

Electron-positron annihilation into heavy leptons at two loops

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Motivation

two loops correction $e^+e^- \rightarrow \mu^+\mu^-$

- Luminosity determination
- Detector calibration
- Precise measurement of hadronic cross sections (e.g. $e^+e^- \rightarrow \pi^+\pi^-$), vital for the $(g-2)_\mu$ anomaly
- Understanding backgrounds for New Physics searches

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two loops correction $e^+e^- \rightarrow \tau^+\tau^-$

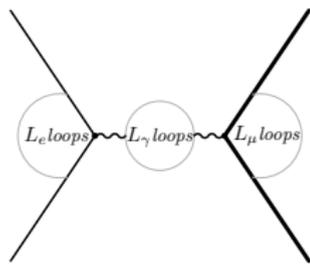
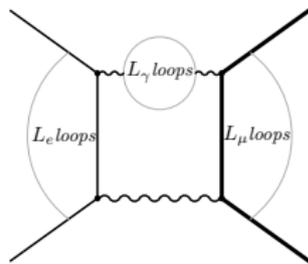
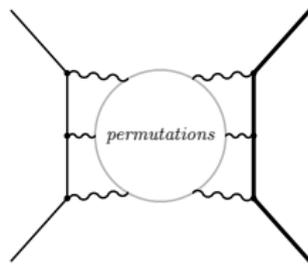
- Test lepton universality
- Precise measurement of τ decay modes

Challenges at NNLO full calculation

- **Two-loop amplitudes:** Massive propagators significantly complicate calculations
- **Divergences:** soft, ultraviolet, and collinear
- **Regularization:** Dimensional regularization $d = 4 - 2\varepsilon$ for UV/IR divergences
- **Electron mass:** Cannot be set to zero (m_e regulates collinear divergences), but power corrections $\frac{m_e^2}{m_\mu^2} \sim 2.3 \times 10^{-5}$ are negligible
- **Muon mass:** Full dependence essential for threshold region $s \sim 4m_\mu^2$
- **Kinematics:** Two dimensionless parameters $\frac{s}{m_\mu^2}, \frac{t}{m_\mu^2}$

Diagram Classification and C-parity

Diagrams

 $\mathcal{A}_{1\gamma}$  $\mathcal{A}_{2\gamma}$  $\mathcal{A}_{3\gamma}$

$$\mathcal{A} = \sum_{L=0}^{\infty} a^{L+1} \sum_{n=1}^{L+1} \mathcal{A}_{n\gamma}^{(L)} \quad \text{where } a = \frac{\alpha}{4\pi}$$

$$C\mathcal{A}_{n\gamma}^{(L)} = (-1)^{n\gamma} \mathcal{A}_{n\gamma}^{(L)}$$

C-even

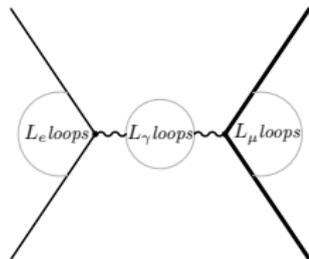
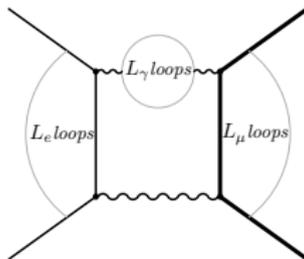
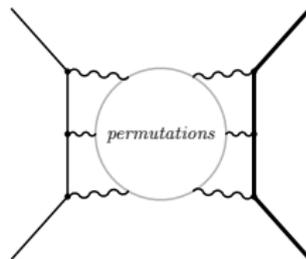
 $(2\gamma, 4\gamma, \dots)$

C-odd

 $(1\gamma, 3\gamma, \dots)$

C-even part of cross section

Cross section

 $A_{1\gamma}$  $A_{2\gamma}$  $A_{3\gamma}$

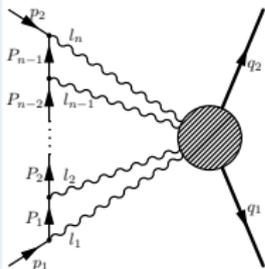
$$d\sigma = |\mathcal{A}|^2 d\Phi = d\sigma_{C\text{-even}} + d\sigma_{C\text{-odd}}$$

