Radiation of a Charge in a Waveguide with a Boundary between Two Dielectrics

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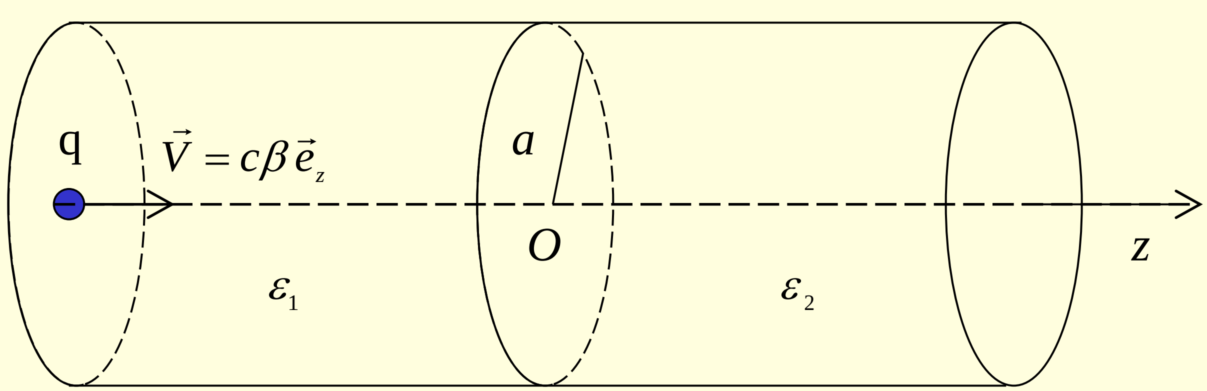
FORMULATION OF THE PROBLEM



$$\vec{E}_{1,2} = \vec{E}^{\,q}_{1,2} + \vec{E}^{\,b}_{1,2}, \qquad (1)$$

The "forced" field (called by V.L. Ginzburg [*]) is the field of the charge in a regular waveguide. It contains **Cherenkov radiation (CR)** if the charge velocity exceeds the Cherenkov threshold $\beta > \beta_{C1,2} = 1\big/\sqrt{\varepsilon_{1,2}}$.

$$E^{q}_{z1,2} = \frac{2iq}{\pi a^2} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\chi_{0n} r}{a}\right)}{J_1^2\left(\chi_{0n}\right)} \int_{-\infty}^{+\infty} \omega\, d\omega \frac{\left(\beta^2 \varepsilon_{1,2} - 1\right) \exp\left[i\omega\left(z/c\beta - t\right)\right]}{\varepsilon_{1,2}\left[\omega^2\left(1 - \beta^2 \varepsilon_{1,2}\right) + \beta^2 \omega_n^2\right]}, \qquad (2)$$

* ***V.L. Ginzburg and V.N. Tsytovich, "Transition Radiation and Transition Scattering", Hilger, London, p. 445 (1990).***

The "free" field is connected with the influence of the boundary. It includes **transition radiation (TR)**.

$$E^b_{z1,2} = \frac{2iq\beta}{\pi a^2 \varepsilon_{1,2}} \sum_{n=1}^{\infty} \omega_n^2 \frac{J_0\left(\frac{\chi_{0n} r}{a}\right)}{J_1^2\left(\chi_{0n}\right)} \int_{-\infty}^{+\infty} d\omega \tilde{B}_{n1,2} \exp\left[-i\left(\omega t - k_{z1,2}|z|\right)\right], \quad (3)$$

$$\tilde{B}_{n1,2} = \frac{\left(\varepsilon_{1,2} - \varepsilon_{2,1}\right)\left[\omega\left(\pm \varepsilon_{1,2}^2 \beta^2 \mp 1\right) - \beta\sqrt{\omega^2 \varepsilon_{2,1} - \omega_n^2}\right]}{\tilde{g}(\omega)\left[\omega^2\left(1 - \varepsilon_{1,2}^2 \beta^2\right) + \beta^2 \omega_n^2\right]\left(\omega \pm \beta\sqrt{\omega^2 \varepsilon_{2,1}^2 - \omega_n^2}\right)}, \quad (4)$$

$$\tilde{g}(\omega) = \varepsilon_1 \sqrt{\omega^2 \varepsilon_2^2 - \omega_n^2} + \varepsilon_2 \sqrt{\omega^2 \varepsilon_1^2 - \omega_n^2}, \quad (5)$$

$$k_{z1,2} = \frac{1}{c}\sqrt{\omega^2 \varepsilon_{1,2} - \omega_n^2}, \qquad \operatorname{Im} k_{z1,2} > 0 \quad (6)$$

$$\omega_n = \frac{\chi_{0n} c}{a},$$

χ_{0n} is the n^{th} zero of the Bessel function $\left(J_0\left(\chi_{0n}\right) = 0\right)$

Methods of analysis

We investigate the exact solution with analytical and computational methods.
Analytical research is an asymptotic investigation using the steepest descent technique.
Computations are based on original algorithm using certain transformation of the initial integration path. Such approaches were applied as well in some papers concerning both boundless homogenous media and problems with interface between two media

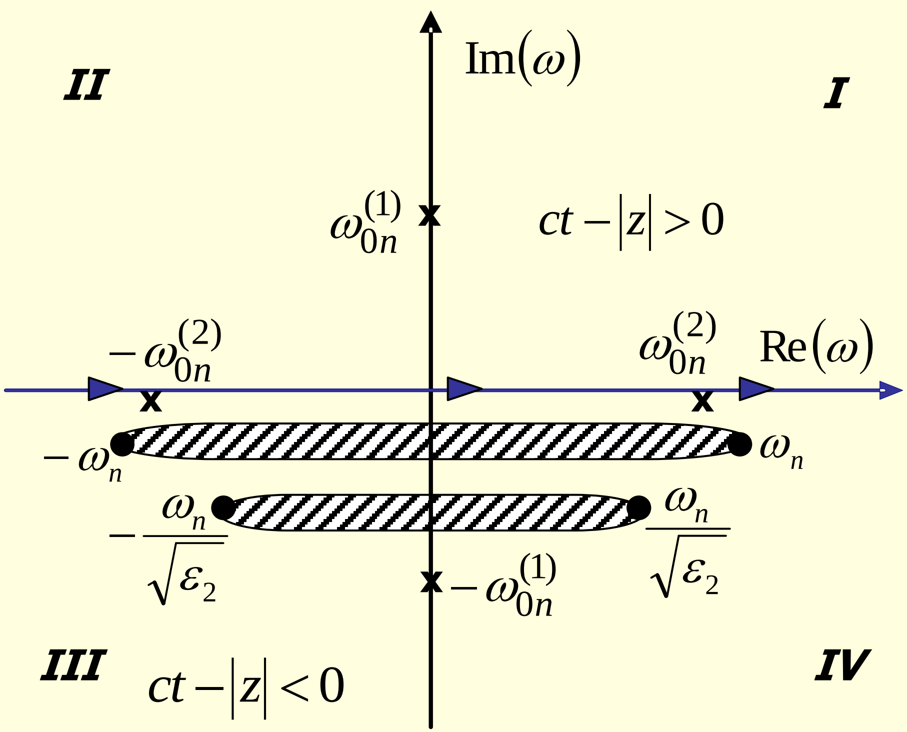
A.V. Tyukhtin and S.N. Galyamin Phys. Rev. E 77 (2008) 066606,
S.N. Galyamin.and A.V. Tyukhtin Phys. Rev. B 81 (2010) 35134

including the case of a waveguide partially filled with cold plasma

T.Yu. Alekhina and A.V. Tyukhtin, Phys. Rev. E 83 (2011) 066401.

