The Discrete Semiclassical Action of Causal Dynamical Triangulations

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Outline:

1. Structure of the Causal Dynamical Triangulations model
   - Path integral for gravitation and Dynamical Triangulations
   - Regularisation of the path integral via discrete geometries
   - Preservation of causality and its benevolences

2. Working with the CDT model
   - Numerical Monte Carlo simulations
   - Fundamental results of numerics

3. Semiclassical solution of CDT
   - The single structure: $(4, 1)$ simplices
   - The double structure: $(4, 1)$ and $(\mathbf{3}, \mathbf{2})$ simplices
   - Final remarks and the triple structure

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Path integral formulation of quantum gravity

The propagator in one-dimensional quantum mechanics

$$G(x'', x'; t'', t') = \int_{\mathrm{Paths}(x'', x')} \mathcal{D}[x(t)] \; \exp\left(\frac{i}{\hbar} S[x(t)]\right)$$

By analogy, for gravitational field

$$G_{G,\Lambda}(g''_{ij}, g'_{ij}; t'', t') := \int_{\mathrm{Geom}(M)} \mathcal{D}[g_{\mu\nu}] \; \exp\left(\frac{i}{\hbar} S_{EH}[g_{\mu\nu}]\right),$$

where

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G} \int_M d^d x \, \sqrt{|\det g|} \, (R - 2\Lambda)$$

and $g_{\mu\nu} \equiv [g_{\mu\nu}] \in \mathrm{Geom}(M)$ denotes the equivalence class of metrics with respect to diffeomorphisms

The illness and a remedy

Problems with the gravitational path integral

- Necessity of a suitable regularisation and renormalisation
- No absolute parametrization of geometries
- Ambiguity of the (needed) Wick rotation

A possible cure – Dynamical Triangulations

- Lattice regularisation (discretisation) of geometries
- Length of lattice links as an UV cutoff
- Renormalisation by taking the continuous limit

Recipe for Dynamical Triangulations

Explicitly, we replace the continuous partition function

$$\mathcal{Z}(G,\Lambda) = \int_{\mathrm{Geom}(M)} \mathcal{D}[g_{\mu\nu}]\ \exp\left(\frac{i}{\hbar} S_{EH}[g_{\mu\nu}]\right)$$

with the discrete

$$\mathcal{Z}(\kappa_0,\kappa_4,\Delta) = \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} \exp\left(\frac{i}{\hbar} S_R[\mathcal{T}]\right),\ C_{\mathcal{T}} \equiv |\mathrm{Aut}(\mathcal{T})|,$$

where the sum is over piecewise linear manifolds of a fixed topology, assembled from intrisically Minkowskian polytopes (most conviently equilateral simplices)

Geometry without coordinates

Curvature is associated with $(d-2)$-subsimplices and expressed by deficit angles around them δ: $\sum_{k\in\triangle^{(d-2)}}\Theta_k = 2\pi - \delta$, hence

$$\frac{1}{2}\int_M d^d x\,\sqrt{|\det g|}\,R \longmapsto \sum_{i\in\mathcal{T}} \mathrm{vol}(\triangle_i^{(d-2)})\,\delta_i,$$

$$\int_M d^d x\,\sqrt{|\det g|} \longmapsto \sum_{i\in\mathcal{T}} \mathrm{vol}(\triangle_i^{(d)})$$

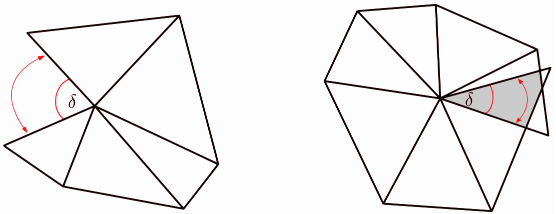

Figure: Curvature at a 2-dim manifold built from non-equilateral triangles

Imposing causality on Dynamical Triangulations

Restrictions on a set of discrete geometries

- Simplicial manifolds must admit a global proper-time foliation
- Consequently, they are built from time-denoted layers
- Furthermore, topology of their spatial slices is preserved
- There exist *d* types of possible simplicial building blocks

