DIS 2012, Bonn, 29-03-12

On the timelike splitting functions at next-to-next-to-leading order and beyond

Andreas Vogt (University of Liverpool)

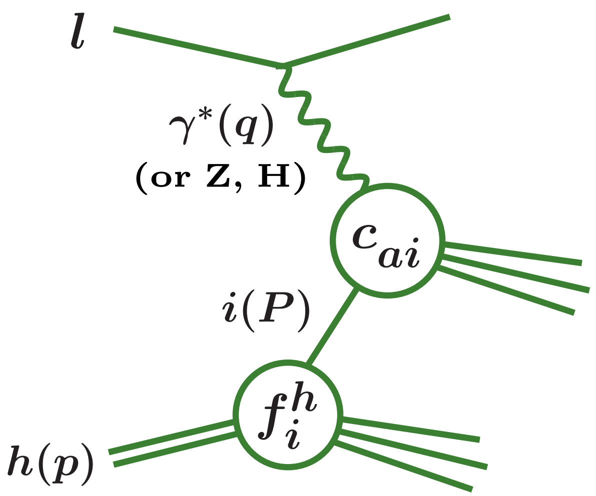
mainly with Andrea Almasy (UoL, now DESY), Sven Moch (DESY)

- Introduction: inclusive DIS and semi-inclusive e^+e^- annihilation
- Unfactorized observables + generalized DMS relation: NNLO $P^T_{\mathrm{qq,\,gg}}$
- Crewther-like structure of physical kernels: $P^T_{\mathrm{qg,\,gq}}$ at NNLO, almost
- Small-x instability, next-to-next-to leading logarithmic resummation

[Mitov, MV, hep-ph/0604053;] MV, arXiv:0709.3899; AMV, 1107.2263; A.V., 1108.2993

Hard lepton-hadron processes in pQCD (I)

Inclusive deep-inelastic scattering (DIS), semi-incl. l^+l^- annihilation (SIA)



Left $\longrightarrow$ right: DIS, q spacelike, $Q^2 = -q^2$

$P = \xi p\,,\ f_i^h =$ **parton distributions**

Top $\longrightarrow$ bottom: l^+l^-, q timelike, $Q^2 = q^2$

$p = \xi P\,,$ **fragmentation distributions**

Drell-Yan (DY) l^+l^- production: bottom $\longrightarrow$ top, 2nd hadron from right ($\{\ldots\}$)

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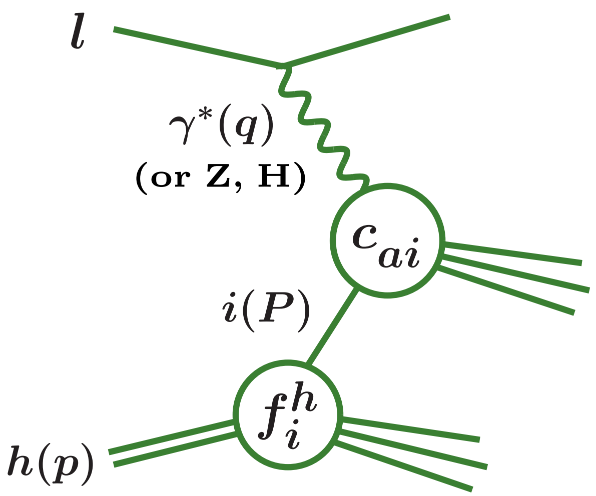

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Structure functions, fragmentation functions etc F_a: coefficient functions

$$F_a(x, Q^2) \; = \; \Big[C_{a,i\{j\}}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2)\{\otimes f_j^{h'}(\mu^2)\}\Big](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables: $x = Q^2/(2p\cdot q)$ in DIS etc. μ: renorm./mass-fact. scale

Hard lepton-hadron processes in pQCD (II)

Parton/fragmentation distributions f_i : (renorm. group) evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) \;=\; \left[P^{S/T}_{ik/K\,ip}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

$\otimes$ = Mellin convolution. Initial conditions incalculable in perturbative QCD.

$\Rightarrow$ **predictions: fit-analyses of reference processes, universality of $f_i(\xi,\mu^2)$**

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Expansion in α_s: splitting functions P, coefficient fct's c_a of observables

$$P \;=\; \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \ldots$$

$$C_a \;=\; \underbrace{\alpha_s^{n_a} \Big[c_a^{(0)} + \alpha_s c_a^{(1)}}_{} + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \ldots \Big]$$

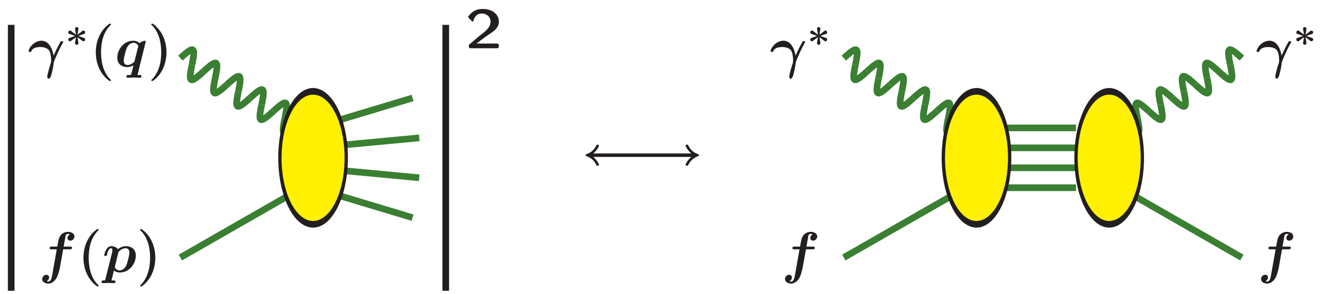
NLO: first real prediction of size of cross sections

NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate of pQCD predictions

$P^{(2)S}_{ik}$ via forward amplitudes: Vermaseren, MV (04). Here: $P^{(2)T}_{ik}$, indirectly

Three-loop calculation of inclusive DIS

Optical theorem: $(\gamma^*, W, H)\,f$ **total cross sections** $\longleftrightarrow$ **forward amplitudes**



Dispersion relation in x**: coefficient of** $(2p\cdot q)^N$ $\leftrightarrow$ N**-th Mellin moment**

$$\hat{F}_a^N \;=\; \int_0^1 dx\, x^{N-1}\, \hat{F}_a(x)$$

α_s^n: $\varepsilon^{-n}\ldots\varepsilon^{-2}$: **lower-order terms,** ε^{-1}: n**-loop splitting functions + . . . ,**
ε^{0}: n**-loop coefficient fct's + . . . ,** ε^{k}, $0 < k < l$: **required for order** $n+l$