The case of flying from vacuum into dielectric $\varepsilon_1 = 1,\ \varepsilon_2 > 1$

Vacuum: analysis of $\tilde{B}_{n1}$ in a complex plane (ω) at $\beta > \beta_{C2}$



Poles: $\pm\omega_{0n}^{(1)} = \pm\frac{i\beta\omega_n}{\sqrt{1-\beta^2}}$

$$\pm\omega_{0n}^{(2)} = \pm\frac{i\beta\omega_n}{\sqrt{1-\varepsilon_2\beta^2}}, \quad \beta < \frac{1}{\sqrt{\varepsilon_2}}$$

$$\pm\omega_{0n}^{(2)} = \pm\frac{\beta\omega_n}{\sqrt{\varepsilon_2\beta^2 - 1}} - i\delta_1, \quad \beta > \frac{1}{\sqrt{\varepsilon_2}}$$

Branch points : $\pm\tilde{\omega}_n^{(1)} = \pm\omega_n - i\delta_2$

$$\pm\tilde{\omega}_n^{(2)} = \pm\omega_n / \sqrt{\varepsilon_2} - i\delta_3$$

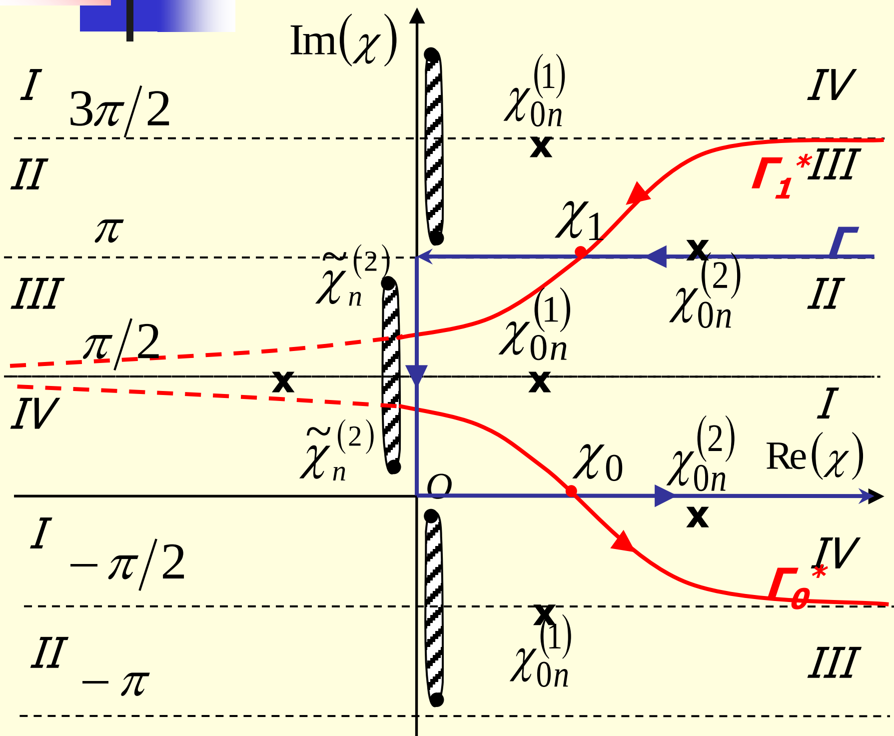
Branch cuts : $\mathrm{Re}\sqrt{\omega^2 - \left(\tilde{\omega}_n^{(1)}\right)^2} = 0$

$$\mathrm{Re}\sqrt{\omega^2 - \left(\tilde{\omega}_n^{(2)}\right)^2} = 0$$

The case of flying from vacuum into dielectric $\varepsilon_1 = 1, \varepsilon_2 > 1$

The steepest descend technique: the change of variable and transformation to a complex plane (χ).

Vacuum



The change of variable:

$$\omega = \omega_n\, ch\chi, \sqrt{\omega^2 - \omega_n^2} = \omega_n\, sh\chi$$

Singularities of integrands $\tilde{B}_{n1}$ in a complex plane (ω): branch points $\tilde{\omega}_n^{(2)}$ and poles $\omega_{0n}^{(1)}$, $\omega_{0n}^{(2)}$ turn into $\tilde{\chi}_n^{(2)}, \chi_{0n}^{(1)}$ and $\chi_{0n}^{(2)}$ correspondingly.

Saddle points: $\chi_0 = arch(ct/R)$,

$$\chi_1 = \chi_0 + i\pi,\ R = \sqrt{(ct)^2 - z^2}$$

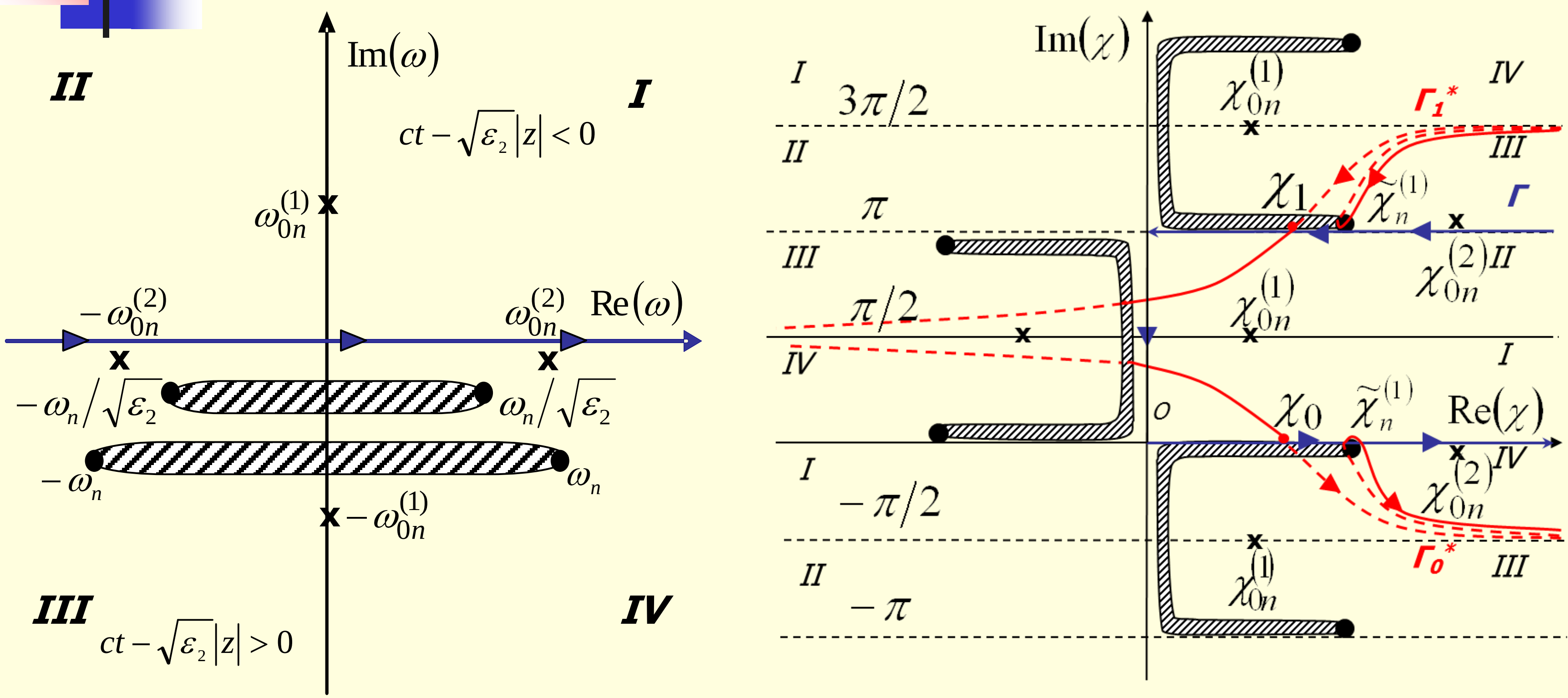
SDP (***Γ*₀***, ***Γ*₁***): $\cos(\chi'') = \dfrac{\pm 1}{ch(\chi' - \chi_0)}$,

