform define the splitting functions. Similarly one can consider transverse moment weighting starting with the light-front TMD in Eq. 4,

$$p_T^\alpha \Phi^{[\pm]}(x, p_T; n) = \int \frac{d\xi \cdot P d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \langle P | \bar{\psi}(0) U_{[0, \pm\infty]}^n U_{[0_T, \xi_T]}^T iD_T^\alpha(\pm\infty) U_{[\pm\infty, \xi]}^n \psi(\xi) | P \rangle \Big|_{LF}^{(12)}.$$

Integrating over  $p_T$  gives the lowest transverse moment, which appears in the  $q_T$ -weighted result of Eq. 10. This moment involves twist-3 (or higher) collinear multi-parton correlators, in particular the quark-quark-gluon correlator

$$\Phi_F^{n\alpha}(x - x_1, x_1 | x) = \int \frac{d\xi \cdot P d\eta \cdot P}{(2\pi)^2} e^{i(p-p_1) \cdot \xi} e^{i p_1 \cdot \eta} \langle P | \bar{\psi}(0) U_{[0, \eta]}^n F^{n\alpha}(\eta) U_{[\eta, \xi]}^n \psi(\xi) | P \rangle \Big|_{LC}^{(13)}.$$

In terms of this correlator and the similarly defined correlator  $\Phi_D^\alpha(x - x_1, x_1|x)$  one finds

$$\int d^2 p_T p_T^\alpha \Phi^{[U]}(x, p_T) = \tilde{\Phi}_\partial^\alpha(x) + C_G^{[U]} \pi \Phi_G^\alpha(x), \quad (14)$$

$$\tilde{\Phi}_\partial^\alpha(x) = \Phi_D^\alpha(x) - \Phi_A^\alpha(x) = \int dx_1 \Phi_D^\alpha(x - x_1, x_1|x) - \int dx_1 PV \frac{1}{x_1} \Phi_F^{n\alpha}(x - x_1, x_1|x),$$

$$\Phi_G^\alpha(x) = \Phi_F^{n\alpha}(x, 0|x).$$

The latter is referred to as a gluonic pole or ETQS-matrix element [5, 6]. They are multiplied with gluonic pole factors  $C_G^{[U]}$  (e.g.  $C_G^{[\pm]} = \pm 1$ ), that tell us that new functions are involved with characteristic process dependent behavior [7, 8]. This behavior is for the single transverse moments also coupled to the behavior under time reversal. While  $\tilde{\Phi}_\partial^\alpha$  is T-even,  $\Phi_G^\alpha$  is T-odd. Since time reversal is a good symmetry of QCD, the appearance of T-even or T-odd functions in the parametrization of the correlators is linked to specific observables with this same character. In particular single spin asymmetries are T-odd observables.

The analogous treatment for fragmentation functions is simpler because the gluonic pole matrix elements vanish in that case [9, 10]. Nevertheless, there exist T-odd fragmentation functions, but their QCD operator structure is T-even, similar as the structure of  $\tilde{\Phi}_\partial^\alpha$ . There is thus no process dependence, which comes from the factors  $C_G^{[U]}$  multiplying the gluonic poles.

The use of transverse moments in the description of azimuthal asymmetries via transverse momentum weighting of the cross section can be extended to higher moments involving higher harmonics such as  $\cos(2\varphi)$ . Also here process dependence may come in from double gluonic pole matrix elements  $\Phi_{GG}^{\alpha\beta}$ , which are twist four operators. This affects studies that involve the quark TMD  $h_{1T}^{\perp q}(x, p_T)$  (Pretzelosity distribution) or the gluon Boer-Mulders function  $h_1^{\perp g}(x, p_T)$  (linear gluon polarization in unpolarized targets).

A largely unexplored territory is that of TMD factorization, the evolution of TMDs [11] and the possible link to kT-factorization as used for small- $x$  physics [12]. It will be addressed in some of the other talks in this session.

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