

Solving the Euclidean Hamiltonian Constraint with Spin-Foam Methods

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Quantum Gravity Colloquium 6

¹With Thomas Thiemann and Emanuele Alesci [arXiv:1109.1290]

Plan of the Talk

- ① Motivation
 - Canonical LQG
 - The Rigging Map
 - Spin-Foams
- ② Spin-Foam Projector
 - Construction
 - SF-Projector
- ③ Hamiltonian Constraint
 - Quantization
 - Action on trivalent nodes
- ④ One-vertex Expansion
- ⑤ Discussion

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Canonical LQG

Dirac Program: $\mathcal{H}_{kin} \xrightarrow{\hat{C}\psi=0} \mathcal{H}_{phys}$

Constraints

Hamiltonian/
Diffeomorphismn: H_a, H

Gauss-constraint : G_a

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Hilbert space

$$\mathcal{H}_{kin} = \bigoplus_{\Gamma} \mathcal{H}_{kin,\Gamma}$$

where

$$\mathcal{H}_{\Gamma} \simeq L^2 \left(SU(2)^{|\Gamma^{(1)}|}, d\mu_H \right)$$

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The Rigging Map I

- $(C_I)_{I \in \mathcal{I}}$ self-adjoint set of first-class, densely defined operators on \mathcal{D}_{kin} ; Lie-algebra closed
- Null in general not in the point spectrum of C_I
- Consider Gelfand Triple

$$\mathcal{D}_{kin} \hookrightarrow \mathcal{H}_{kin} \hookrightarrow \mathcal{D}_{kin}^*$$

Generalized solutions: $I \in \mathcal{D}_{kin}^*$ such that

$$[(C_I)' I] (f) := I(C_I^\dagger f) = 0 \quad \forall f \in \mathcal{H}_{kin}$$

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The Rigging Map II

- Equip \mathcal{D}_{phys}^* with Hilbert space structure

$$\mathcal{D}_{phys} \hookrightarrow \mathcal{H}_{phys} \hookrightarrow \mathcal{D}_{phys}^*$$

Define rigging map $\eta : \mathcal{H}_{kin} \rightarrow \mathcal{D}_{phys}^*$ such that

① $\langle \eta(f), \eta(f') \rangle_{phys} = [\eta(f')](f)$

② $[\hat{O}'\eta(f)](f') = [\eta(f)](\hat{O}^\dagger f')$

- Physical Hilbert space: Completion of $\eta(\mathcal{D}_{kin})/\ker(\eta)$

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- Works for *Diff* constraint
- Heuristically: $\eta(f) = \overline{\int dt_I \exp(t^i C_i) f}$

[Thiemann]

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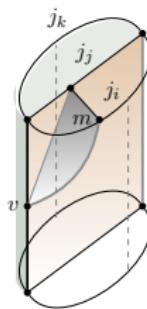
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Spin-Foams I

Time evolution of spin-network produces a two-complex



Define a Partition-function

$$\int \exp(tH) \rightsquigarrow \sum_{\kappa} \underbrace{Z[\kappa]}_{\text{Feynman Diagramms}}$$

[Reisenberger, Rovelli]

Spin-Foams II

- GR constrained BF theory $\xrightarrow{\text{Discretization}}$ Spin-Foam model

$$Z[\kappa] = \prod_v \mathcal{A}_v \prod_e \mathcal{A}_e \prod_f \mathcal{A}_f$$

- First quantize then constrain
- Problems:
 - Second-class system

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$$[\eta(T)] = \sum_{T'} \sum_{\kappa: T \rightarrow T'} Z[\kappa] \langle T' |$$

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BF-Action

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$$S^{BF} = \int_{\mathcal{M}} \text{Tr} B \wedge F(A)$$

A SO(4) connection on \mathcal{M} , $F = dA + [A, A]$ curvature, B Lie-algebra valued 2-form

Obtain GR if $B = e \wedge e \rightsquigarrow$ Simplicity constraints

BF-partition function:

$$\begin{aligned} Z(\mathcal{M}) &= \int \int \mathcal{D}A \mathcal{D}B \exp \left(i \int_{\mathcal{M}} \text{Tr} B \wedge F \right) \\ &= \int \mathcal{D}A \delta(F) \end{aligned}$$

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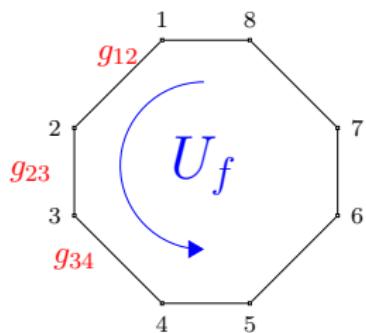
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Discretization I