$$sh(\chi' - \chi_0)\sin(\chi'') < 0$$

If $\mathrm{Re}\,\chi_{0,1} < \mathrm{Re}\,\chi_{0n}^{(2)}$ the poles can be crossed in transformation of contour ***Γ*** into the new contour ***Γ**** passing through the saddle points, and the contributions from the corresponding singularities should be included in asymptotic expressions.

The case of flying from vacuum into dielectric $\varepsilon_1 = 1,\ \varepsilon_2 > 1$

Dielectric: analysis of $\tilde{B}_{n2}$ in a complex planes of (ω) (left) and of (χ) (right) at $\beta > \beta_{C2}$



The change of variable: $\omega = \omega_n ch\chi / \sqrt{\varepsilon_2}\,,\ \sqrt{\omega^2\varepsilon_2 - \omega_n^2} = \omega_n sh\chi/\sqrt{\varepsilon_2}$

The poles (if $\mathrm{Re}\,\chi_{0,1} < \mathrm{Re}\,\chi_{0n}^{(2)}$) and the branch points (if $\mathrm{Re}\,\chi_{0,1} < \mathrm{Re}\,\tilde{\chi}_n^{(2)}$) can be crossed in the transformation of ***Γ*** into ***Γ****.

The case of flying from vacuum into dielectric $\varepsilon_1 = 1, \varepsilon_2 > 1$

The field in vacuum

Contributions of poles gives **Cherenkov transition radiation (CTR)**:

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$$E_{z1}^{CTR} = 0, \tag{7}$$

The field in dielectric

$$E_{z2}^{CTR} = \frac{4q}{a^2 \varepsilon_2} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\chi_{0n} r}{a}\right)}{J_1^2\left(\chi_{0n}\right)} \cos\left[\frac{\omega_n}{c\sqrt{\varepsilon_2 \beta^2 - 1}}\left(ct\beta - z\right)\right] \Theta\left(z_2 - z\right), \qquad z_2 = \frac{ct}{\beta \varepsilon_2} \tag{8}$$

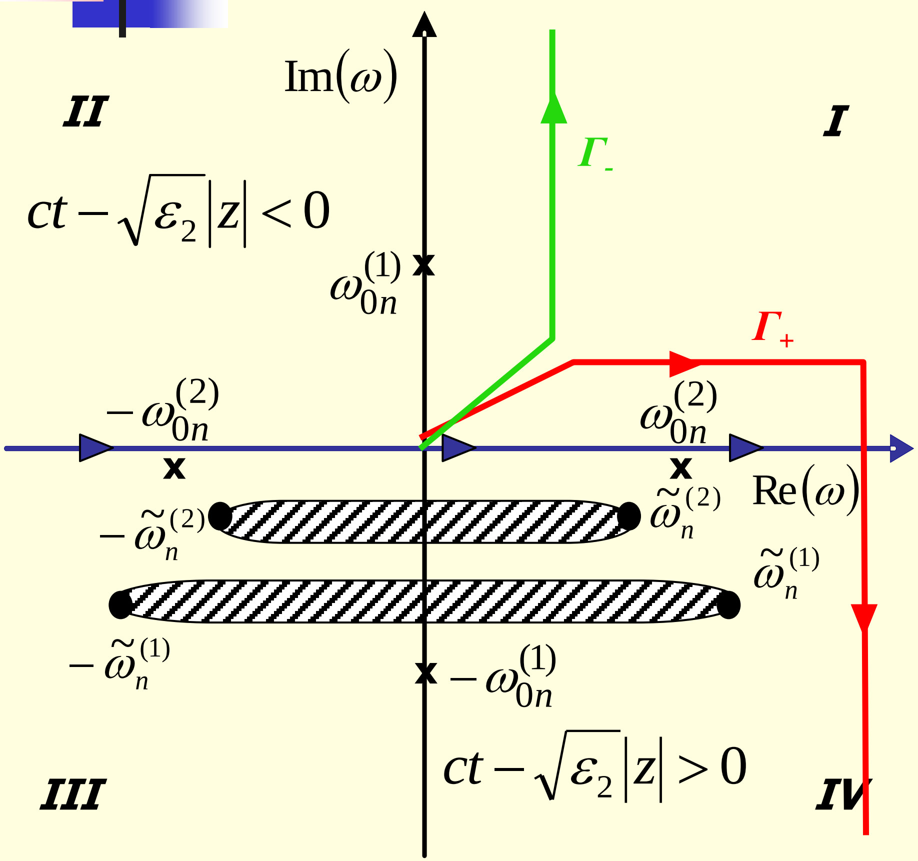
CTR is the exact expressions for the forced field taken with opposite sign. So, there is a compensation for the forced wave field (so-called "wakefield") with a part of free one in some domain $z < z_2$.