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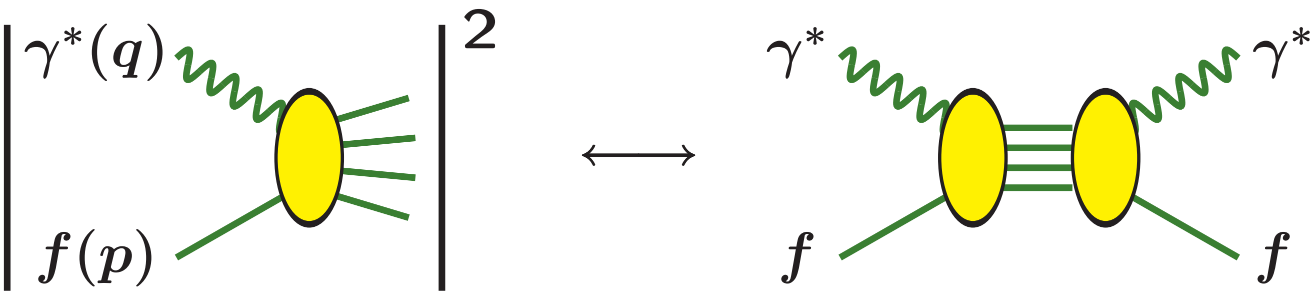

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$\gamma^*\mathrm{qq}$, $H\mathrm{gg}$ **form factors: 'reverse-engineered' to** ε^{-1} **at order** α_s^3 **MVV (05)**

Confirmed and extended to ε^{0} **by diagram calculations**

Baikov et al. (09); Lee et al., Gehrmann et al. (10)

From spacelike to timelike quantities (I)

DIS $\longrightarrow$ semi-incl. l^+l^-: crossing, $x \rightarrow 1/x$ relation for bare tree diagrams

Unrenormalized 'diagonal' $H\mathrm{g}$ structure function $F^{\mathrm{b}}_{H,\mathrm{g}}$ for $D = 4 - 2\varepsilon$

$$F^{\mathrm{b}}_{H,g}(a_{\mathrm{s}}^{\mathrm{b}}, Q^2) \;=\; \delta(1-x) \;+\; \sum_{n=1} (a_{\mathrm{s}}^{\mathrm{b}})^n \left(Q^2/\mu^2\right)^{-n\varepsilon} F^{\mathrm{b}}_{H,n}$$

Iterative decomposition in $H\mathrm{gg}$ form factors $\mathcal{F}_n$ and real-emission parts $\mathcal{R}_n$

$$F^{\mathrm{b}}_{H,1} \;=\; 2\,\mathcal{F}_1\,\delta(1-x) + \mathcal{R}_1$$

cf. Matsuura, van Neerven (88)

$$F^{\mathrm{b}}_{H,2} \;=\; (2\,\mathcal{F}_2 + \mathcal{F}_1^2)\delta(1-x) + 2\,\mathcal{F}_1\mathcal{R}_1 + \mathcal{R}_2$$

$$F^{\mathrm{b}}_{H,3} \;=\; (2\,\mathcal{F}_3 + 2\,\mathcal{F}_1\mathcal{F}_2)\delta(1-x) + (2\,\mathcal{F}_2 + \mathcal{F}_1^2)\mathcal{R}_1 + 2\,\mathcal{F}_1\mathcal{R}_2 + \mathcal{R}_3$$

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Analytic cont. of $\mathcal{R}_n$: q^2 [$\rightarrow (i\pi)^k$], phase-space factor $x^{1-2\varepsilon}$, $x \rightarrow 1/x$

$$\ln(1-x) \;\rightarrow\; \ln(1-x) - \ln x + i\pi$$

Curci et al. (80); Floratos et al. (81); Stratmann, Vogelsang (96); ...

Only $\mathcal{R}_1$ from trees only (same $\mathcal{R}_1^T$ from '$i = 0$') $\longrightarrow$ 'small' 3-loop problem

From spacelike to timelike quantities (II)

Reassemble for timelike case ($\mathcal{F}_n^T$ known), α_s and $G^a_{\mu\nu}G_a^{\mu\nu}$ renormalization:

Diagonal splitting and coefficient functions from mass-factorization relations

$$
\begin{aligned}
F^{(1)T}_{H,\mathrm{g}} &= -\frac{1}{\varepsilon}P^{(0)}_{\mathrm{gg}} + c^{(1)T}_{H,\mathrm{g}} + \varepsilon\, a^{(1)T}_{H,\mathrm{g}} + \varepsilon^2\, b^{(1)T}_{H,\mathrm{g}} + \ldots \\
F^{(2)T}_{H,\mathrm{g}} &= \frac{1}{2\varepsilon^2}\left\{\left(P^{(0)}_{\mathrm{gi}} + \beta_0\delta_{\mathrm{gi}}\right)P^{(0)}_{\mathrm{ig}}\right\} - \frac{1}{2\varepsilon}\left\{P^{(1)T}_{\mathrm{gg}} + 2P^{(0)}_{\mathrm{gi}}c^{(1)T}_{H,\mathrm{i}}\right\} \\
&\quad + c^{(2)T}_{H,\mathrm{g}} - P^{(0)}_{\mathrm{gi}}a^{(1)T}_{H,\mathrm{i}} + \varepsilon\left\{a^{(2)T}_{H,\mathrm{g}} - P^{(0)}_{\mathrm{gi}}b^{(1)T}_{H,\mathrm{i}}\right\} + \ldots \\
F^{(3)T}_{H,\mathrm{g}} &= -\frac{1}{6\varepsilon^3}\left\{P^{(0)}_{\mathrm{gi}}P^{(0)}_{\mathrm{ij}}P^{(0)}_{\mathrm{jg}} + \ldots\right\} + \frac{1}{6\varepsilon^2}\left\{2P^{(0)}_{\mathrm{gi}}P^{(1)T}_{\mathrm{ig}} + \ldots\right\} \\
&\quad - \frac{1}{6\varepsilon}\left\{2P^{(2)T}_{\mathrm{gg}} + 3P^{(1)T}_{\mathrm{gi}}c^{(1)T}_{H,\mathrm{i}} + 6P^{(0)}_{\mathrm{gi}}c^{(2)T}_{H,\mathrm{i}}\right. \\
&\qquad \left. -3P^{(0)}_{\mathrm{gi}}\left(P^{(0)}_{\mathrm{ij}} + \beta_0\delta_{\mathrm{ij}}\right)a^{(1)T}_{H,\mathrm{j}}\right\} + c^{(3)T}_{H,\mathrm{g}} - \frac{1}{2}P^{(1)T}_{\mathrm{gi}}a^{(1)T}_{H,\mathrm{i}} + \ldots
\end{aligned}
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F_{H,g}^{(2)T} &= \frac{1}{2\varepsilon^2} \Big\{ \Big(P_{gi}^{(0)} + \beta_0 \delta_{gi} \Big) P_{ig}^{(0)} \Big\} - \frac{1}{2\varepsilon} \Big\{ P_{gg}^{(1)T} + 2 P_{gi}^{(0)} c_{H,i}^{(1)T} \Big\} \\
&\quad + c_{H,g}^{(2)T} - P_{gi}^{(0)} a_{H,i}^{(1)T} + \varepsilon \Big\{ a_{H,g}^{(2)T} - P_{gi}^{(0)} b_{H,i}^{(1)T} \Big\} + \ldots \\
F_{H,g}^{(3)T} &= -\frac{1}{6\varepsilon^3} \Big\{ P_{gi}^{(0)} P_{ij}^{(0)} P_{jg}^{(0)} + \ldots \Big\} + \frac{1}{6\varepsilon^2} \Big\{ 2 P_{gi}^{(0)} P_{ig}^{(1)T} + \ldots \Big\} \\
&\quad - \frac{1}{6\varepsilon} \Big\{ 2 P_{gg}^{(2)T} + 3 P_{gi}^{(1)T} c_{H,i}^{(1)T} + 6 P_{gi}^{(0)} c_{H,i}^{(2)T} \\
&\qquad - 3 P_{gi}^{(0)} \Big(P_{ij}^{(0)} + \beta_0 \delta_{ij} \Big) a_{H,j}^{(1)T} \Big\} + c_{H,g}^{(3)T} - \frac{1}{2} P_{gi}^{(1)T} a_{H,i}^{(1)T} + \ldots
\end{aligned}
$$

