Twistorial phase space for complex Ashtekar variables

Quantum Gravity Colloquium 6

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Motivation

- Dupuis, Freidel, Livine, Speziale and Tambornino developed a twistorial formulation for $SU(2)$ Ashtekar–Barbero variables.
- What about the case of $SL(2,\mathbb{C})$, i.e. complex Ashtekar variables?

Outline

Three points:

1. Spinors for classical $SL(2,\mathbb{C})$ phase space on a fixed graph.
2. An application: Spinorial version of the simplicity constraints.
3. How the spinorial formalism naturally reveals the Dupuis–Livine map.

1. Spinorial decomposition of the $SL(2,\mathbb{C})$ phase space on a fixed graph

$SL(2,\mathbb{C})$ phase space

Start from the selfdual decomposition of the Holst action:

$$S_{\text{Holst}} = \frac{\beta + \mathrm{i}}{\mathrm{i}\beta} \int_M \Sigma^\alpha{}_\beta \wedge R^\beta{}_\alpha[A] + \text{cc.} \tag{1}$$

Where $\Sigma^\alpha{}_\beta = \Sigma^i \tau^\alpha{}_{\beta i} = \big(\frac{1}{2}\epsilon_l{}^i{}_m e^l \wedge e^m + \mathrm{i} e^0 \wedge e^i\big)\tau^\alpha{}_{\beta i}$ are the selfdual components of the Plebanski 2-form $\Sigma^{IJ} = e^I \wedge e^J$. Performing a 3+1 decomposition $M = \Sigma \times \mathbb{R}$ we identify the symplectic structure, e.g.

$$\big\{\Pi_i{}^a(p), A^j{}_b(q)\big\} = \delta_i^j \delta_b^a \tilde{\delta}(p,q) \tag{2}$$

Where

$$\Pi_i{}^a = -\frac{\beta + \mathrm{i}}{4\mathrm{i}\beta} \tilde{\eta}^{abc} \Sigma_{ibc} \tag{3}$$

is the momentum conjugate of the $SL(2,\mathbb{C})$ connection $A^i{}_a = \Gamma^i{}_a + \mathrm{i} K^i{}_a$.

Smeared phase space on a fixed graph

A fixed graph Γ consists of oriented links $\gamma, \gamma', \dots$, to each of which we assign a dual face $f, f', \dots$. Introduce smeared variables:

$$SL(2,\mathbb{C}) \ni g[f] = \mathrm{Pexp} \int_\gamma A \tag{4a}$$

$$\mathfrak{sl}(2,\mathbb{C}) \ni \Pi[f] = \int_f \Pi \tag{4b}$$

Holonomy flux algebra

For a single link:

$$\{g[f], g[f']\} = 0 \tag{5a}$$

$$\{\Pi_i[f], g[f]\} = g[f]\tau_i \tag{5b}$$

$$\{\Pi_i[f], \Pi_j[f]\} = \epsilon_{ij}{}^m \Pi_m[f] \tag{5c}$$

Preliminaries: Twistorial phase space

1. A twistor Z is a bispinor $Z = (\omega^\mu, \bar{\pi}_{\bar{\mu}}) \in \mathbb{C}^2 \oplus (\bar{\mathbb{C}}^2)^*$.
2. $SL(2,\mathbb{C})$ acts in the obvious way:

$$\omega^\mu \xrightarrow{g} +g^\mu{}_\nu \omega^\nu \tag{6a}$$

$$\bar{\pi}_{\bar{\mu}} \xrightarrow{g} -\bar{g}_{\bar{\mu}}{}^{\bar{\nu}} \bar{\pi}_{\bar{\nu}} \tag{6b}$$

3. There is an $SL(2,\mathbb{C})$ invariant symplectic structure available:

$$\left\{\pi_\mu, \omega^\nu\right\} = \delta^\nu_\mu \tag{7}$$

$$\left\{\bar{\pi}_{\bar{\mu}}, \bar{\omega}^{\bar{\nu}}\right\} = \bar{\delta}^{\bar{\nu}}_{\bar{\mu}} \tag{8}$$

Decomposition of the phase space on a graph

Use Twistors to decompose the phase space of smeared variables. In this disussion we restrict ourselves to one single link.

Step 1 Attach a twistor to both final and initial point, to have a pair $(\underset{\sim}{Z}, Z) = (\underset{\sim}{\omega}^{\mu}, \underset{\sim}{\bar{\pi}}_{\bar{\mu}}, \omega^{\mu}, \bar{\pi}_{\bar{\mu}})$.