$$d\sigma_{C\text{-even}} = a^2 \left[|\mathcal{A}_{1\gamma}^{(0)}|^2 + 2\Re \mathcal{A}_{1\gamma}^{(1)} \mathcal{A}_{1\gamma}^{(0)*} a + \left(|\mathcal{A}_{1\gamma}^{(1)}|^2 + |\mathcal{A}_{2\gamma}^{(1)}|^2 + 2\Re \mathcal{A}_{1\gamma}^{(2)} \mathcal{A}_{1\gamma}^{(0)*} + 2\Re \mathcal{A}_{3\gamma}^{(2)} \mathcal{A}_{1\gamma}^{(0)*} \right) a^2 \right] d\Phi$$

$$d\sigma_{C\text{-odd}} = 2a^3 \Re \left[\mathcal{A}_{2\gamma}^{(1)} \mathcal{A}_{1\gamma}^{(0)*} + \left(\mathcal{A}_{2\gamma}^{(1)} \mathcal{A}_{1\gamma}^{(1)*} + \mathcal{A}_{2\gamma}^{(2)} \mathcal{A}_{1\gamma}^{(0)*} \right) a \right] d\Phi.$$

Collinear divergences

Diagrams without photons connecting two points on the electron line

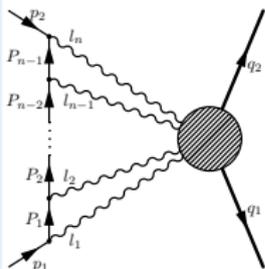


$$S = \int d^d l_1 \dots d^d l_n \delta \left(p_1 + p_2 - \sum_{k=1}^n l_k \right) \frac{j^{\mu_n \dots \mu_1} J_{\mu_n \dots \mu_1}}{P_{n-1}^2 \dots P_1^2 l_n^2 \dots l_1^2}$$

$$j^{\mu_1 \dots \mu_n} = \bar{v} \gamma^{\mu_n} \hat{P}_{n-1} \gamma^{\mu_{n-1}} \dots \hat{P}_1 \gamma^{\mu_1} u, \quad P_k = p_1 - \sum_{r=1}^k l_r,$$

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The collinear divergences may come from $l_1, \dots, l_k \propto p_1$ and/or $l_r, \dots, l_n \propto p_2$ with $1 \leq k < r \leq n$

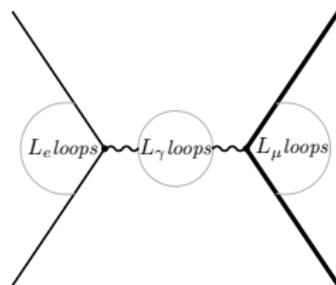
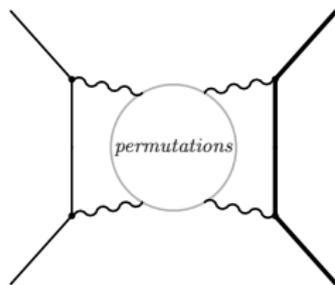
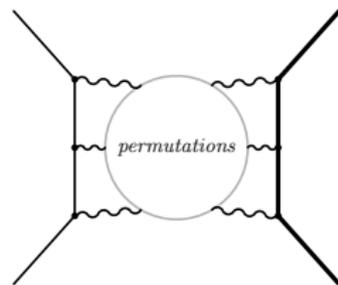
Numerator is suppressed due to $\hat{p}_1 u = \bar{v} \hat{p}_2 = 0$ and transversality condition:

$$l_k^{\mu_k} J_{\mu_n \dots \mu_1} = 0, \quad (k = 1, \dots, n)$$

In such diagrams the collinear divergences do not appear.

C-even part of cross section

Cross section


 $\mathcal{A}_{1\gamma} \quad m_e \neq 0$

 $\mathcal{A}_{2\gamma}^{(1)} \quad m_e = 0$

 $\mathcal{A}_{3\gamma}^{(2)} \quad m_e = 0$

$$d\sigma_{C\text{-even}} = a^2 \left[|\mathcal{A}_{1\gamma}^{(0)}|^2 + 2\Re \mathcal{A}_{1\gamma}^{(1)} \mathcal{A}_{1\gamma}^{(0)*} a + \left(|\mathcal{A}_{1\gamma}^{(1)}|^2 + |\mathcal{A}_{2\gamma}^{(1)}|^2 + 2\Re \mathcal{A}_{1\gamma}^{(2)} \mathcal{A}_{1\gamma}^{(0)*} + 2\Re \mathcal{A}_{3\gamma}^{(2)} \mathcal{A}_{1\gamma}^{(0)*} \right) a^2 \right] d\Phi$$