Products = convolutions, performed via N-space using FORM Vermaseren

Two-loop ‘off-diagonal’ quantities like $c_{H,q}^{(2)T}$: direct continuation of DIS $F_{H,q}^{b}$

Soar, MVV; Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (09)

Two-loop and diagonal results and checks

Second order including ε^2 terms, all cases: agreement with known results

$\gamma^*\mathrm{q,g}$: Rijken, van Neerven (96, ε^0); Mitov, Moch (06, ε^1). $H\mathrm{q,g}$: Higgs decay

NNLO diagonal splitting functions: $P_{\mathrm{qq}}^{(2)T}(x)$, $P_{\mathrm{gg}}^{(2)T}(x)$ up to coefficients of

$$C_F^3\zeta_2\ln^2 x\,p_{\mathrm{qq}}(x) \quad , \quad C_A^3\zeta_2\ln^2 x\,p_{\mathrm{gg}}(x)$$

Fixed by fermion-number conservation, $n_f = 0$ (QGD) momentum sum rule

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Confirmed by extending NS approach of Dokshitzer, Marchesini, Salam (05)

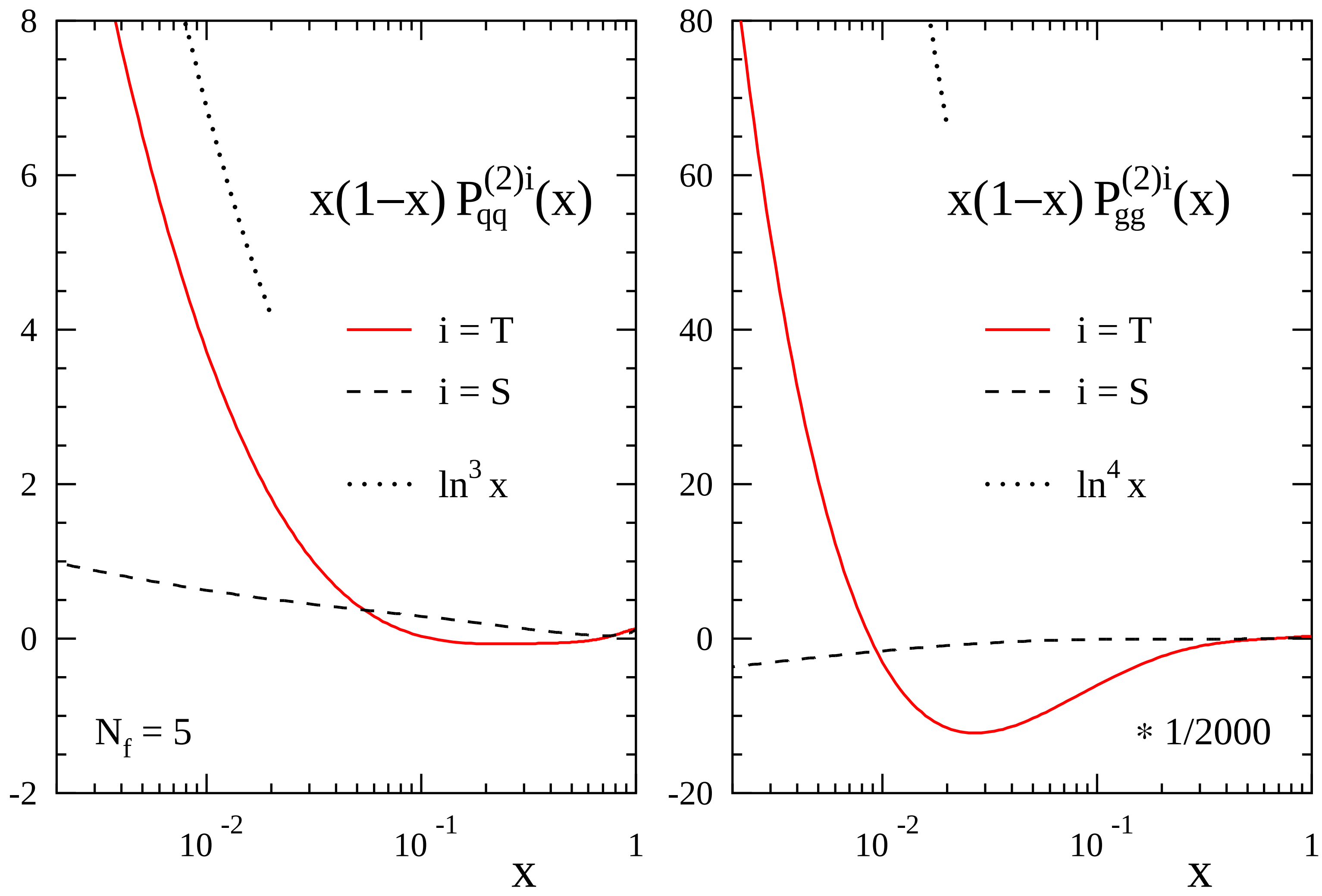
$$P_{\rm gg}^{(2)T-S}\Big|_{C_A^k n_f^{3-k}} = 2 \left[\left\{ \ln x \cdot P_{\text{av.}}^{(1)} \right\} \otimes P_{\rm gg}^{(0)} + \left\{ \ln x \cdot P_{\rm gg}^{(0)} \right\} \otimes P_{\text{av.}}^{(1)} \right]_{C_A^k n_f^{3-k}}$$

Momentum sum rule $\Rightarrow$ $P_{\rm qg}^{(2)T}(N=2)$, $P_{\rm gq}^{(2)T}(N=2)$ completely known

Leading double log $x^{-1} \ln^4 x$ agrees with Mueller (81); Bassetto et al. (82)

No obvious unfact. decomposition/DMS generalization in off-diagonal cases

Third-order diagonal splitting functions



T : extreme small-x rise from $x \gtrsim 10^{-2}$, in gg despite huge cancellations

From spacelike to timelike quantities (III)

Singlet idea: (most of) $T - S$ difference from factorization $\Rightarrow$ physical kernels

Blümlein, Ravindran, van Neerven (00)

Here: transverse/Higgs-exchange DIS & SIA, $F^S = (F_1, F_H)$, $F^T = (F_T, F_H^T)$

$$\frac{d}{d \ln Q^2} F \;=\; K \cdot F \;=\; \left\{ \left(\beta \, \frac{dC}{d\alpha_s} \;+\; C \cdot P \right) \cdot C^{-1} \right\} \cdot F$$