Consider an abstract two-complex κ with boundary $\partial\kappa$

Each face of κ

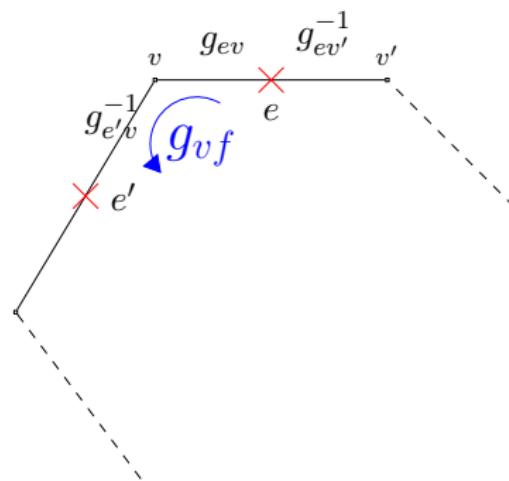


$$\delta(F) \rightsquigarrow \delta \left(\prod_{ef} U_{ef} \right) \quad U_{ef} \in \text{SO}(4)$$

$$Z^{BF}[\kappa] = \int \prod_{e \in \kappa_{int}} dU_e \prod_f \delta \left(\prod_{ef} U_{ef} \right)$$

[Kaminski, Kisielowski, Lewandowski], [Ding, Han, Rovelli]

Discretization II



$$g_{fv} := (g_{ev} g_{e'v}^{-1})^{\epsilon_{ef}}$$

$$Z^{BF}[\kappa] = \int d g_{fv} \prod_f \delta \left(\prod_{v \in \partial f} g_{fv} \right) \prod_v A_v(g_{fv})$$

Discretization II

$$Z^{BF}[\kappa] = \int dg_{fv} \prod_f \delta \left(\prod_{v \in \partial f} g_{fv} \right) \prod_v A_v(g_{fv})$$

Expansion of $A_v(g)$

$$A_v(g_{fv}) = \sum_{\rho_f, I_e} \left(\prod_f \sqrt{\dim \rho_f} \right) \text{Tr}_v \left[\bigotimes_{ev} I_e \right] T_{\Gamma_v, \rho_f, I_e}^{BF}(g_f)$$

where $T_{\Gamma, \rho, I}^{BF}(g) = \underbrace{T_{\Gamma, j^+, \iota^+}(g^+)}_{\text{SU}(2)} \otimes \underbrace{T_{\Gamma, j^-, \iota^-}(g^-)}_{\text{SU}(2)}$

Simplicity Constraint

Simplicity constraint restricts representations to $j^\pm = \frac{|\gamma \pm 1|}{2} j$

Barbero-Immirzi parameter $\gamma \in \mathbb{R}$; here $\gamma = (2n + 1)$ $n \in \mathbb{N}$

Solution space spanned by

$$T_{\Gamma_v, j_f, \iota_e}^E(g_{ef}) = \prod_{f_v} \sqrt{d_{j_{f_v}^+} d_{j_{f_v}^-}} \prod_{e_v} \left[\iota_e^{A_{e1} \dots A_{eF}} \prod_{f \in \partial v} C_{A_{ef}}^{m_{ef}^+ m_{ef}^-} \right] \\ \prod_{(e,f) \in \partial v} \begin{bmatrix} \epsilon^{n_{ef}^+ n_{ef}'^+} & \epsilon^{n_{ef}^- n_{ef}'^-} & R_{m_{ef}^+ n_{ef}^+}^{j_f^+}(g_{ef}^+) & R_{m_{ef}^- n_{ef}^-}^{j_f^+}(g_{ef}^-) \end{bmatrix}$$

Projection of vertex Amplitude

$$\mathcal{A}_v^E(g_f) = \sum_{j_f, \iota_e} \langle T_{\Gamma_v, j_f, \iota_e}^E | \mathcal{A}_v \rangle T_{\Gamma_v, j_f, \iota_e}^E(g_f)$$

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EPRL-Partition Function I

Integration only over bulk variables \implies spin-network structure on $\partial\kappa$

Note: $T^E(g)$ reduces to SU(2) spin-network functions
when $g \rightarrow h \in \text{SU}(2)$

$Z[\kappa]$ defines operator on \mathcal{H}_{kin} with matrix elements

$$\langle \psi_{out} | Z[\kappa] | \psi_{in} \rangle = \sum_{j_f, \iota_e} \prod_f d_{j_f^+} d_{j_f^-} \prod_{l \in \partial\kappa^{(1)}} \frac{1}{\sqrt{d_{j_l}}} \prod_{v \in \mathcal{V}_{int}} \mathcal{A}_v^E(j_f, \iota_e).$$

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SF-Projector

Use the matrix elements to define a '**rigging map**'

$$\eta[T_{\Gamma, \iota_e, j_l}] := \sum_{\Gamma'} \sum_{T'_{\Gamma'} \in \mathcal{H}_{kin, \Gamma'}} \sum_{\kappa: \Gamma \rightarrow \Gamma'} \langle T_{\Gamma, \iota_e, j_l} | Z[\kappa] | T'_{\Gamma', \iota'_e, j'_l} \rangle \langle T_{\Gamma', \iota'_e, j'_l} |$$