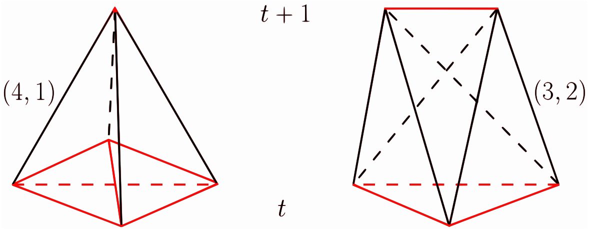

Figure: Types of simplices in CDT for $d = 4$ (without the dual ones)

Transition to the Euclidean framework

Let us introduce anisotropy between time and space in a triangulation with a cutoff a,

$$l^2_{\text{timelike}} = -\alpha\, a^2, \; l^2_{\text{spacelike}} = a^2,$$

Then there is a straightforward Wick rotation $\alpha \mapsto -\alpha$ transforming

$$\sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} \exp\left(\frac{i}{\hbar} S_R[\mathcal{T}]\right) \longmapsto \sum_{\mathcal{T}'} \frac{1}{C_{\mathcal{T}'}} \exp\left(-\frac{1}{\hbar} S_R^E[\mathcal{T}']\right),$$

where

$$S_R^E[\mathcal{T}'] = -(\kappa_0 + 6\Delta) N_0 + \kappa_4 (N_4^{(\mathbf{4},\mathbf{1})} + N_4^{(\mathbf{3},\mathbf{2})}) + \Delta(2N_4^{(\mathbf{4},\mathbf{1})} + N_4^{(\mathbf{3},\mathbf{2})}),$$

if $\alpha > \frac{7}{12}$ (in case of $d = 4$)

CDT research

Towards numerical implementation

- Wick rotated, the CDT-model becomes a kind of statistical system of random geometry
- Thus far, it is analytically solvable only for $d = 2$
- Might be investigated numerically through the MC simulations
- An appropriate program for $d = 4$ has been written by J. Jurkiewicz and A. Görlich and improved recently by J. Gizbert-Studnicki

Applied numerical setup

The method

- We start from some simple, ad hoc constructed configuration
- Consecutive triangulations are generated using MC moves
- There are 7 kinds of such allowed moves (in case of $d = 4$)

Technical choices

- Topology of the simplicial manifold $S^1 \times S^3$ (time is periodic)
- Fixed number of time-denoted layers / spatial slices
- Number of simplices kept restricted due to a modified action $S_R^E(\mathcal{T}) + \varepsilon\,(\langle N_4^{(4,1)}\rangle - \mathcal{N}_4)^2$

Phase structure of CDT

Phases

- A (sparse) – unphysical
- B (squashed) – unphysical
- C (extended) – physical

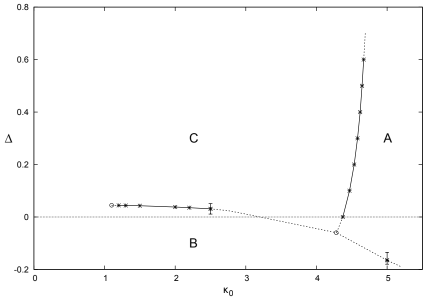

Figure: Phase diagram of 4-dim CDT

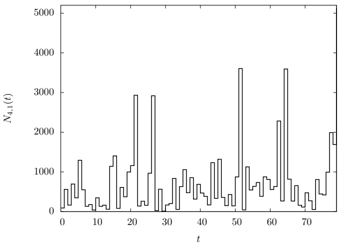

Figure: An individual configuration from phase A

Phase structure of CDT

Phases

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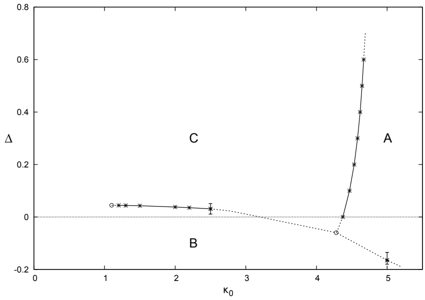
- B (squashed) – unphysical
- C (extended) – physical