Analytic continuation ($\mathcal{AC}$): $f(x) \rightarrow x f(1/x)$, colour-factor ratios for $K_{i \neq j}$

$$\mathcal{AC} \left[K^{(n)S}(x) \right] \;=\; K^{(n)T}(x) \quad \text{for} \quad n \;=\; 0, 1 \quad \text{(LO, NLO)}$$

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NNLO, diagonal cases – subscripts $1, 0$ for $i\pi$/no $i\pi$ in cont'n of $\ln(1-x)$

$$\mathcal{AC}_0\; [K^{(2)}_{11/HH}(x)] \;-\; K^{(2)}_{TT/HH}(x) \;=\; 12\, C_{F/A}\; \zeta_2\, \beta_0\, P^{(0)}_{\rm qq/gg}(x)\; \ln x$$

$\mathcal{AC}_1$: old $\zeta_2 C^3_{F/A}$ splitting function error + $P^{(0)}_{\rm qq/gg}$ with and without $\ln x$

Structure of 'Crewther relation' of GLS sum rule in DIS and Adler fct. in e^+e^-

Broadhurst, Kataev (93); Crewther (97); Baikov, Chetyrkin, Kühn (10)

From spacelike to timelike quantities (IV)

Second moments (momentum sum rule) of off-diagonal kernels at NNLO

$$\Big[\, \mathcal{AC}_0 \; \big[K^{(2)}_{1H}\big] - K^{(2)}_{TH}\Big]_{N=2} \;=\; \zeta_2\, \beta_0\, C_F \Big(- \frac{212}{3}\, C_A - 16 C_F \Big) \;,$$

$$\Big[\, \mathcal{AC}_0 \; \big[K^{(2)}_{H1}\big] - K^{(2)}_{HT}\Big]_{N=2} \;=\; \zeta_2\, \beta_0\, n_f \Big(9 C_A - \frac{38}{3}\, C_F + 4\beta_0 \Big) \;.$$

Generalization to all N/x-space: assume same $\{1, \ln x\} P^{(0)}_{ij}(x)$ structure

– higher-weight HPLs impossible, no reason for $\ln(1\pm x)$ (SUSY, large x) –

$\ln^k(1-x)$ in $P^{(2)}_{ij}$ also by Grunberg (11)

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$\Rightarrow$ NNLO splitting functions $P^{(n)T}_{\rm gq}$, $P^{(n)T}_{\rm qg}$ (but see next slide for latter) with

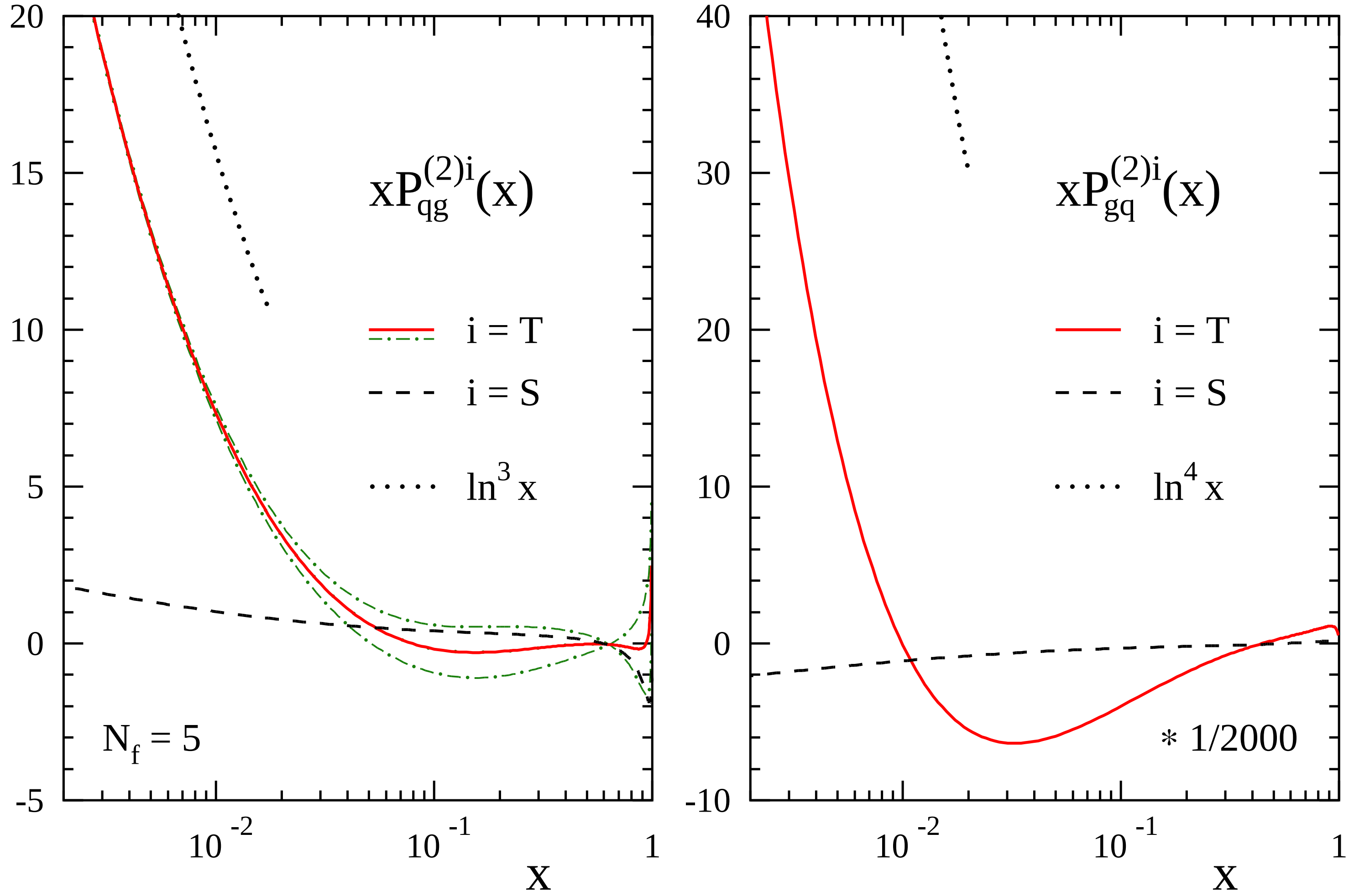
$$\mathcal{AC}_0\big[K^{(2)}_{1H}\big] - K^{(2)}_{TH} \;=\; \zeta_2\,\beta_0\,P^{(0)}_{\rm gq}(x)\,[12\,C_A\,(\ln x-1)\;-\;6\,C_F]$$

$$\mathcal{AC}_0\big[K^{(2)}_{H1}\big] - K^{(2)}_{HT} \;=\; \zeta_2\,\beta_0\,P^{(0)}_{\rm qg}(x)\,[8(C_A-C_F) - 12\ln x\,(C_A-2C_F) + 6\beta_0]$$