Numerical approach

The dielectric area

$$E_{z2}^{b} = \frac{2iq\beta c^{2}}{\pi a^{4}\varepsilon_{2}} \sum_{n=1}^{\infty} \chi_{0n}^{2} \frac{J_{0}\left(\frac{\chi_{0n} r}{a}\right)}{J_{1}^{2}\left(\chi_{0n}\right)} I_{2}^{b},$$

$$I_{2}^{b} = \int_{-\infty}^{+\infty} d\omega \tilde{B}_{n2} \exp\left[-i\left(\omega t - k_{z2}|z|\right)\right],$$



$$\varepsilon_1 = 1,\ \varepsilon_2 > 1,\ \ \beta > \beta_{C2} = 1\Big/\sqrt{\varepsilon_2}$$

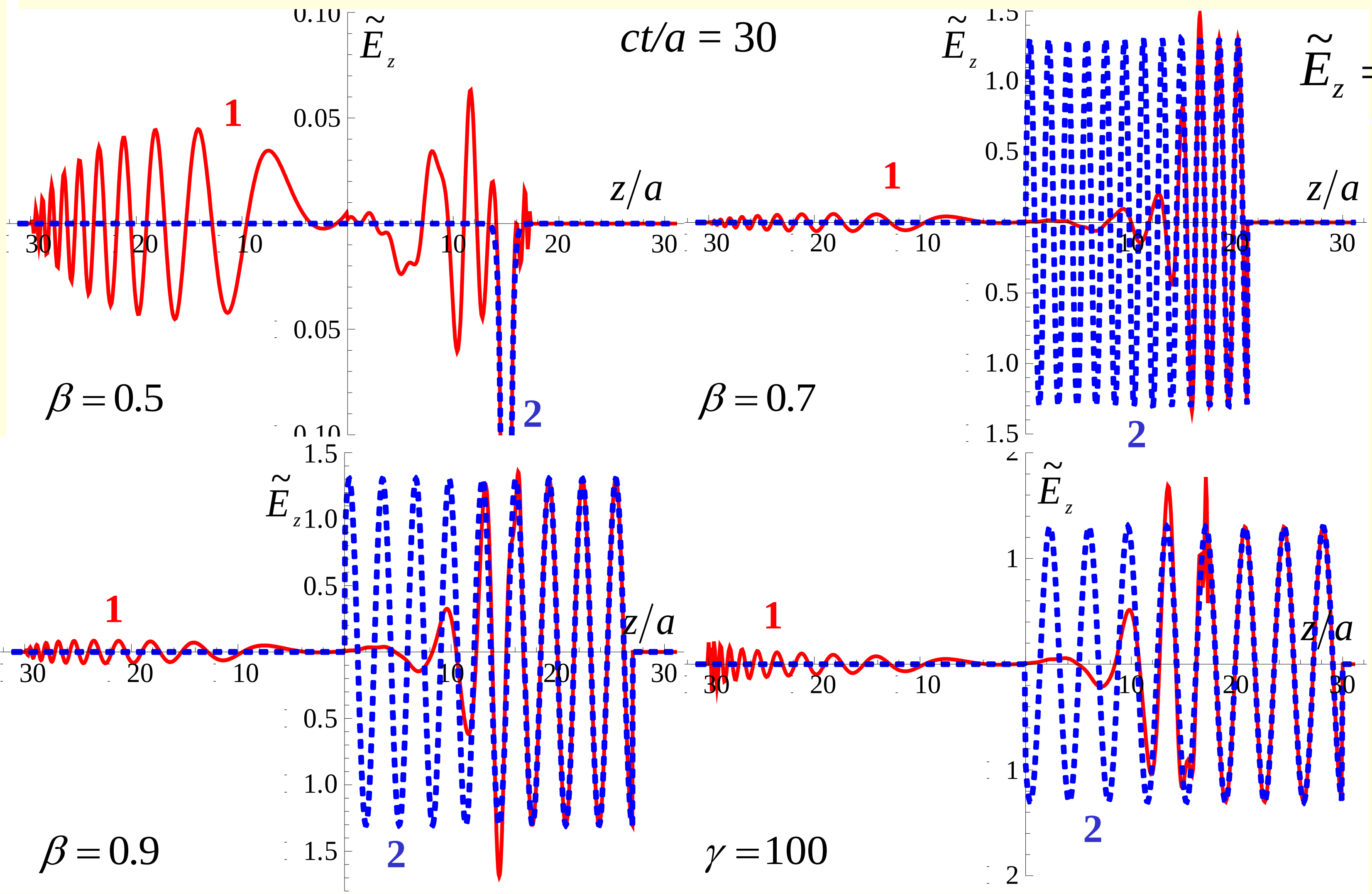
Integrands $\tilde{B}_{n2}$ decrease in the *I* and *II* quadrants of a complex plane (ω) at $ct - \sqrt{\varepsilon_2}|z| < 0$ and in the *III* and *IV* quadrants – at $ct - \sqrt{\varepsilon_2}|z| > 0$

Transformation of the initial contour in an upper half-plane into contour Γ_- **(green one)** before "wave front" and in a lower half-plane into contour Γ_+ **(red one)** behind "wave front".

The case of flying from vacuum into dielectric

$\varepsilon_1 = 1,\ \varepsilon_2 = 3$

Dependence of the first mode of the field $\tilde{E}_z$ in vacuum and dielectric on *z/a* for different velocities of the charge motion β.



$$\tilde{E}_z = E_z \frac{\pi c^2}{2q\omega_0^2}$$

$n = 1,$

$a = 5\ mm,$

$r = 0$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$z < z_2,$

$$z_2 = \frac{ct}{\beta\varepsilon_2}$$

$\omega_0 = 2\pi \cdot 10 GHz$

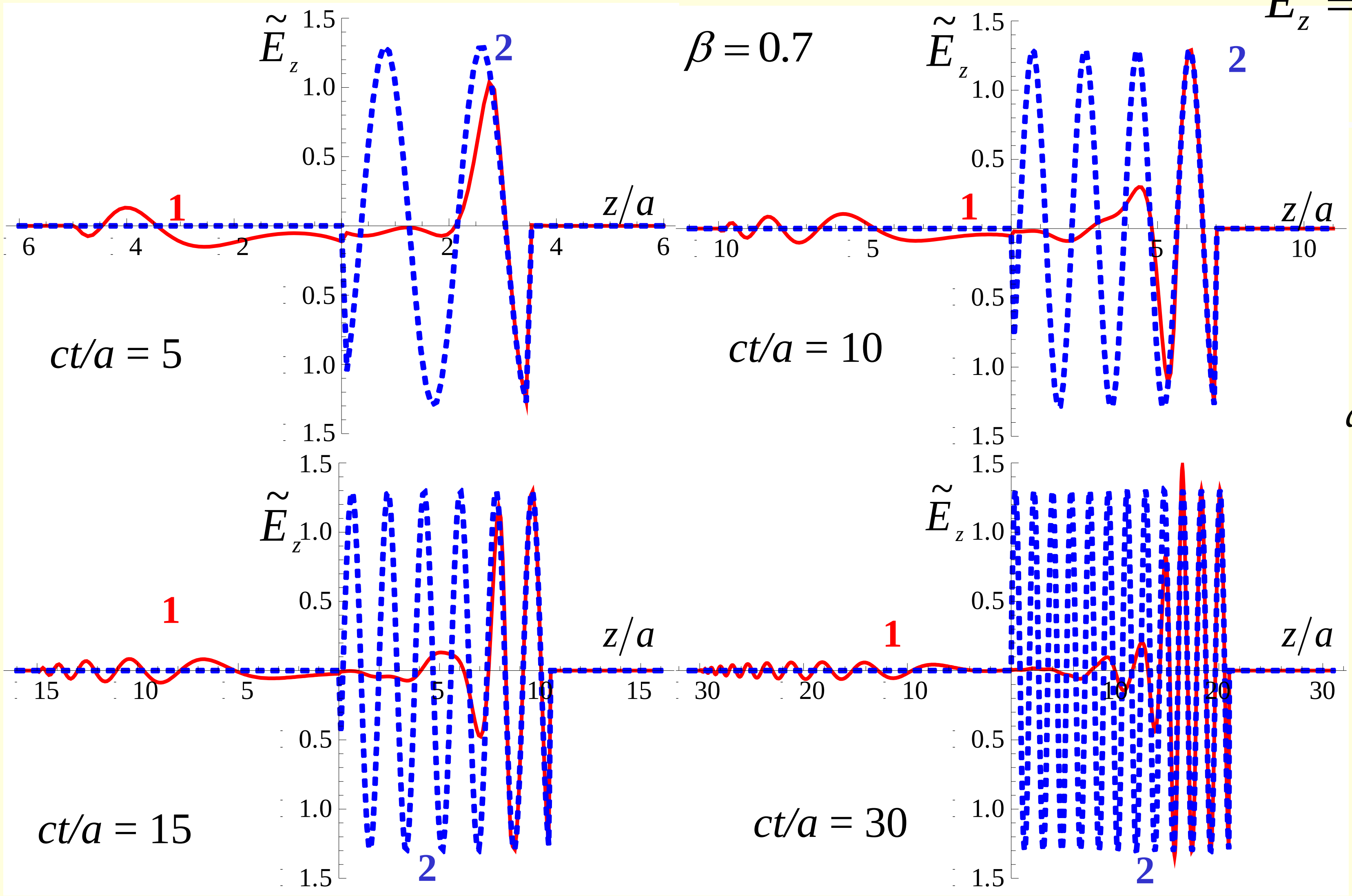
1 – the whole field, 2 – the forced field