$d\sigma_{C\text{-even}}$ - symmetric with $\mu^+ \leftrightarrow \mu^-$

$$d\sigma_{C\text{-even}}(\cos \theta) = d\sigma_{C\text{-even}}(-\cos \theta)$$

Soft divergences

Factorization of soft divergences [Yennie, Frautschi, Suura, 1961]

$$\mathcal{A} = e^{\mathcal{V}} \mathcal{H}, \quad \mathcal{V} = -\sum_{i<j} Q_i Q_j V(p_i, p_j)$$

$$V(p_i, p_j) = -\frac{e^2}{2} \int \frac{d^d k}{i(2\pi)^{d/2}} \frac{1}{k^2 + i0} \left(\frac{2p_i - k}{k^2 - 2kp_i + i0} + \frac{2p_j + k}{k^2 + 2kp_j + i0} \right)^2$$

Hard amplitude \mathcal{H} is finite, so we can set $\varepsilon = \frac{4-d}{2} = 0$

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Cross section

$$d\sigma_{\text{incl}}(\omega_0) = e^{\mathcal{W}(\omega_0) + 2\Re\mathcal{V}} |\mathcal{H}|^2 d\Phi.$$

$$W(p_i, p_j | \omega_0) = -e^2 \int_{\omega < \omega_0} \frac{d^{d-1}k}{(2\pi)^{d-1} 2\omega} \left(\frac{p_i}{k \cdot p_i} - \frac{p_j}{k \cdot p_j} \right)^2$$

While $\mathcal{W}(\omega_0)$ and \mathcal{V} are divergent, sum $\mathcal{W}(\omega_0) + 2\Re\mathcal{V}$ is finite

Tensor decomposition

Helicity amplitudes $\mathfrak{H}_{\lambda_e - \lambda_{e^+} \lambda_{\mu^-} - \lambda_{\mu^+}}$

For $m_e = 0$, helicity conservation require $\mathfrak{H}_{\pm\pm\lambda_{\mu^-}\lambda_{\mu^+}} = 0$

$$\begin{aligned} \mathfrak{H}_{+-++} \stackrel{P}{=} \mathfrak{H}_{-+--}, \quad \mathfrak{H}_{+--+} \stackrel{P}{=} \mathfrak{H}_{-+-+}, \quad \mathfrak{H}_{+---} \stackrel{P}{=} \mathfrak{H}_{-+++}, \quad \mathfrak{H}_{+----} \stackrel{P}{=} \mathfrak{H}_{-++++}, \\ \mathfrak{H}_{++++} \stackrel{C}{=} \mathfrak{H}_{+---}, \quad \mathfrak{H}_{+---} \stackrel{C}{=} \mathfrak{H}_{++++} \end{aligned}$$

We have 3 independent helicity amplitudes: $\mathfrak{H}_{+-++}, \mathfrak{H}_{+--+}, \mathfrak{H}_{+---}$

Tensor decomposition

Helicity amplitudes $\mathfrak{H}_{\lambda_e - \lambda_{e+} \lambda_\mu - \lambda_{\mu+}}$

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$$\begin{aligned} \mathfrak{H}_{+---+} &\stackrel{P}{=} \mathfrak{H}_{-+-+}, & \mathfrak{H}_{+--+} &\stackrel{P}{=} \mathfrak{H}_{-+-+}, & \mathfrak{H}_{+---+} &\stackrel{P}{=} \mathfrak{H}_{-+-+}, & \mathfrak{H}_{+----} &\stackrel{P}{=} \mathfrak{H}_{-++++}, \\ & & \mathfrak{H}_{++++-} &\stackrel{C}{=} \mathfrak{H}_{+---+}, & \mathfrak{H}_{+---+} &\stackrel{C}{=} \mathfrak{H}_{+----} \end{aligned}$$

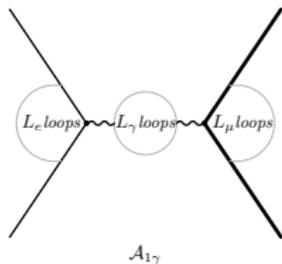
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Basis

$$\begin{aligned} \mathcal{T}_1 &= \bar{U} V \bar{v} (\hat{q}_1 - \hat{q}_2) u, \\ \mathcal{T}_3 &= \bar{U} (\hat{p}_1 - \hat{p}_2) V \bar{v} (\hat{q}_1 - \hat{q}_2) u, \\ \mathcal{T}_4 &= \bar{U} \gamma_\mu V \bar{v} \gamma^\mu u \end{aligned}$$