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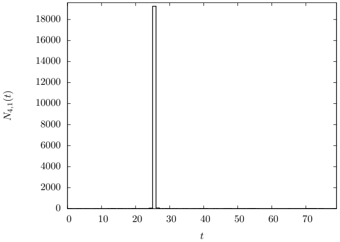
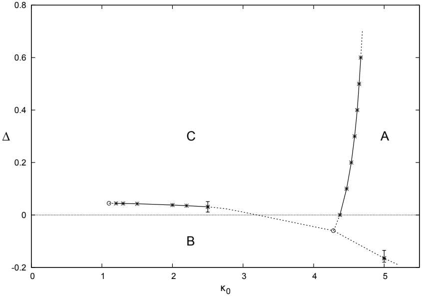


Figure: An individual configuration from phase B

Phase structure of CDT

Phases

- A (sparse) – unphysical
- B (squashed) – unphysical
- C (extended) – physical



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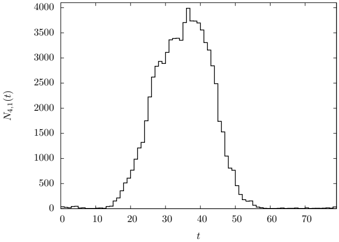


Figure: An individual configuration from phase C

Tools of investigation

Data characteristics

- Our numerical setup yields the "grand canonical" ensemble
- Yet all generated configurations have $\sim$ the same weights
- Hence we can easily calculate observables

Basic observables

- The average distribution of the number of (p, q) simplices $\langle N_{p,q}(t) \rangle$
- The covariance matrix of the number of (p, q) simplices $C_{p,q}(t', t'') = \big\langle (N_{p,q}(t) - \langle N_{p,q}(t) \rangle)(N_{p,q}(t') - \langle N_{p,q}(t') \rangle) \big\rangle$

Semiclassical approximation of the model

Let us consider only $(4,1)$ simplices; it turns out behaviour of them can be described using a typical semiclassical approximation,

$$S_{\mathrm{dis}}[\bar{N}_{4,1}(t)+\delta N_{4,1}(t)] = S_{\mathrm{dis}}[\bar{N}_{4,1}(t)]+$$
$$\frac{1}{2!}\sum_{j,i=0}^{T-1}\partial_{N_{4,1}(j)}\partial_{N_{4,1}(i)}S_{\mathrm{dis}}[\bar{N}_{4,1}(t)]\,\delta N_{4,1}(i)\delta N_{4,1}(j)+O(\delta N_{4,1}(t)^3),$$

where S_{dis} is a discretised action of the Wick-rotated mini-superspace model

$$S_{\mathrm{cont}} = \frac{1}{24\pi G}\int dt\,\sqrt{g_{tt}}\left(\frac{g^{tt}\dot{V}_3(t)^2}{V_3(t)} + k_2 V_3(t)^{\frac{1}{3}} - \lambda V_3(t)\right),$$

where $V_3(t) = 2\pi^2 a(t)^3$ ($a(t)$ is the scale factor of a homogenous, isotropic universe)

Emergence of the de Sitter universe

The average distribution $\langle N_{4,1}(i)\rangle$ in the blob is well approximated by a discrete version of the classical solution of S_{cont},

$$N_0 \cos^3(\omega(t - t_0))$$

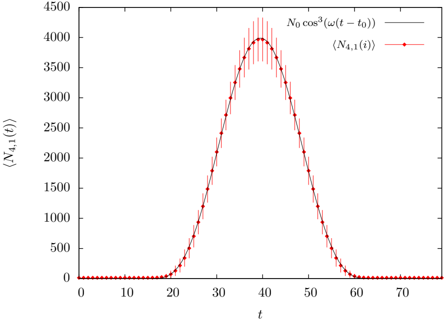


Figure: The average distribution $\langle N_{4,1}(i)\rangle$ with the (semi)classical solution fitted and the average quantum fluctuations

Role of the (inverse) covariance matrix

The inverse of the covariance matrix $P_{4,1} := (C_{4,1})^{-1}$ satisfies a relation

$$(P_{4,1})_{i,j} = \partial_{N_{4,1}(j)} \partial_{N_{4,1}(i)} S_{\mathrm{dis}}[\langle N_{4,1}(t) \rangle]$$