The case of flying from vacuum into dielectric

$\varepsilon_1 = 1,\ \varepsilon_2 = 3$

Dependence of the first mode of the field $\tilde{E}_z$ in vacuum and dielectric on *z/a* at different moments *ct/a*.

$$\tilde{E}_z = E_z \frac{\pi c^2}{2q\omega_0^2}$$



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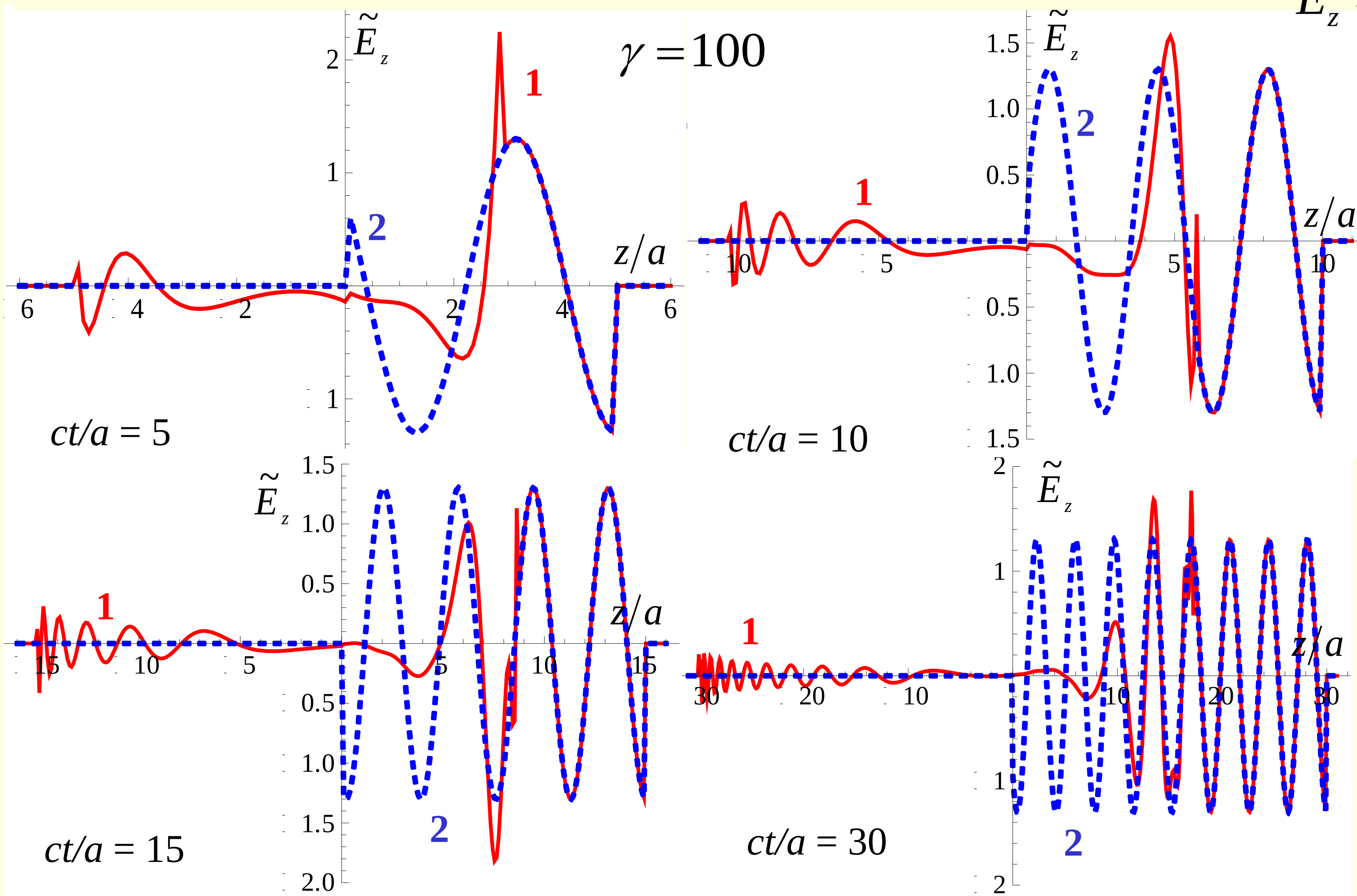


$n = 1,$

$a = 5\ mm,$

$r = 0$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$z < z_2,$$

$$z_2 = \frac{ct}{\beta\varepsilon_2}$$

$\omega_0 = 2\pi \cdot 10\,GHz$

1 – the whole field, 2 – the forced field

The case of flying from dielectric into vacuum $\varepsilon_1 > 1, \varepsilon_2 = 1$

The field in dielectric

Contributions of poles give the reflected wave of CR (CTR). CTR exists at $\beta_{C1} < \beta < \beta_{CT1}$ in the area $|z| < z_1$, $z_1 = ct/\beta\varepsilon_1$:

$$E_{z1}^{CTR} = \frac{4q}{a^2\varepsilon_1} \frac{\left(\varepsilon_1\beta^2 - 1 - \sqrt{1-\beta^2(\varepsilon_1 - 1)}\right)}{\left(1+\sqrt{1-\beta^2(\varepsilon_1-1)}\right)\left(1+\varepsilon_1\sqrt{1-\beta^2(\varepsilon_1-1)}\right)} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\chi_{0n} r}{a}\right)}{J_1^2(\chi_{0n})} \cos\left[\frac{\chi_{0n}(ct\beta - |z|)}{a\sqrt{\varepsilon_1\beta^2 - 1}}\right] \Theta(z_1 - |z|), \quad (9)$$