$$\mathcal{A} = A_1 \mathcal{T}_1 + A_3 \mathcal{T}_3 + A_4 \mathcal{T}_4.$$

1 γ reducible diagram



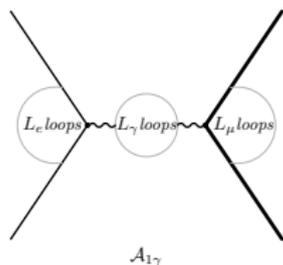
$$\mathcal{A}_{1\gamma} = \frac{a}{s[1 - \Pi_{tot}(s)]} \bar{U}(\Gamma^{\nu})_{(\mu)} V \bar{v}(\bar{\Gamma}_{\nu})_{(e)} u$$

$$\mathcal{A}_{1\gamma} = A_{1,1\gamma} \mathcal{T}_1 + A_{4,1\gamma} \mathcal{T}_4$$

$$A_{1,1\gamma} = -\frac{aF_{1,(e)}F_{2,(\mu)}}{2Ms[1 - \Pi_{tot}(s)]}$$

$$A_{4,1\gamma} = \frac{aF_{1,(e)}(F_{1,(\mu)} + F_{2,(\mu)})}{s[1 - \Pi_{tot}(s)]}$$

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Polarization operator

$$\Pi_{tot}(q^2) = \sum_{\ell} \Pi_{\ell}(q^2) + \Pi^{(\text{had})}(q^2).$$

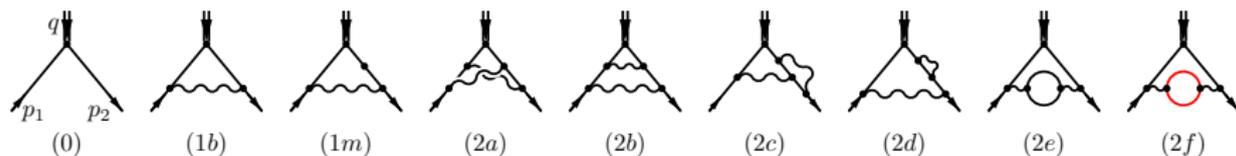
$$\Pi^{(\text{had})}(q^2) = -\frac{\alpha q^2}{3\pi} \int_{s_0}^{\infty} \frac{ds R(s)}{s(s - q^2)} \quad R(s) = \frac{\sigma_{e^+e^- \rightarrow \text{hadrons}}}{\sigma_{e^+e^- \rightarrow \mu^+\mu^-, \text{Born}, m_{\mu}=0}}$$

$\Pi_{\ell}(q^2) = \Pi(q^2/m_{\ell}^2)$ are known up to two loops [Kallen, Sabry 1955]

$\Pi^{(\text{had})}(q^2)$ can be found here [Ignatov 2008] from experimental data

Form factor

QED contributions



$$\Gamma^\nu = \gamma^\nu F_1(s) - \frac{\sigma^{\nu\rho} q_\rho}{2m} F_2(s)$$

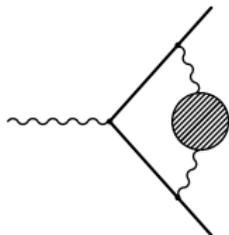
For $F_2(s) \propto m^2/s$ at $s \gg m^2$, we can neglect $F_2(s)$ for electron

Lepton form-factors were obtained in [Mastrolia, Remiddi 2003], [Bonciani, Mastrolia, Remiddi 2004]

We have recalculated it and found some typos in $F_1(s)$

Form factor

Hadronic vacuum polarization insertion



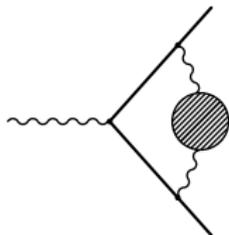
$$F_k^{(\text{had})}(s) = \frac{16a^2}{3} \int_{s_0}^{\infty} \frac{ds_1}{s_1} K_k \left(\frac{s}{m^2}, \frac{s_1}{m^2} \right) R(s_1)$$

integral kernels $K_{1,2}$ are expressed in terms of dilogarithms

$$R(s) = \frac{\sigma_{e^+e^- \rightarrow \text{hadrons}}}{\sigma_{e^+e^- \rightarrow \mu^+\mu^-, \text{Born}, m_\mu=0}} \text{ can be taken from exp data + QCD}$$

Form factor

Hadronic vacuum polarization insertion

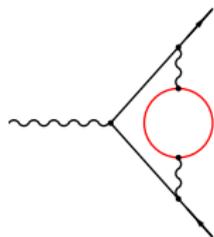


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Another lepton flavor vacuum polarization insertion