Hence it is used for further investigation of the discrete semiclassical action, having in particular very simple structure,

$$P_{4,1} = \begin{pmatrix} a_1 & b_1 & * & * & b_T \\ b_1 & a_2 & \iddots & * & * \\ * & \ddots & \ddots & \ddots & * \\ * & * & \iddots & a_{T-1} & b_{T-1} \\ b_T & * & * & b_{T-1} & a_T \end{pmatrix},$$

where asterisks denote numerical noise

Form of the discrete action

The discrete action may be decomposed into

$$S_{\mathrm{dis}} = k_1 \sum_{t=0}^{T-1} \left(\tilde{S}_k(t) + \tilde{S}_p(t) \right) + \varepsilon \left(\langle N_{4,1} \rangle - \mathcal{N}_4 \right)^2,$$

where $k_1 \sum_t \tilde{S}_k(t)$, $k_1 \sum_t \tilde{S}_p(t)$ are its kinetic and potential part, respectively; investigation of their precise form favours the more involved kinetic term

$$\tilde{S}_k(t) = \frac{(N_{4,1}(t+1) - N_{4,1}(t))^2}{N_{4,1}(t) + N_{4,1}(t+1)} \left(1 + \xi_1 \left(\frac{N_{4,1}(t+1) - N_{4,1}(t)}{N_{4,1}(t) + N_{4,1}(t+1)} \right)^2 \right)$$

and the simplest possible potential term

$$\tilde{S}_p(t) = \tilde{k}_2 N_{4,1}(t)^{\frac{1}{3}} - \tilde{\lambda} N_{4,1}(t)$$

Comparison with the data

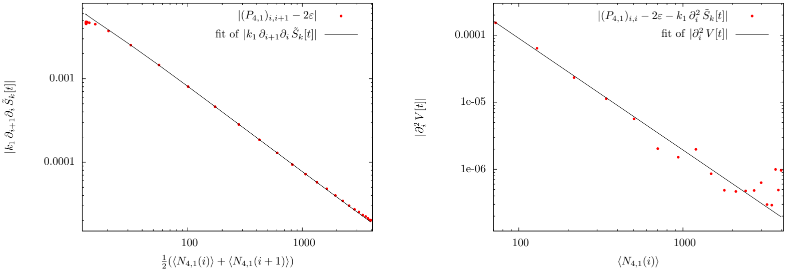
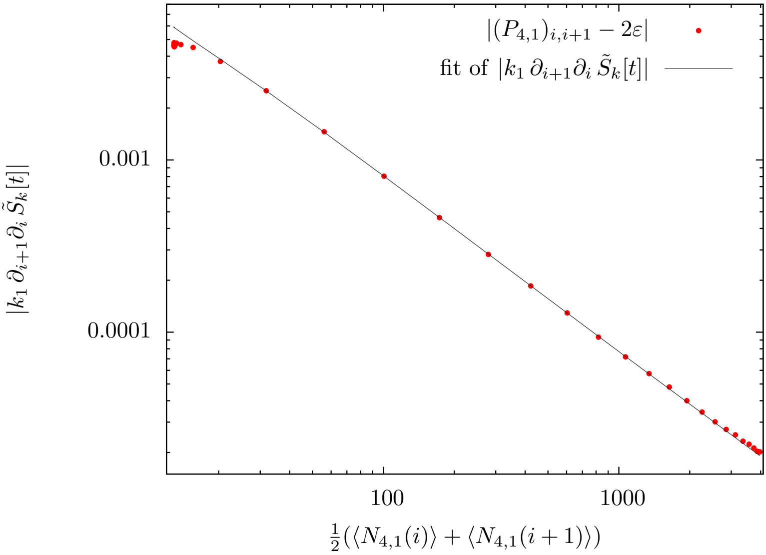


Figure: $P_{4,1}$ and second derivatives of the kinetic term (left) and the potential term (right)