The field in vacuum

Contributions of poles give the transmitted wave of CR (CTR) at $\beta_{C1} < \beta < \beta_{CT1}$ in the area $z < z_3$:

$$E_{z2}^{CTR} = \frac{8q}{a^2\varepsilon_2} \frac{1}{\left(1+\varepsilon_1\sqrt{1-\beta^2(\varepsilon_1-1)}\right)} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\chi_{0n} r}{a}\right)}{J_1^2(\chi_{0n})} \cos\left[\frac{\chi_{0n}}{\sqrt{\varepsilon_1\beta^2-1}}\left(\frac{ct}{a}\beta - \frac{z}{a}\sqrt{1-\beta^2(\varepsilon_1-1)}\right)\right] \Theta(z_3 - z), \quad (10)$$

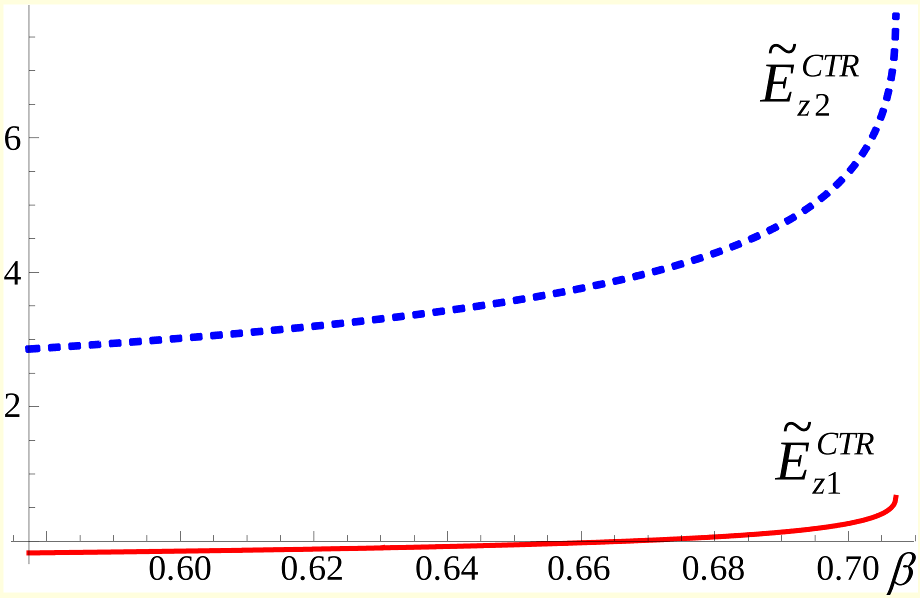
At $\beta > \beta_{CT1}$ the field decreases exponentially with the distance from the boundary.

$$z_3 = \frac{ct\sqrt{\left|1-\beta^2(\varepsilon_1-1)\right|}}{\beta}, \quad \beta_{C1} = 1/\sqrt{\varepsilon_1}, \quad \beta_{CT1} = 1/\sqrt{\varepsilon_1 - 1}$$

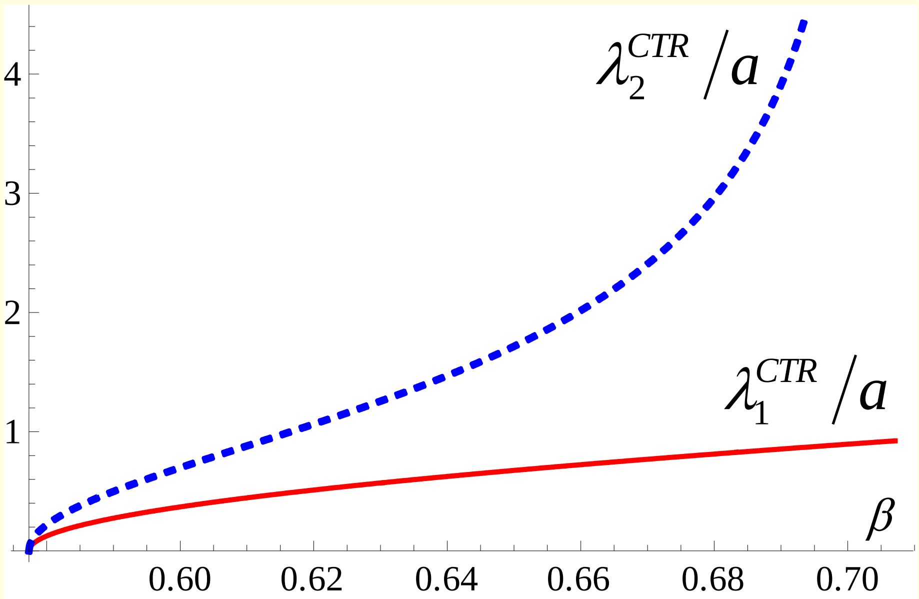
The case of flying from dielectric into vacuum $\varepsilon_1 = 3, \varepsilon_2 = 1$

Dependence of the amplitude $\tilde{E}_z^{CTR}$ and the wave length λ^{CTR}/a of the first mode of CTR in vacuum and dielectric on the velocity of the charge motion β.