Step 2 Decompose both holonomy and flux into these variables:

$$g[f]^{\alpha}{}_{\beta} = \frac{\underset{\sim}{\pi}^{\alpha}\pi_{\beta} + \underset{\sim}{\omega}^{\alpha}\omega_{\beta}}{\sqrt{\underset{\sim}{\pi}_{\mu}\underset{\sim}{\omega}^{\mu}\pi_{\nu}\omega^{\nu}}} \in SL(2, \mathbb{C}) \tag{9a}$$

$$\Pi[f]^{\alpha}{}_{\beta} = \frac{1}{4}\left(\pi^{\alpha}\omega_{\beta} + \pi_{\beta}\omega^{\alpha}\right) \in \mathfrak{sl}(2, \mathbb{C}) \tag{9b}$$

This decomposition is

- possible unless $\pi_{\alpha}\omega^{\alpha} = \underset{\sim}{\pi}_{\alpha}\underset{\sim}{\omega}^{\alpha} = 0$.
- unique up to complex rescailings of the spinors.

Step 3 But we can also construct

$$\underset{\sim}{\Pi}^{\alpha}{}_{\beta} = \frac{1}{4}\left(\underset{\sim}{\pi}^{\alpha}\underset{\sim}{\omega}_{\beta} + \underset{\sim}{\pi}_{\beta}\underset{\sim}{\omega}^{\alpha}\right) \tag{10}$$

Corresponding to $\underset{\sim}{\Pi}[f] = \Pi[f^{-1}] = -g[f]\Pi[f]g[f]^{-1}$, i.e. flux through oppositely oriented face. On the level of spinors this implies a constraint:

$$\underset{\sim}{\Pi}[f] = -g[f]\Pi[f]g[f]^{-1} \Leftrightarrow C = \pi_{\mu}\omega^{\mu} - \underset{\sim}{\pi}_{\mu}\underset{\sim}{\omega}^{\mu} = 0 \tag{11}$$

Step 4 Check if the canonical commutation relations are recovered:

$$\left\{\Pi_i, \Pi_j\right\} = \epsilon_{ij}{}^{m}\Pi_m \qquad \mathit{OK} \tag{12a}$$

$$\left\{\Pi_i, g\right\} = g\tau_i \qquad \mathit{OK} \tag{12b}$$

$$\text{if } C = 0 : \left\{g^{\alpha}{}_{\beta}, g^{\mu}{}_{\nu}\right\} = 0 \qquad \mathit{OK} \tag{12c}$$

Intermediate summary

1. Any point in the reduced phase space on graph can be parametrised by a set of twistors—unless $\Pi_\alpha{}^\beta[f]\Pi_\beta{}^\alpha[f] = 0$.
2. There are two twistors to each link.
3. The symplectic structure of the holonomy flux algebra is recovered on the constraint hypersurface $C = \pi_\alpha\omega^\alpha - \underset{\sim}{\pi}_\alpha\underset{\sim}{\omega}^\alpha$=0.
4. Performing a symplectic reduction the original phase space is recovered. Already plausible from counting $3 \times 2 = 2 \times 2 \times 2 - 2$ complex degrees of freedom.
5. The symplectic structure simplifies. In the holonomy-flux algebra momenta don't commute, here they do: $\{\pi_\alpha, \pi_\beta\} = 0$.

2. Linear simplicity constraints

Spinorial version of the linear simplicity constraint

Consider the linear simplicity constraints

$$\exists n^I : \forall f : \Sigma_{IJ}[f] n^I = 0 \tag{13}$$

- In the Hamiltonian formalism these emerge as reality conditions on the momentum $\Pi_i{}^a$.
- In the covariant picture they guarantee the simplicity $\Sigma^{IJ} = e^I \wedge e^J$ of the Plebanski 2-form around a 4-simplex.

In the spinorial language this equation becomes:

$$\Sigma_{\alpha\beta}[f] \bar{\epsilon}_{\bar{\alpha}\bar{\beta}} n^{\beta\bar{\beta}} + \text{cc.} = -\frac{2\mathrm{i}\beta}{\beta + \mathrm{i}} \Pi_{\alpha\beta}[f] \bar{\epsilon}_{\bar{\alpha}\bar{\beta}} n^{\beta\bar{\beta}} + \text{cc.} = 0 \tag{14}$$

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Linear simplicity constraints in terms of spinors

In terms of spinorial variables:

$$\frac{\mathrm{i}\beta}{\beta+\mathrm{i}}\big(\omega_\alpha\pi_\beta+\omega_\beta\pi_\alpha\big)\bar{\epsilon}_{\bar{\alpha}\bar{\beta}}n^{\beta\bar{\beta}}+\mathrm{cc.}=0 \tag{15}$$

- This equations has two free spinor indices.
- But the pair $\omega^\alpha, n^{\alpha\bar{\alpha}}\bar{\omega}_{\bar{\alpha}}$ is (unless $\omega=0$) a complete basis in $\mathbb{C}^2$.