SUSY limit $C_A = C_F = n_f$ for $\Delta^{(n)}_{\rm A} = P^{(n)A}_{\rm qq} + P^{(n)A}_{\rm gq} - P^{(n)A}_{\rm qg} - P^{(n)A}_{\rm gg}$

$$\Delta^{(1)}_{\rm S} + \Delta^{(1)}_{\rm T} = 12\,p_{\rm qg}(x)\quad,\quad \Delta^{(2)}_{\rm S} + \Delta^{(2)}_{\rm T} = -\,24\,\zeta_2\,p_{\rm qg}(x)\;+\;\text{non-}\zeta_2\text{ terms}$$

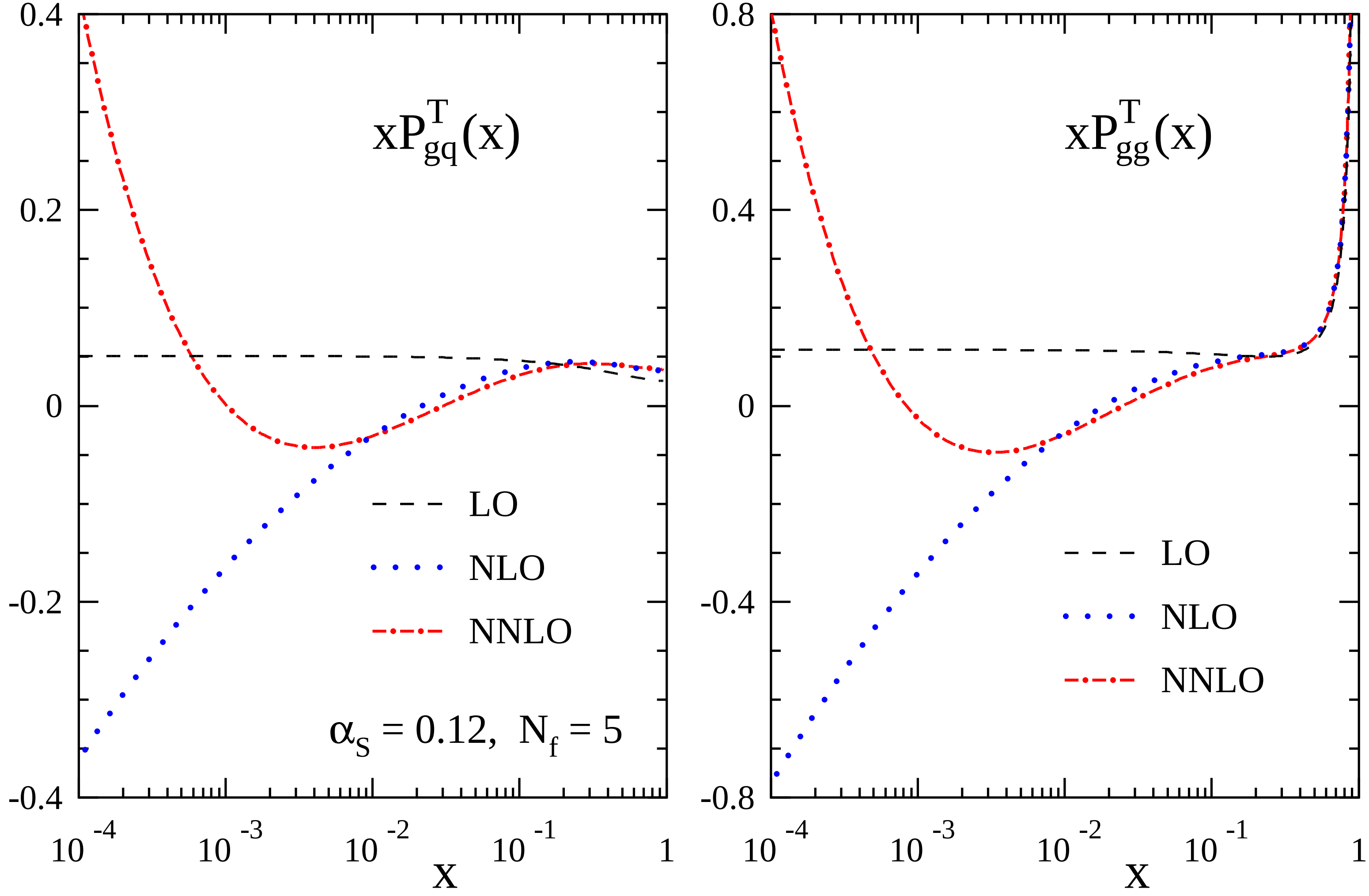
Third-order off-diagonal splitting functions



$q \to g$: **not entirely fixed by Crewther-like ST-relation, $N = 2$, SUSY limit**

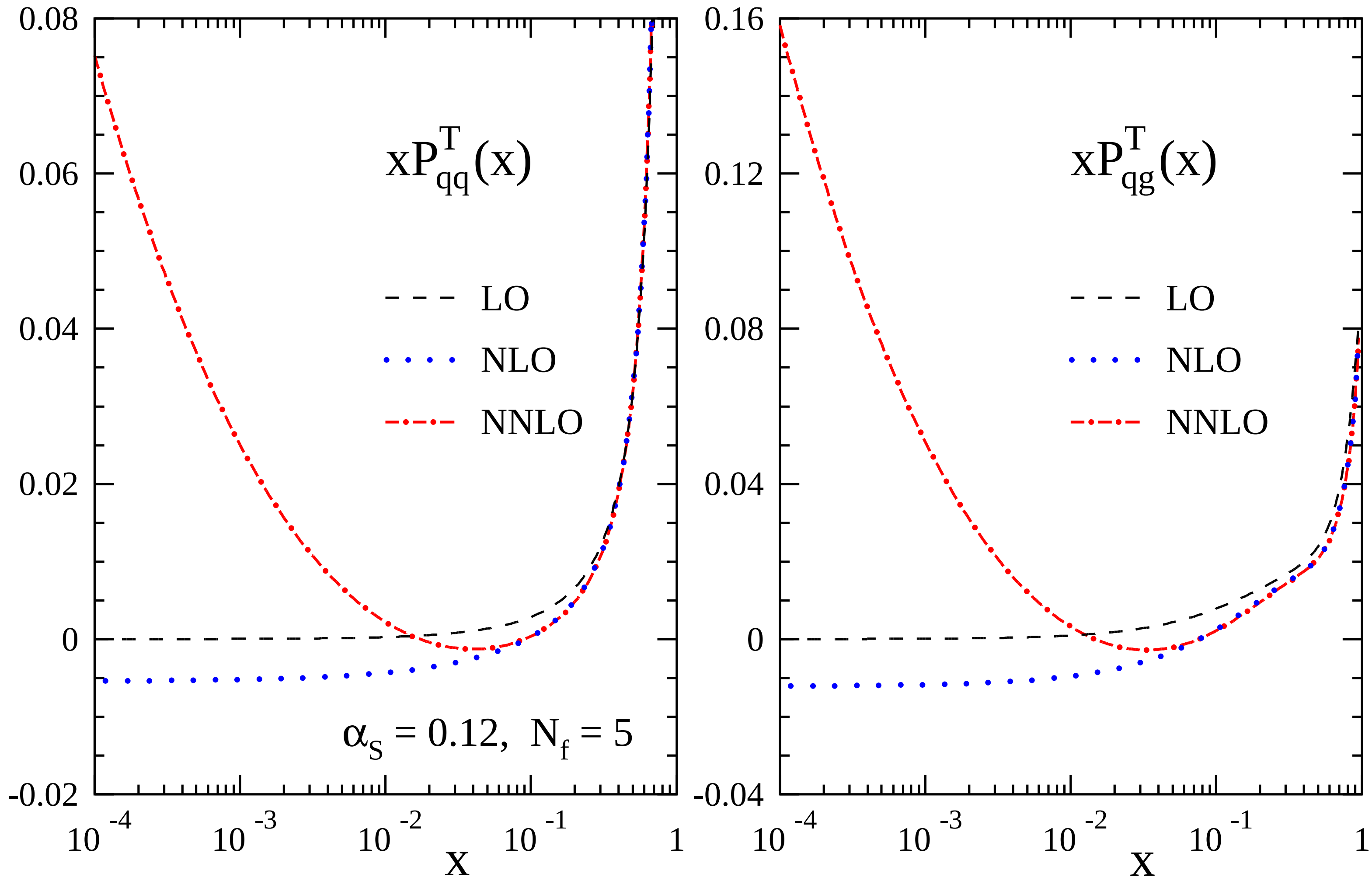
Dash-dotted: $\delta P_{qg}^{(2)T}(x) = \pm\, 2\zeta_2\beta_0\,(C_A - C_F)\,(11 + 24\ln x)\,P_{qg}^{(0)T}(x)$