The amplitude



The wave length



$$\beta_{C1} < \beta < \beta_{CT1},$$

$$\beta_{C1} = 1/\sqrt{\varepsilon_1} \approx 0.56,$$

$$\beta_{CT1} = 1/\sqrt{\varepsilon_1 - 1} \approx 0.71$$

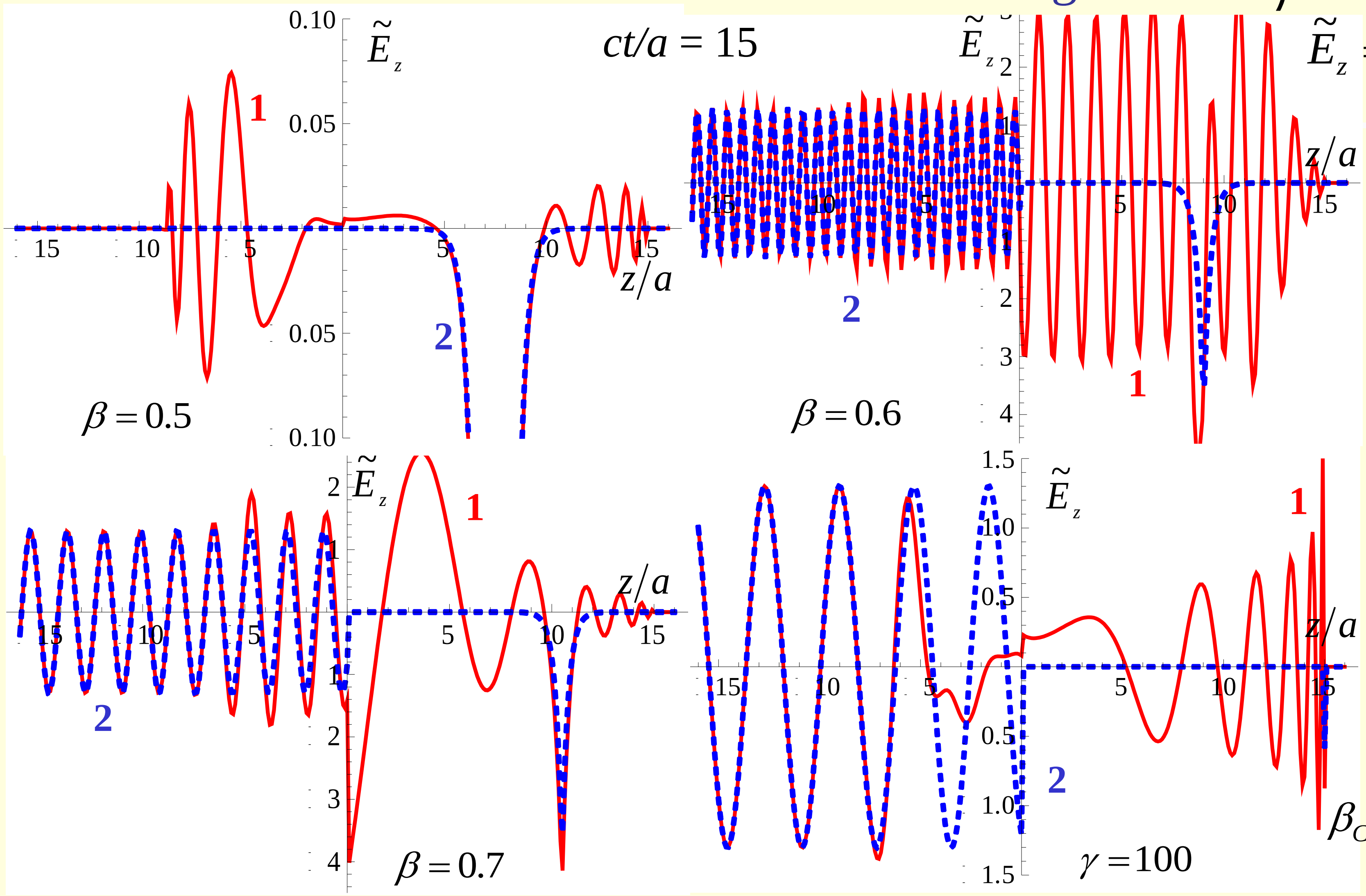
$$\tilde{E}_z = E_z \frac{\pi c^2}{2q\omega_0^2}, \quad \omega_0 = 2\pi \cdot 10GHz$$

$$n = 1,\ a = 5\ mm,\ r = 0$$

The case of flying from dielectric into vacuum

$\varepsilon_1 = 3,\ \varepsilon_2 = 1$

Dependence of the first mode of the field $\tilde{E}_z$ in vacuum and dielectric on z/a for different velocities of the charge motion β.



$$\tilde{E}_z = E_z \frac{\pi c^2}{2q\omega_0^2}$$

$n = 1$

$a = 5\ mm,$

$r = 0$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$|z| < z_1,$

$$z_1 = \frac{ct}{\beta\varepsilon_1}$$

$$\beta_{C1} = \frac{1}{\sqrt{\varepsilon_1}},$$

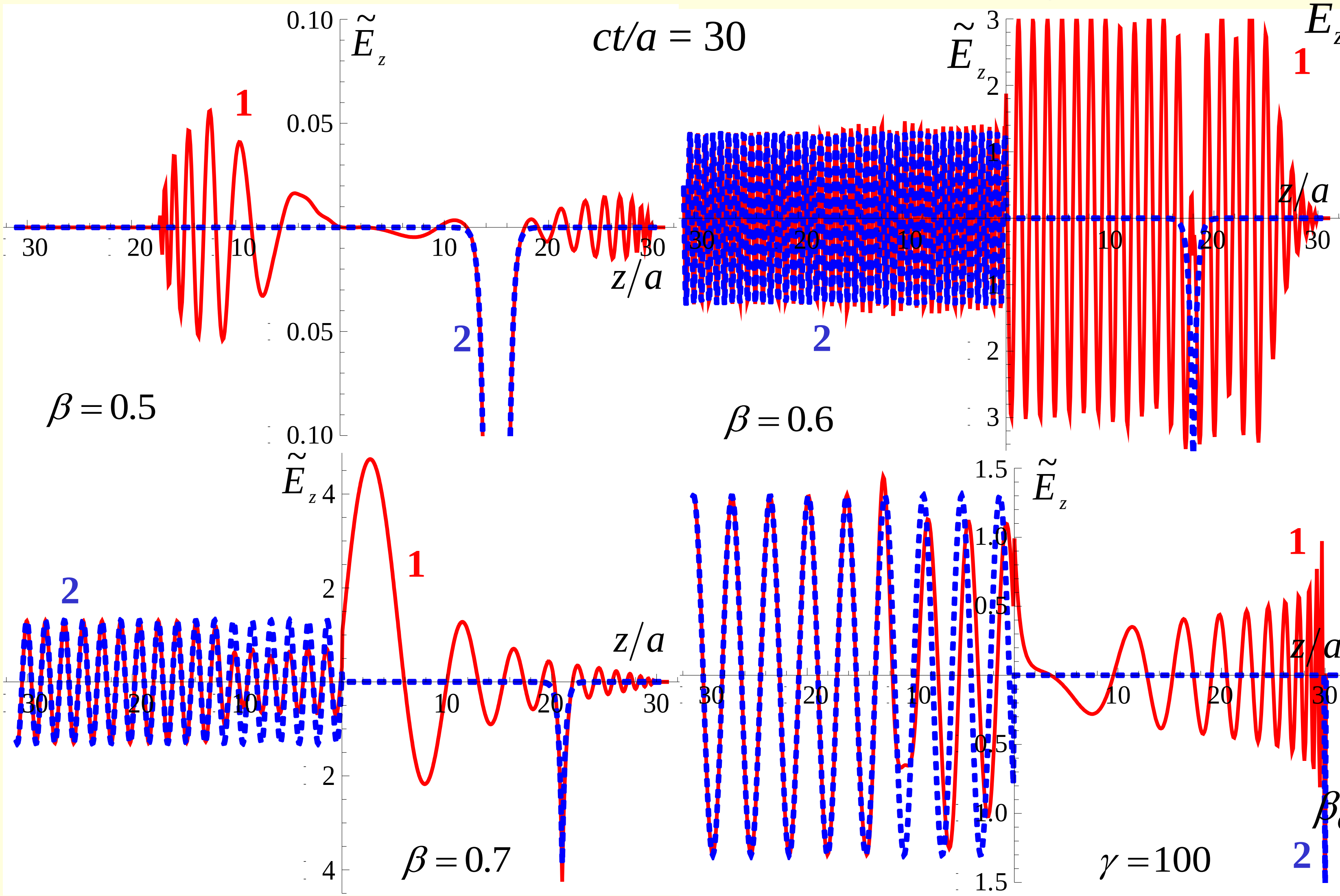
$$\beta_{CT1} = \frac{1}{\sqrt{\varepsilon_1 - 1}}$$