Contraction with this basis elements reveals the following two constraints:

$$F_1=\frac{\mathrm{i}}{\beta+\mathrm{i}}\omega^\alpha\pi_\alpha+\mathrm{cc.}=0 \tag{16a}$$

$$F_2=n^{\alpha\bar{\beta}}\pi_\alpha\bar{\omega}_{\bar{\beta}}=0 \tag{16b}$$

Notice that F_1 is real but F_2 is complex.

Constraint algebra and master constarint

The corresponding constraint algebra is:

$$\{F_1, F_2\} = -\frac{2\mathrm{i}\beta}{\beta^2+1} F_2 \tag{17a}$$

$$\{F_1, \bar{F}_2\} = +\frac{2\mathrm{i}\beta}{\beta^2+1} \bar{F}_2 \tag{17b}$$

$$\{F_2, \bar{F}_2\} = \pi_\alpha \omega^\alpha - \bar{\pi}_{\bar{\alpha}} \bar{\omega}^{\bar{\alpha}} \tag{17c}$$

F_1 is of first class, but F_2 is second class. Define the master constraint:

$$\mathbf{M} = \bar{F}_2 F_2 \tag{18}$$

And observe

$$\{F_1, \mathbf{M}\} = 0 \tag{19}$$

Right hand side is identically zero!

3. Quantisation

Canonical quantisation of the simplicity constraints

Performing canonical quantisation, e.g.:

$$\big(\pi_\mu f\big)(\omega) = -\mathrm{i}\frac{\partial}{\partial\omega^\mu}f(\omega) \tag{20}$$

and choosing a normal ordering we find:

$$\widehat{F}_1 = \frac{1}{\beta^2+1}\Big[(\beta-\mathrm{i})\omega^\alpha\frac{\partial}{\partial\omega^\alpha} - (\beta+\mathrm{i})\bar{\omega}^{\bar{\alpha}}\frac{\partial}{\partial\bar{\omega}^{\bar{\alpha}}} - 2\mathrm{i}\Big] \tag{21a}$$

$$\widehat{F}_2 = -\mathrm{i}n^{\alpha\bar{\alpha}}\bar{\omega}_{\bar{\alpha}}\frac{\partial}{\partial\omega^\alpha} \tag{21b}$$

$$\widehat{\mathbf{M}} = \widehat{F}_2^\dagger\widehat{F}_2 = \frac{1}{2}\omega^\mu\frac{\partial}{\partial\omega^\mu}\frac{\partial}{\partial\bar{\omega}^{\bar{\mu}}}\bar{\omega}^{\bar{\mu}} - \frac{1}{2}\Big(\widehat{L}^2 - \widehat{K}^2\Big) + \widehat{L}^2 \tag{21c}$$

Homogenous functions

We are imposing these constraints on homogenous functions $f(\omega)$:

$$\omega^{\mu}\frac{\partial}{\partial\omega^{\mu}}f^{(\rho,j_o)}(\omega)=(-j_o-1+\mathrm{i}\rho)f^{(\rho,j_o)}(\omega) \tag{22a}$$

$$\bar{\omega}^{\bar{\mu}}\frac{\partial}{\partial\bar{\omega}^{\bar{\mu}}}f^{(\rho,j_o)}(\omega)=(+j_o-1+\mathrm{i}\rho)f^{(\rho,j_o)}(\omega) \tag{22b}$$

On these functions $SL(2,\mathbb{C})$ acts unitarily. There is an invariant inner product, and a canonical orthonormal basis:

$$\begin{aligned}\left\langle f^{(\rho,j_o)}_{j,m},f^{(\rho,j_o)}_{j',m'}\right\rangle&=\frac{\mathrm{i}}{2}\int_{\mathrm{P}\mathbb{C}^2}\omega_\alpha\mathrm{d}\omega^\alpha\wedge\bar{\omega}_{\bar{\alpha}}\mathrm{d}\bar{\omega}^{\bar{\alpha}}\,\overline{f^{(\rho,j_o)}_{j,m}(\omega)}f^{(\rho,j_o)}_{j',m'}(\omega)=\\&=\delta_{jj'}\delta_{mm'}\end{aligned} \tag{23}$$

And $j=j_o,j_o+1,\dots$ and $m=-j,\dots,j$ are spin quantum numbers.