Comparison with the data



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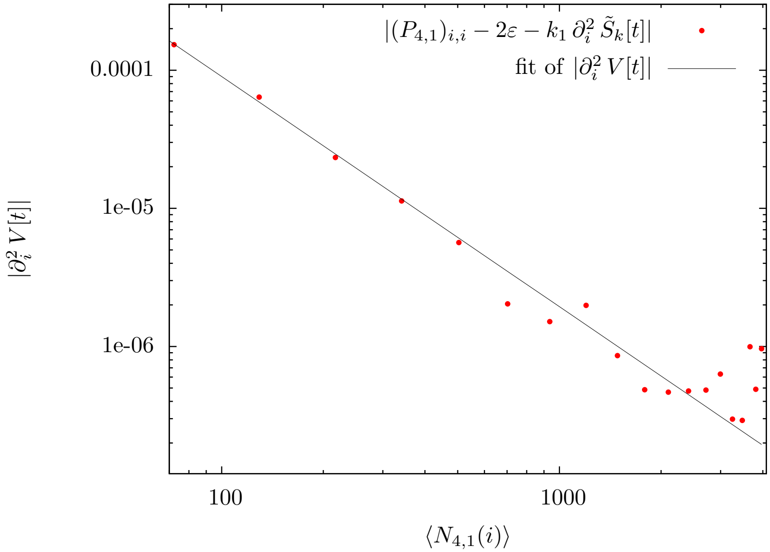


Spatial slices of (**3**, **2**) simplices

In the blob $\langle N_{\mathbf{3},\mathbf{2}}(i+\frac{1}{2})\rangle$ is proportional to $\langle N_{4,1}(i)\rangle$ and after rescaling their joint distribution can automatically be fitted with the semiclassical solution

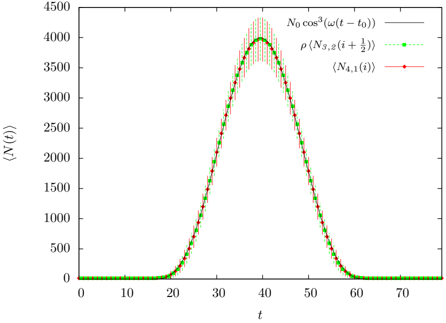


Figure: The joint distribution of $\langle N_{4,1}(i)\rangle$, $\rho\langle N_{\mathbf{3},\mathbf{2}}(i+\frac{1}{2})\rangle$ with the classical solution fitted and the average quantum fluctuations

The double inverse covariance matrix

To study the extended discrete action we introduce the joint covariance matrix for $(4,1)$ and $(\mathbf{3},\mathbf{2})$ simplices, which inverse has the form

$$P_{\mathrm{dbl}} = \begin{pmatrix} c_1 & d_1 & * & * & d_T & f_1 & * & * & * & f_T \\ d_1 & c_2 & \ddots & * & * & f_{T-1} & f_2 & * & * & * \\ * & \ddots & \ddots & \ddots & * & * & \ddots & \ddots & * & * \\ * & * & \ddots & c_{T-1} & d_{T-1} & * & * & \ddots & f_{T-1} & * \\ d_T & * & * & d_{T-1} & c_T & * & * & * & f_1 & f_T \\ & & & & & & & & & \\ f_1 & f_{T-1} & * & * & * & e_1 & * & * & * & * \\ * & f_2 & \ddots & * & * & * & e_2 & * & * & * \\ * & * & \ddots & \ddots & * & * & * & \ddots & * & * \\ * & * & * & f_{T-1} & f_1 & * & * & * & e_{T-1} & * \\ f_T & * & * & * & f_T & * & * & * & * & e_T \end{pmatrix}$$

Discrete action for the double slice structure

The discrete action generalizes to

$$S_{\mathrm{dis}}^{(\mathrm{dbl})} = k_1^{(\mathrm{d})} \sum_{t=0}^{T-1} \left(\tilde{S}_k^{(\mathrm{dbl})}(t) + \tilde{S}_p^{(\mathrm{dbl})}(t) \right) + \varepsilon \left(\langle N_{4,1} \rangle - \mathcal{N}_4 \right)^2$$