NNLO approximations for $P^T_{\mathrm{gi}}(x, \alpha_s)$



NLO/NNLO: terms up to $x^{-1} \ln^2 x$ / $x^{-1} \ln^4 x$. Unstable at $x \lesssim 0.005$

NNLO approximations for $P_{qi}^{T}(x, \alpha_s)$



NLO: no $x^{-1} \ln x$ terms. NNLO: up to $x^{-1} \ln^3 x$. Unstable at $x \lesssim 0.02$

Small-x resummation via unfactorized SIA

Phase-space integrations: $x^{a\varepsilon}$ **terms analogous to** $(1-x)^{b\varepsilon}$ **large-**x **factors**

2nd order: Matsuura, van Neerven (88), Rijken, vN (95)

Decomposition of the D**-dim. partonic fragmentation functions for** $a = T, \phi$

$$\widehat{F}^{(n)}_{a,\mathrm{g}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\varepsilon} \Big\{ A^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon B^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon^2 C^{(\ell,n)}_{a,\mathrm{g}} + \ldots \Big\}$$

Leading log: terms of the form $x^{-1} \ln^{n+\delta-1} x$ **at all orders** $\varepsilon^{-n+\delta}$ **with** $\delta = 0,\ 1,\ 2,\ \ldots$**, and** $\widehat{F}^{(n)}_{a,\mathrm{g}}$ **is decomposed into** n **contributions of the form**

$$\varepsilon^{-2n+1}\, x^{-1-k\,\varepsilon} \;=\; \varepsilon^{-2n+1}\, x^{-1} \Big[1 \;-\; k\,\varepsilon \ln x \;+\; \frac{1}{2} (k\,\varepsilon)^2 \ln^2 x \;+\; \ldots \Big], \qquad k = 2,\ 4,\ \ldots,\ 2n$$

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Leading log: terms of the form $x^{-1} \ln^{n+\delta-1} x$ at all orders $\varepsilon^{-n+\delta}$ with $\delta = 0,\ 1,\ 2,\ \ldots$, and $\widehat{F}^{(n)}_{a,\mathrm{g}}$ is decomposed into n contributions of the form

$$\varepsilon^{-2n+1}\, x^{-1-k\,\varepsilon} \;=\; \varepsilon^{-2n+1}\, x^{-1} \Big[1 \,-\, k\,\varepsilon \ln x \,+\, \frac{1}{2} (k\,\varepsilon)^2 \ln^2 x \,+\, \ldots \Big], \qquad k = 2,\ 4,\ \ldots,\ 2n$$

$n-1$ KLN-type cancellations – $\widehat{F}^{(n)}_{a,\mathrm{g}}$ starts at order $1/\varepsilon^n$ – plus 3 constraints from the NNLO results $\Rightarrow$ $n+2$ linear equations for n coefficients $A^{(\ell,n)}_{a,\mathrm{g}}$

Thus: N$^{\text{n}}$LO known $\Rightarrow$ highest $n+1$ (N$^{\text{n}}$LL) double logs fixed at all orders

'All-order' mass factorization: NNLL timelike splitting & coefficient functions

Resummed timelike splitting functions

$$\frac{C_A}{C_F}\, P^T_{\mathrm{gq}}(N,\alpha_s) \;\stackrel{\mathrm{LL}}{=}\; P^T_{\mathrm{gg}}(N,\alpha_s) \;\stackrel{\mathrm{LL}}{=}\; \frac{1}{4}(N-1)\left\{(1-4\xi)^{1/2}-1\right\}, \quad \xi = -\frac{8C_A a_s}{(N-1)^2}$$

Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82)

NNL contributions to the $\overline{\mathrm{MS}}$ splitting functions: only partially in closed form

$$\left[P^T_{\mathrm{gg}}\right]^{\mathrm{NLL}}_{C_F=0} \;=\; \left\{(1-4\xi)^{-1/2}+1\right\} a_s\left(\frac{11}{6}C_A+\frac{1}{3}n_f\right)$$

$$\left[\frac{C_A}{C_F}\, P^T_{\mathrm{gq}}\right]^{\mathrm{NLL}}_{C_F=0} \;=\; \left[P^T_{\mathrm{gg}}\right]^{\mathrm{NLL}}_{C_F=0} + \left\{(1-4\xi)^{1/2}-1\right\}\frac{1}{24}(N-1)^2(1+n_f/C_A)$$

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LL coefficient functions for F_T and F_ϕ [also: Albino, Bolzoni, Kniehl, Kotikov (11)]

$$C^{LL}_{T,g} = \frac{C_F}{C_A}\left(C^{T,LL}_{\phi,g} - 1\right) = \frac{C_F}{C_A}\left\{(1-4\xi)^{-1/4}-1\right\} \quad \text{in } \overline{\text{MS}}$$

'Everything else', including all of P^T_{qq}, P^T_{qg}, the quark coefficient fct's, $C_{L,i}$:

Tables of coefficients to order α_s^{16} – numerically sufficient for $x \gtrsim 10^{-4}$

$$P^{(n)T}_{gg,NLL}(N) = -\frac{(-8)^n C_A^{n-1}}{3(N-1)^{2n}}\left[(11C_A^2 + 2C_A n_f)B^{(n)}_{gg,1} - 2C_F n_f B^{(n)}_{gg,2}\right]$$