Dupuis–Livine map

The constraints are diagonal on the orthonormal basis:

$$\widehat{F}_1 f^{(\rho,j_o)}_{j,m} = \frac{2}{\beta^2+1}\big(-\beta j_o + \rho\big) f^{(\rho,j_o)}_{j,m} \tag{24a}$$

$$\widehat{\mathbf{M}} f^{(\rho,j_o)}_{j,m} = \big(j(j+1) - j_o(j_o+1)\big) f^{(\rho,j_o)}_{j,m} \tag{24b}$$

There is just one solution possible:

$$\boxed{\rho = \beta j_o, \quad \text{and} \quad j = j_o} \tag{25}$$

The solutionspace is isomorphic to the j-th $SU(2)$ representation space:

$$|j,m\rangle = f^{(\beta j,j)}_{j,m} \tag{26}$$

This is essentially the Dupuis–Livine map.

Discussion

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- The phase space of smeared holonomy-flux variables on a fixed graph was decomposed in terms of twistors. To each link belongs a pair of twistors—one for each of its boundary points.
- This decomposition works as long as $\Pi[f]^{\mu}{}_{\nu}\Pi[f]^{\nu}{}_{\mu} \neq 0$, that is unless f is null.
- In terms of twistors the linear simplicity constraint reduces to $F_1 = 0$ and $\mathbf{M} = \bar{F}_2 F_2 = 0$.
- In quantum theory both F_1 and $\mathbf{M}$ can be imposed strongly.
- The solution space picks the states $|j, m\rangle = f^{(\beta j, j)}_{j,m}$ in the irreducible $(\rho = \beta j, j_o = j)$ unitary representation space of $SL(2, \mathbb{C})$.
- Moreover $\widehat{F}_2 |j, m\rangle = 0$ but $\widehat{F}_2^{\dagger} |j, m\rangle \neq 0$.
- The spinorial method allows for a clean and simple derivation of the Dupuis–Livine map.
- This result questions the addiditional quantum number sometimes introduced when imposing the constraints weakly.

Vielen Dank für Eure Aufmerksamkeit.

Selected References:

- M. Dupuis, L. Freidel, E. Livine and S. Speziale; Holomorphic Lorentzian Simplicity Constraints; `arXiv:1107.5274`.
- L. Freidel and S. Speziale; Twistors to twisted geometries; `arXiv:1006.0199`.
- E. Livine, S. Speziale, J. Tambornino; Twistor Networks and Covaraint Twisted Geometries; `arXiv:1108.0369`.
- W. Wieland; Twistorial phase space for complex Ashtekar variables; `arXiv:1107.5002`.

Excursus: $F_2 = 0$ in quantum theory

We first found the spinorial version of the simplicity constraint:

$$\widehat{F}_1 f^{(\beta j,j)}_{j,m} = 0 \tag{27a}$$

$$\widehat{\mathbf{M}} f^{(\beta j,j)}_{j,m} = \widehat{F}_2^\dagger \widehat{F}_2 f^{(\beta j,j)}_{j,m} = 0 \tag{27b}$$

It's now not very suprising that $\widehat{F}_2 f^{(\beta j,j)}_{j,m}$ is already annihilated by $\widehat{F}_2$.

$$\widehat{F}_2 f^{(\beta j,j)}_{j,m} = 0, \quad \text{but} \quad \widehat{F}_2^\dagger f^{(\beta j,j)}_{j,m} \neq 0 \tag{28}$$

This can explicially be seen as follows.

Remember first:

$$\widehat{F}_2 = -\mathrm{i} n^{\alpha\bar{\alpha}} \bar{\omega}_{\bar{\alpha}} \frac{\partial}{\partial \omega^{\alpha}} \tag{29}$$

The operator $\widehat{F}_2$ changes the homogeneity weights according to:

$$\begin{array}{l} a \xrightarrow{\widehat{F}_2} a - 1 \\ b \xrightarrow{\widehat{F}_2} b + 1 \end{array} \Leftrightarrow \begin{array}{l} \rho \xrightarrow{\widehat{F}_2} \rho \\ j_o \xrightarrow{\widehat{F}_2} j_o + 1 \end{array} \tag{30}$$

But $\widehat{F}_2$ commutes with the generators of rotations that leave $n^{\alpha\bar{\alpha}}$ invariant.

$$\left[\widehat{F}_2, \widehat{L}_i\right] = 0 \tag{31}$$

By Schur's lemma we thus get:

$$\widehat{F}_2 f_{j,m}^{(\rho,j_o)} = c(\rho, j_o, j) f_{j,m}^{(\rho,j_o+1)} \tag{32}$$

Therefore it must be that:

$$\widehat{F}_2 f_{j,m}^{(\beta j,j)} = 0 \tag{33}$$