Fair agreement with the data is yielded by the following kinetic term

$$\tilde{S}_k^{(\mathrm{dbl})}(t) = -a \frac{2(N_{4,1}(t+1) - N_{4,1}(t))^2}{N_{4,1}(t) + N_{4,1}(t+1)} + \frac{2(\rho N_{\mathbf{3},\mathbf{2}}(t+\frac{1}{2}) - N_{4,1}(t))^2}{N_{4,1}(t) + \sigma N_{\mathbf{3},\mathbf{2}}(t+\frac{1}{2})} + \frac{2(N_{4,1}(t) - \rho N_{\mathbf{3},\mathbf{2}}(t-\frac{1}{2}))^2}{\sigma N_{\mathbf{3},\mathbf{2}}(t-\frac{1}{2}) + N_{4,1}(t)}$$

and the intuitive potential term

$$\tilde{S}_p^{(\mathrm{dbl})}(t) = -\tilde{k}_2^{11} N_{4,1}(t)^{\frac{1}{3}} + \tilde{\lambda}^{11} N_{4,1}(t) + \tilde{k}_2^{22} N_{\mathbf{3},\mathbf{2}}(t+\tfrac{1}{2})^{\frac{1}{3}} - \tilde{\lambda}^{22} N_{\mathbf{3},\mathbf{2}}(t+\tfrac{1}{2})$$

Comparison with the data

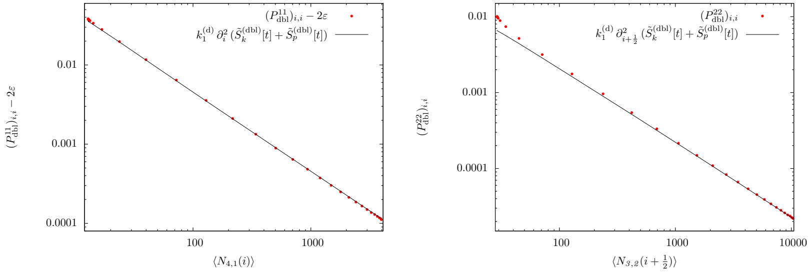
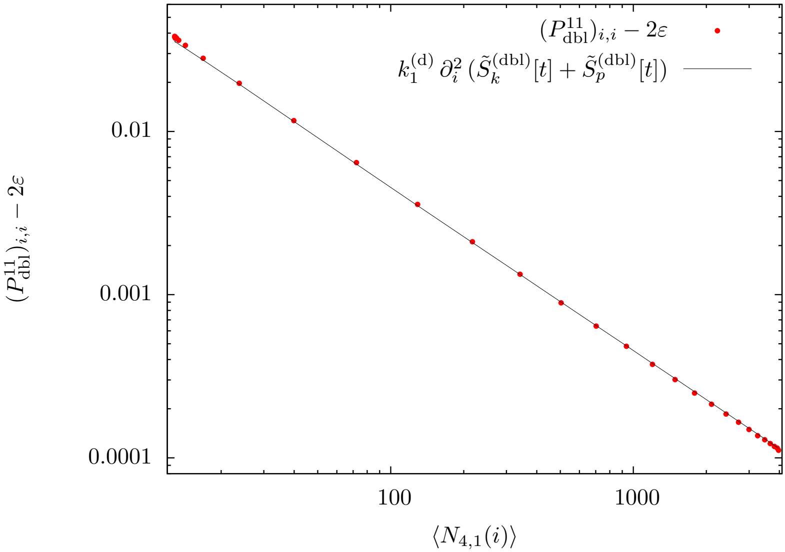


Figure: Second derivatives of the action and diagonals of blocks P^{11}_{dbl} (left) and P^{22}_{dbl} (right)