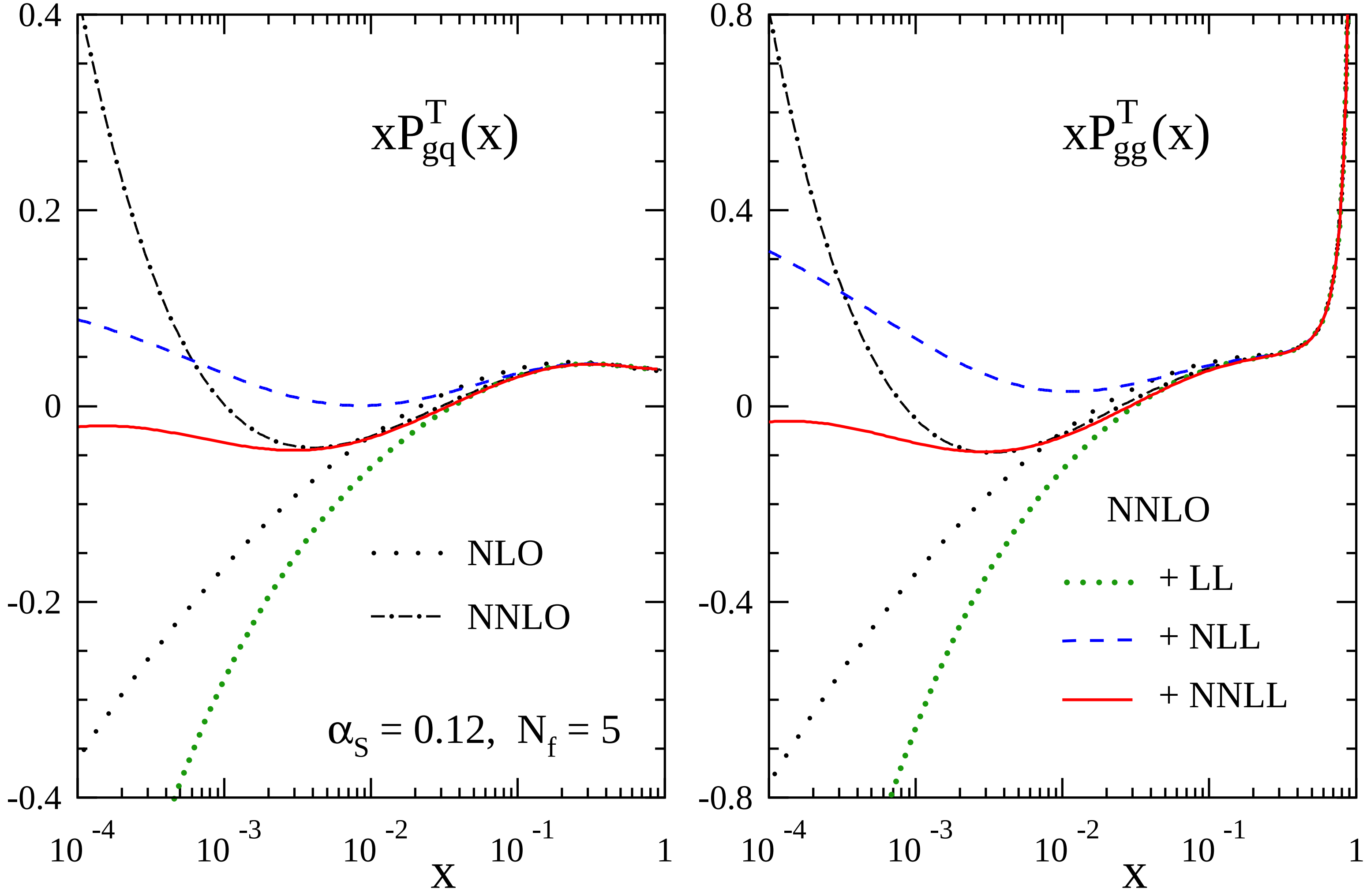
Normalized LL, NLL splitting-fct. coefficients

n	$A_{\rm gi}^{(n)}$	$B_{\rm gg,1}^{(n)}$	$B_{\rm gg,2}^{(n)}$	$B_{\rm gq,1}^{(n)}$	$B_{\rm gq,2}^{(n)}$	$B_{\rm gq,3}^{(n)}$	$A_{\rm qi}^{(n)}$
0	1	1	–	9	–	–	–
1	1	1	2	9	–	–	–
2	2	3	5	29	1	1	1
3	5	10	$\frac{49}{3}$	100	5	$\frac{19}{3}$	$\frac{11}{3}$
4	14	35	$\frac{347}{6}$	357	21	$\frac{179}{6}$	$\frac{73}{6}$
5	42	126	$\frac{6353}{30}$	1302	84	$\frac{3833}{30}$	$\frac{1207}{30}$
6	132	462	$\frac{11839}{15}$	4818	330	$\frac{7879}{15}$	$\frac{2021}{15}$
7	429	1716	$\frac{624557}{210}$	18018	1287	$\frac{444377}{210}$	$\frac{96163}{210}$
8	1430	6435	$\frac{316175}{28}$	67925	5005	$\frac{236095}{28}$	$\frac{44185}{28}$
9	4862	24310	$\frac{54324719}{1260}$	257686	19448	$\frac{42072479}{1260}$	$\frac{6936481}{1260}$

All integer series known, $B_{\rm gg,2}^{(n)} - B_{\rm gq,3}^{(n)} = 2A_{\rm gi}^{(n)}, \; A_{\rm qi}^{(n)} + B_{\rm gg,2}^{(n)} = 2B_{\rm gg,1}^{(n)}$

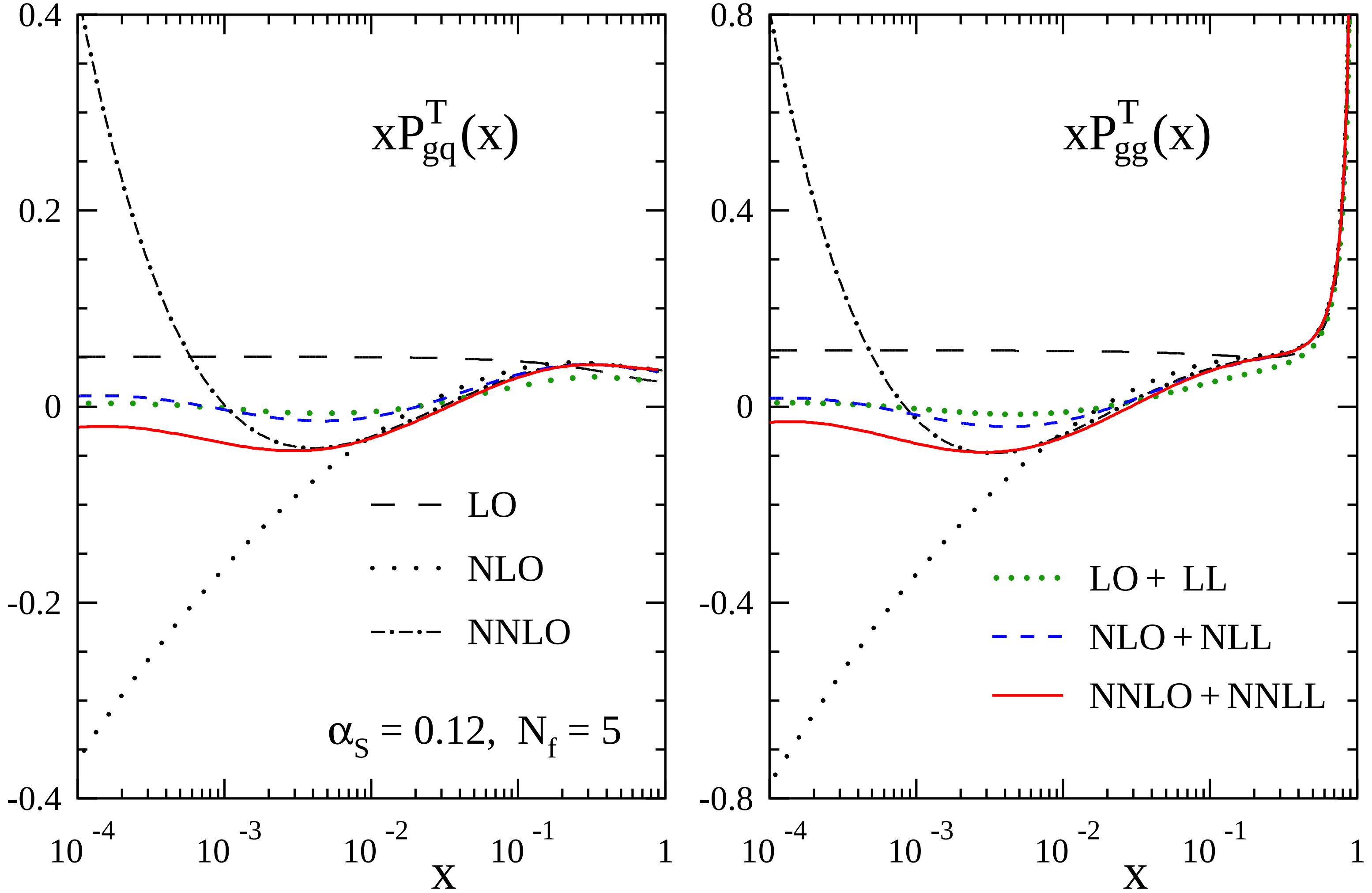
Solution of one non-integer series: analytic structure of all NLL contributions