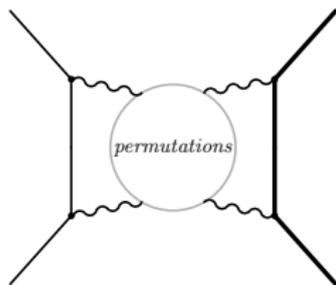
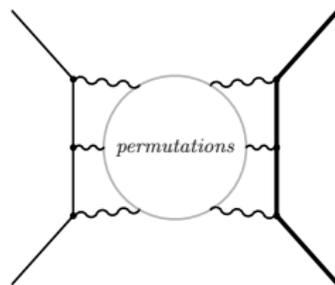


Was obtained in [Ahmed, et. al 2024] but it valid only in the kinematic region: $m_i < m$, $q^2 < 4(m^2 - m_i^2) < 0$

We recalculated it and got a simpler result that works in the entire kinematic domain

2γ and 3γ reducible diagram

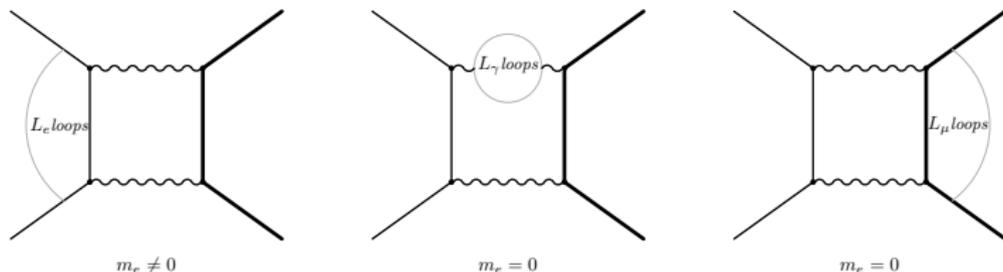
Gauge-invariant sum of diagrams


 $\mathcal{A}_{2\gamma}^{(1)}$

 $\mathcal{A}_{3\gamma}^{(2)}$

- We can set $m_e = 0$
- $\mathcal{A}_{3\gamma}$ integrals from [Mingulov, Lee 2019]
- Result: Goncharov polylogarithms up to weight 4

C-odd part of cross section

Angular asymmetry



First diagram integrals: Frobenius series in m_e [Lee 2024]

$$d\sigma_{C\text{-odd}} = 2a^3 \Re \left[\mathcal{A}_{2\gamma}^{(1)} \mathcal{A}_{1\gamma}^{(0)*} + \left(\mathcal{A}_{2\gamma}^{(1)} \mathcal{A}_{1\gamma}^{(1)*} + \mathcal{A}_{2\gamma}^{(2)} \mathcal{A}_{1\gamma}^{(0)*} \right) a \right] d\Phi$$

$d\sigma_{C\text{-odd}}$ corresponds to charge asymmetry in $\mu^+ \leftrightarrow \mu^-$

$$d\sigma_{C\text{-odd}}(\cos \theta) = -d\sigma_{C\text{-odd}}(-\cos \theta)$$

Final Result: Structure of the Cross Section

C-even differential cross section

$$\frac{d\sigma_{C\text{-even}}}{d\Omega} = \frac{d\sigma_0}{d\Omega} [1 + \delta^{(1)} + \delta^{(2)}]$$

- $\delta^{(1)}$, $\delta^{(2)}$: NLO and NNLO corrections
- They are polynomials in logarithms:

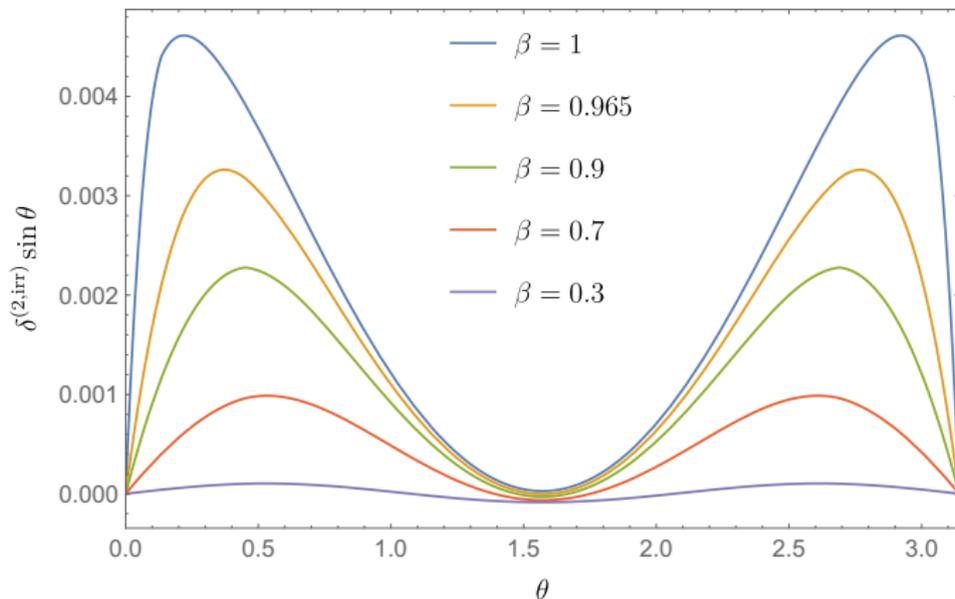
$$\delta^{(2)} = \sum_{k,n=0}^2 \delta_{kn}^{(2)} L_\omega^k L^n + \delta_{03}^{(2)} L^3$$