Comparison with the data



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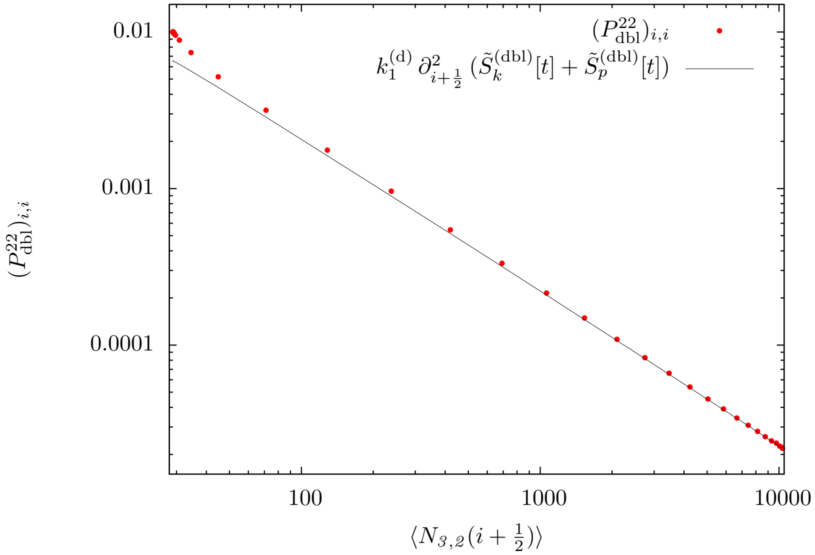

Reduction to the single structure ((4, 1) simplices only)

Using one of two versions of Boltz-Banachiewicz inversion formula one may retrieve $P_{4,1}$ from P_{dbl}:

$$P_{4,1} = \left((P_{\mathrm{dbl}}^{11})^{-1} + Cov_{\mathrm{dbl}}^{12} \, P_{\mathbf{3},\mathbf{2}} \, (Cov_{\mathrm{dbl}}^{12})^{\mathrm{T}} \right)^{-1}$$

Integrating out with some approximations $(\mathbf{3}, \mathbf{2})$ simplices we obtain the single structure coupling constants from the double ones:

$$k_1 = k_1^{(\mathrm{d})} \Big(- a + \frac{\rho}{\rho + \sigma} \Big),$$
$$\tilde{k}_2 = \big(- \tilde{k}_2^{11} + \rho^{-\frac{1}{3}} \, \tilde{k}_2^{22} \big) \frac{\rho + \sigma}{\rho - a(\rho + \sigma)},$$
$$\tilde{\lambda} = \big(- \tilde{\lambda}^{11} + \rho^{-1} \, \tilde{\lambda}^{22} \big) \frac{\rho + \sigma}{\rho - a(\rho + \sigma)},$$

which agrees fairly with the data

Summary

Crucial points

- In the CDT-model gravitational path integral is approximated by the sum over causally well-behaving simplicial manifolds
- After being Wick-rotated, the model for $d = 4$ has been explored using numerical MC simulations
- In one of the observed phases of CDT the semiclassical solution emerges
- The semiclassical description involves the discrete structure

Outlook – the triple structure

The ultimate extension of the discrete framework includes all types of simplices and due to the invariance under time inversion $\langle N_{3,2}(i+\frac{1}{3})\rangle \approx \langle N_{2,3}(i+\frac{2}{3})\rangle \approx \frac{1}{2}\langle N_{\mathbf{3},\mathbf{2}}(i+\frac{1}{2})\rangle$

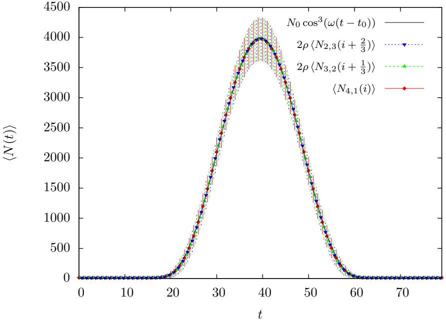

Figure: The joint distribution of $\langle N_{4,1}(i)\rangle$, $2\rho\langle N_{3,2}(i+\frac{1}{3})\rangle$, $2\rho\langle N_{2,3}(i+\frac{2}{3})\rangle$ with the classical solution fitted and the average quantum fluctuations

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J. Ambjørn, A. Görlich, J. Jurkiewicz, R. Loll, J. Gizbert-Studnicki & T.T., *The Semiclassical Limit of Causal Dynamical Triangulations*, Nuclear Physics B **849** (2011) 144-165, arXiv:1102.3929 [hep-th]

T.T., *Analysis of the Semiclassical Solution of CDT*, arXiv:1102.4643 [hep-th]

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