Small-x gluon-parton splitting functions (I)



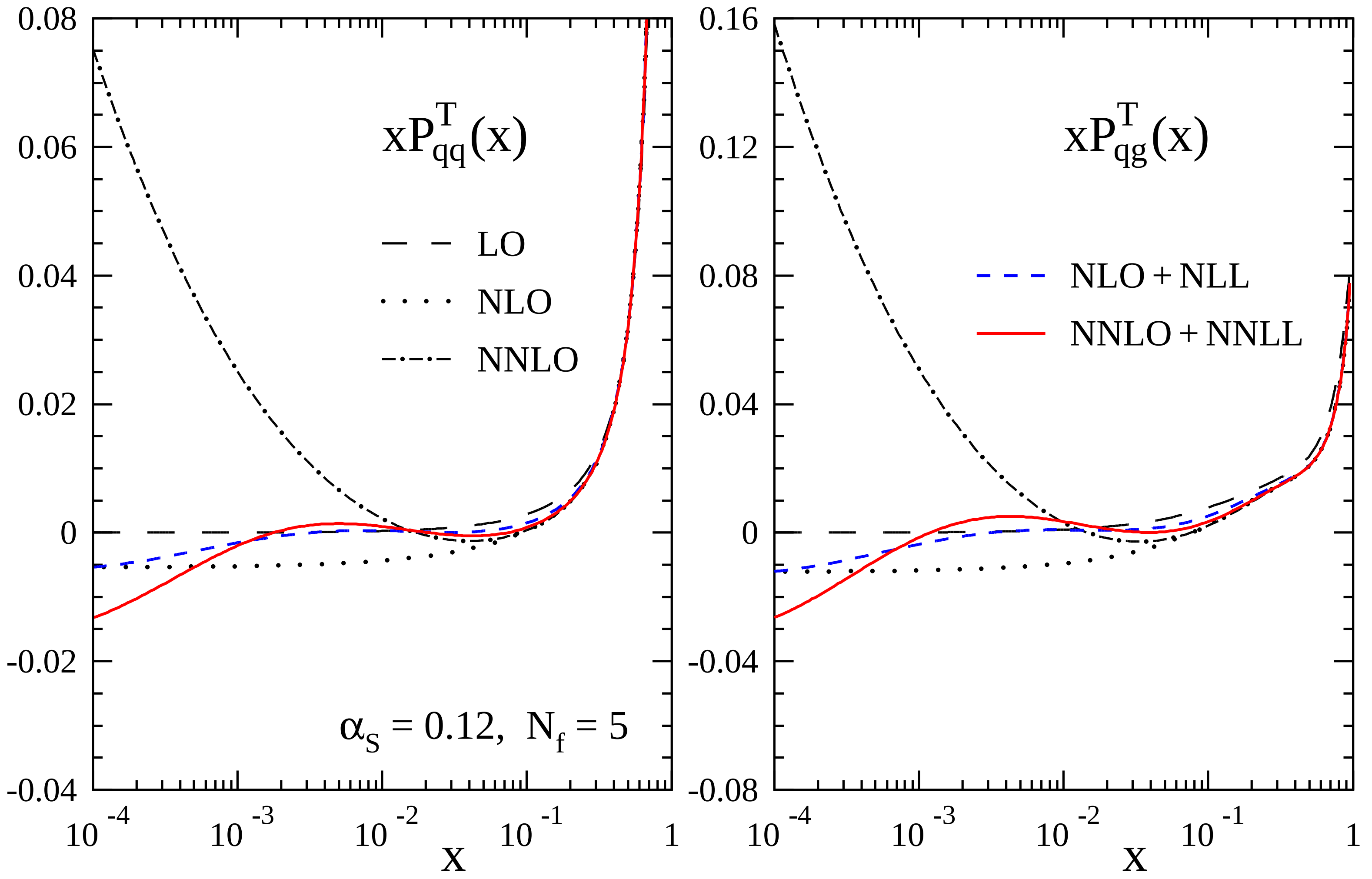
LL insufficient, near-perfect cancellation of NNLO rise by NNLL resummation

Small-x gluon-parton splitting functions (II)



Approximation sequence LO+LL, NLO+NLL, NNLO+NNLL rather stable to very small x

Small-x quark-parton splitting functions



Also consistent with $xP^T_{ji} \approx 0$ at $x < 10^{-2}$ (N^3LL corr's known and positive)

Summary and Outlook

- NNLO timelike splitting functions derived, almost, from spacelike results
  D-dim. cont'n, DMS generalization, DIS$/$SIA phys. kernels, SUSY limit
- Further calculations required to check $P_{\rm gq}^{(2)T}$ and completely fix $P_{\rm gq}^{(2)T}$
  Barring surprises: respective n_f and n_f^2 parts should be sufficient
- Fixed-order small-x behaviour far worse than in spacelike case
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- NNLL small-x resummation obtained from D-dim. unfactorized SIA
  LO+LL, NLO+NLL, NNLO+NNLL approximations reasonably stable
- All-order analytic forms only for LL and part of NLL (no $N=1$ poles)
  x-range limited by knowledge of series esp. at NNLL, now to $x \gtrsim 10^{-4}$
  (order $\alpha_{\rm s}^{16}$, helped by FORM wizardry by Jos Vermaseren)
- $N=1$ at NNLL (multiplicities): more work needed
  Paolo Bolzoni, next talk