$$L = \ln(s/m_e^2), \quad L_\omega = \ln(\sqrt{s}/(2\omega_0))$$

- $\delta^{(2)} = \delta^{(2,\text{red})} + \delta^{(2,\text{irr})}$
 - Reducible $\delta^{(2,\text{red})}$: From 1γ diagrams. Dominant, mostly angle-independent
 - Irreducible $\delta^{(2,\text{irr})}$: From 2γ & 3γ diagrams. Smaller, but has strong angular dependence

Numerical result

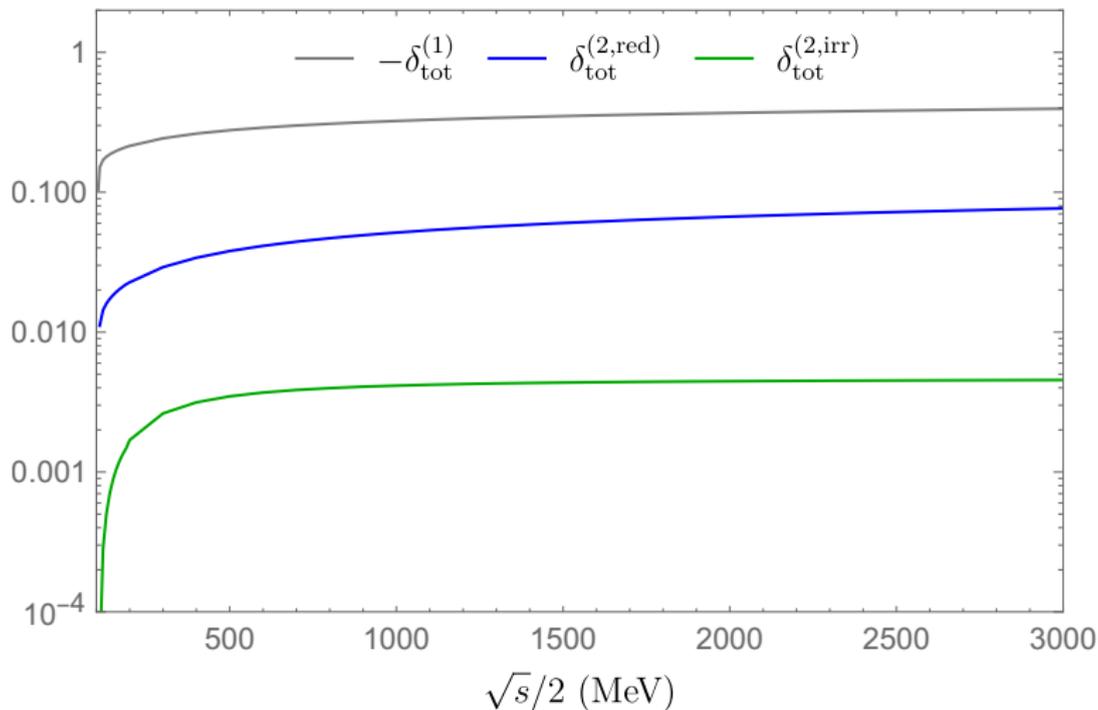
Angular Dependence of the NNLO Correction at $2\omega_0/\sqrt{s} = 0.01$



- $\delta^{(2,\text{irr})}$ shows a strong dependence on the scattering angle θ .
- It becomes large at small angles ($|c| \rightarrow 1$), behaving like $\ln^4(1 - \beta^2 c^2)$.