$z < z_3,\quad z_3 = ct\sqrt{\left|1-\beta^2(\varepsilon_1 - 1)\right|}\Big/\beta,$ 1 – the whole field, 2 – the forced field

The case of flying from dielectric into vacuum

$\varepsilon_1 = 3, \varepsilon_2 = 1$

Dependence of the first mode of the field $\tilde{E}_z$ in vacuum and dielectric on z/a for different velocities of the charge motion β



$$\tilde{E}_z = E_z \frac{\pi c^2}{2q\omega_0^2}$$

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$a = 5\ mm,$

$r = 0$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$|z| < z_1,$

$$z_1 = \frac{ct}{\beta\varepsilon_1}$$

$$\beta_{C1} = \frac{1}{\sqrt{\varepsilon_1}},$$

$$\beta_{CT1} = \frac{1}{\sqrt{\varepsilon_1 - 1}}$$

$z < z_3, \quad z_3 = ct\sqrt{\left|1-\beta^2(\varepsilon_1 - 1)\right|}\Big/\beta,$ 1 – the whole field, 2 – the forced field

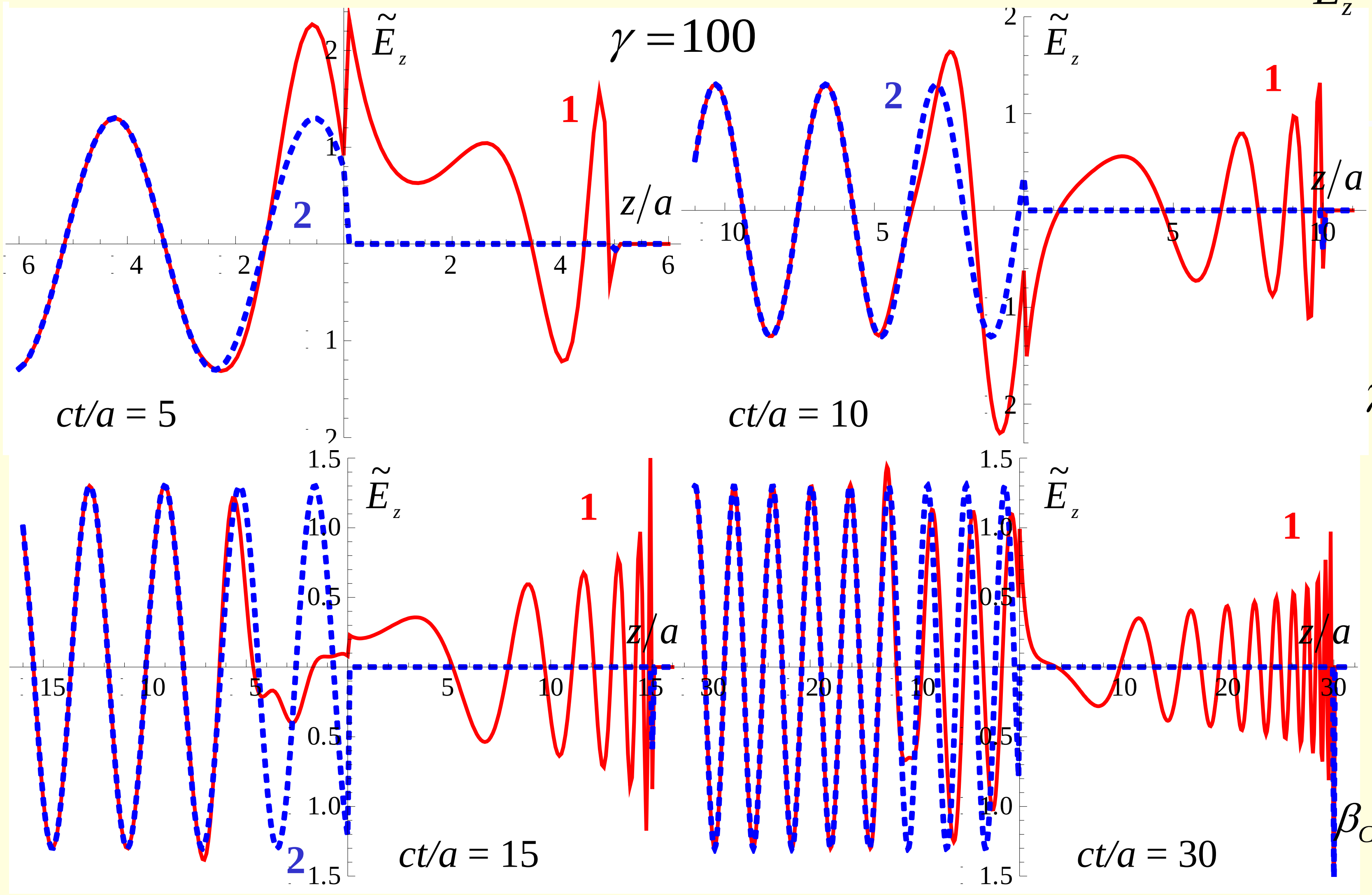
The case of flying from dielectric into vacuum

$\varepsilon_1 = 3,\ \varepsilon_2 = 1$

Dependence of the first mode of the field $\tilde{E}_z$ in vacuum and plasma on *z/a* at different moments *ct/a*.

$\omega_0 = 2\pi \cdot 10GHz$

$$\tilde{E}_z = E_z \frac{\pi c^2}{2q\omega_0^2}$$



$n = 1,$
$a = 5\ mm,$
$r = 0$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$|z| < z_1,$$

$$z_1 = \frac{ct}{\beta\varepsilon_1}$$

$$\beta_{C1} = \frac{1}{\sqrt{\varepsilon_1}},$$

$$\beta_{CT1} = \frac{1}{\sqrt{\varepsilon_1 - 1}}$$

$z < z_3,\quad z_3 = ct\sqrt{\left|1-\beta^2(\varepsilon_1 - 1)\right|}\Big/\beta,$ 1 – the whole field, 2 – the forced field

Conclusions :

1. When the charge flies from vacuum into dielectric there is the area (at least, at $ct/\beta\varepsilon_2 < z < c\beta t$) where the wave field practically coincides with the wakefield in a regular waveguide and the area ($z < ct/\beta\varepsilon_2$) where the boundary influence is principal. «The wakefield area» is large if the dielectric permittivity takes on a large value. This is important for the wakefield acceleration technique.
2. When the charge flies from dielectric into vacuum with certain velocity a large quasi monochromatic radiation is generated in the vacuum region. So, Cherenkov radiation escapes from dielectric into a vacuum at $1/\sqrt{\varepsilon_1} < \beta < 1/\sqrt{\varepsilon_1 - 1}$. CTR is dominant for these velocities in the area $z < ct\sqrt{\left|1-\beta^2(\varepsilon_1-1)\right|}/\beta$. This conclusion is of interest for development of new methods of generation of electromagnetic radiation which is similar to maser one.

Thanks for your attention!