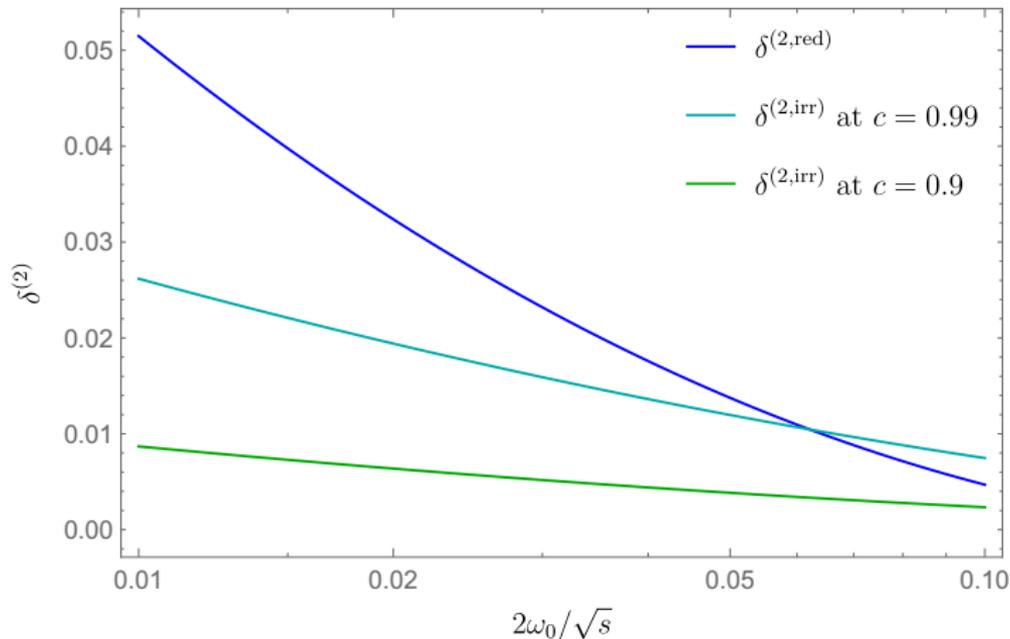
Numerical result

Total Cross Section at $2\omega_0/\sqrt{s} = 0.01$



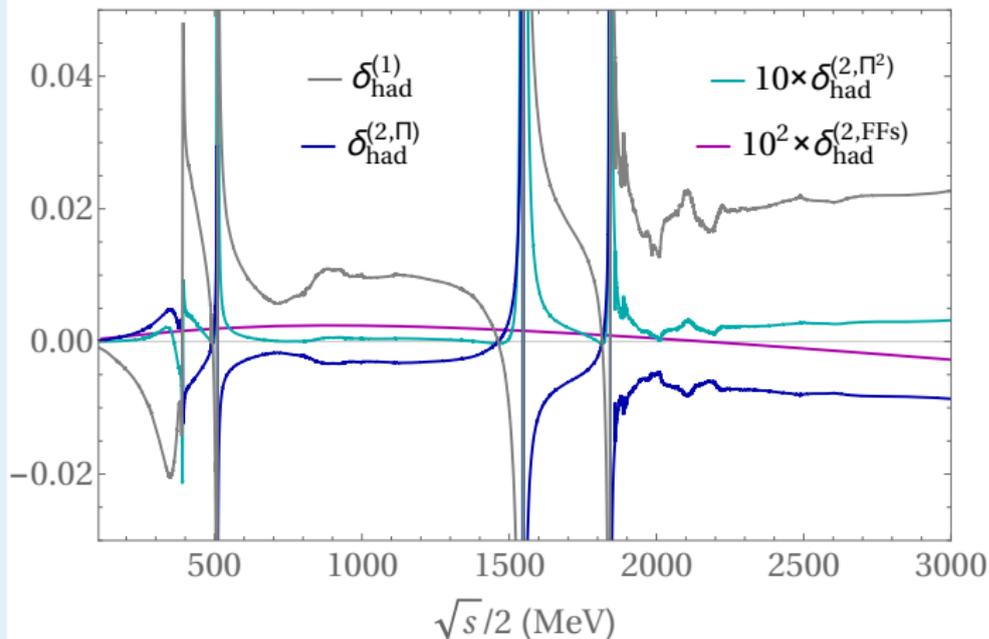
Numerical result

Dependence on soft photon cutoff ω_0 at $\sqrt{s} = 2$ GeV



The size of the NNLO correction is highly sensitive to the soft photon energy cutoff.

Hadronic contribution



Final Result: Structure of the Cross Section

C-odd differential cross section

$\frac{d\sigma_{\text{Codd}}}{d\Omega}$ corresponds to angular asymmetry $\frac{d\sigma_{\text{Codd}}}{d\Omega}(\cos\theta) = -\frac{d\sigma_{\text{Codd}}}{d\Omega}(-\cos\theta)$

$$\frac{d\sigma_{\text{Codd}}}{d\Omega} = \frac{d\sigma_0}{d\Omega} \left[1 + \delta_{\text{asym}}^{(1)} + \delta_{\text{asym}}^{(2)} \right]$$

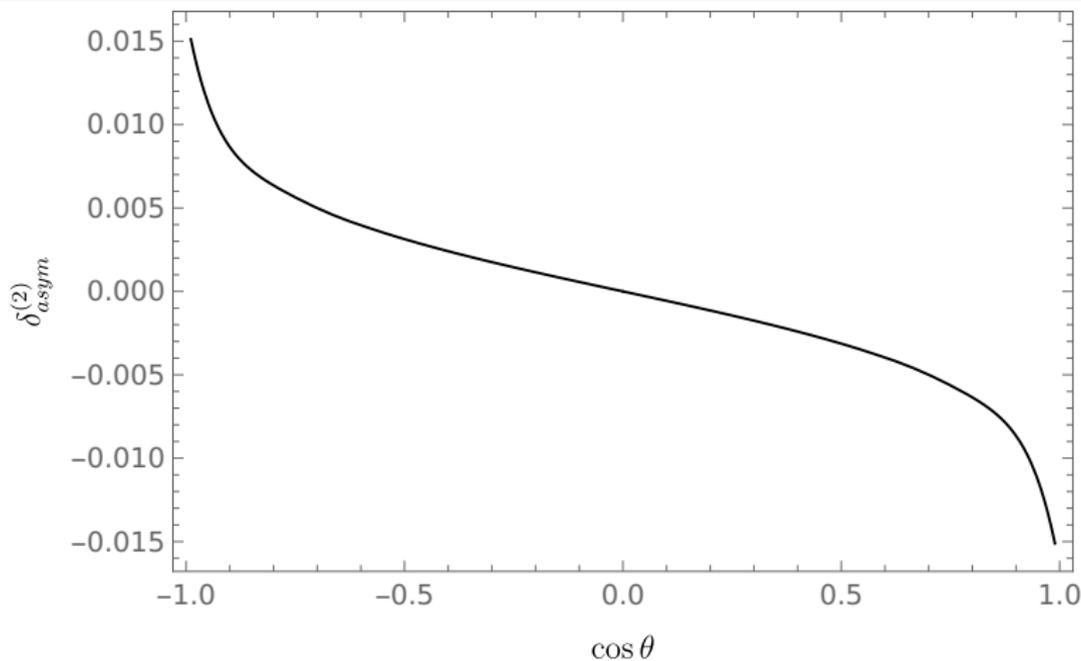
- $\delta_{\text{asym}}^{(1)}$, $\delta_{\text{asym}}^{(2)}$: NLO and NNLO corrections
- They are polynomials in logarithms:

$$\delta^{(2)} = \sum_{k,n=0}^2 \delta_{kn}^{(2)} L_\omega^k L^n$$

$$L = \ln(s/m_e^2), \quad L_\omega = \ln(\sqrt{s}/(2\omega_0))$$

Numerical result

Two loop asymmetry at $\sqrt{s} = 2$ GeV



$\delta_{\text{asym}}^{(2)} \sim 1\%$ at $\sqrt{s} = 2$ GeV, while $\delta_{\text{asym}}^{(1)} \sim 10\%$ and has an opposite sign

Summary

We have done:

- Complete NNLO corrections to $e^+e^- \rightarrow \mu^+\mu^-$ diff. cross section:
 - **exact result:** contains all QED and hadronic corrections
 - **exact at m_μ :** works in threshold region $s \gtrsim 4m_\mu^2$
 - **neglect:** power correction $(m_e/m_\mu)^2$, m_e appears only in $L = \ln(s/m_e^2)$
 - **IR and UV finite:** contain virtual and soft photon corrections
 - can be applied to arbitrary lepton polarization

See result in JHEP08(2025)118 or ArXiv: 2503.09245 and in ancillary files in it

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 - **IR and UV finite:** contain virtual and soft photon corrections
 - can be applied to arbitrary lepton polarization
- As a byproduct we recompute:
 - ordinary two loop form factor and find some typos
 - form factor with another lepton flavor vacuum polarization insertion valid for all kinematic region
 - hadronic contribution to the form factor
- Our results can be easily applied to the process $e^+e^- \rightarrow \tau^+\tau^-$

See result in JHEP08(2025)118 or ArXiv: 2503.09245 and in ancillary files in it

Conclusion

Need to do:

- **Include hard γ radiation:** real radiation correction at $\omega_0 \sim \sqrt{s}/2$
- **Extend to EW sector:** Electroweak corrections for $\sqrt{s} > 10$ GeV

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Thank you for attention!