$\{T'_{\Gamma'}\}$ is a basis in $\mathcal{H}_{kin, \Gamma'}$

κ is a two-complex with boundary $\partial\kappa = \Gamma \cup \Gamma'$

Sum over κ also includes a sum over all colorings of κ

Do the constraint vanish?

$$0 \stackrel{!}{=} \eta[T](\hat{C}T')$$

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Hamiltonian Constraint

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$$\mathcal{C} = -\frac{2}{\kappa} \text{Tr}[(^{(\gamma)}F - (\gamma^2 - s)K \wedge K) \wedge e]$$

e inverse triad, K extrinsic curvature, F curvature of A

Thiemann's Trick

$$H[N] = \int_{\Sigma} d^3x N(x) H(x) = -2 \int_{\Sigma} N \text{Tr}(F \wedge \{A, V\})$$

N Lapse function, V 3d Volume

[Thiemann:QSD]

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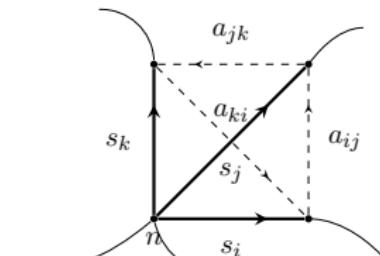
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Regularization

Regularized Hamiltonian



$$H_T[N] := \sum_{\Delta \in T} H_\Delta[N]$$

with

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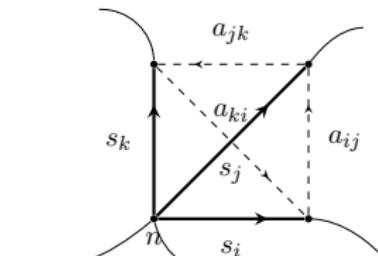
Elementary tetrahedron $\Delta \in T$
adapted to a graph Γ

Curvature regularized along the loop

$$\alpha_{ij} := s_i \circ a_{ij} \circ s_j^{-1}$$

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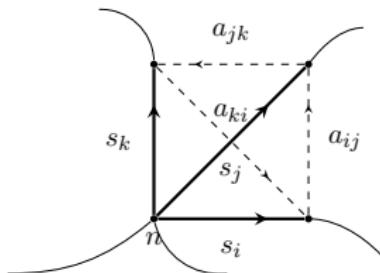
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$$\hat{H}_\Delta^m[N] := \frac{N(\mathfrak{n})}{N_m^2} \epsilon^{ijk} \text{Tr} \left[\hat{h}_{\alpha_{ij}}^{(m)} \hat{h}_{s_k}^{(m)} \left[\hat{h}_{s_k}^{(m)-1}, \hat{V} \right] \right]$$

[Gaul, Rovelli]

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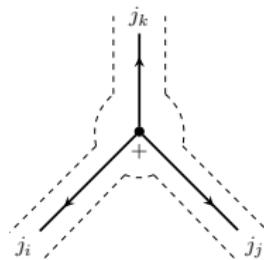
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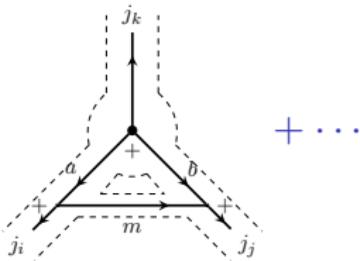
Trivalent Node I

Tri-valent node



$$\hat{H}_{\Delta}^m[N] := \frac{N(n)}{N_m^2} \epsilon^{ijk} \text{Tr} \left[\hat{h}_{\alpha ij}^{(m)} \hat{h}_{s_k}^{(m)} \left[\hat{h}_{s_k}^{(m)-1}, \hat{V} \right] \right]$$

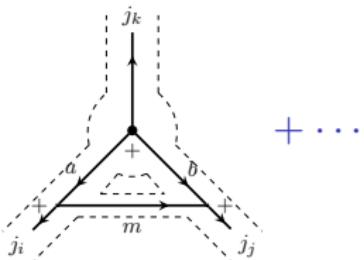
$$\sum_{a,b} A^{(m)}(j_i, a|j_j, b|j_k)$$



+ ⋯

Trivalent Node II

$$\sum_{a,b} A^{(m)}(j_i, a|j_j, b|j_k)$$


 $+ \dots$

$$A^{(m)}(j_i, a|j_j, b|j_k) := \sum_c \Lambda \sum_{\beta(j_i, j_j, m, c)} V_{j_k}^{\beta}(j_i, j_j, m, c)$$

$$\times \left[(-)^x \left\{ \begin{smallmatrix} a & j_j & c \\ \beta & m & j_i \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a & b & j_k \\ m & c & j_j \end{smallmatrix} \right\} - (-)^y \left\{ \begin{smallmatrix} j_i & b & c \\ m & \beta & j_j \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a & b & j_k \\ c & m & j_i \end{smallmatrix} \right\} \right]$$

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One-vertex Expansion I

We want to show:

$$\sum_S \langle T_{out} | Z[\kappa] | S \rangle \langle S | \hat{H}^{(m)} | T_{in} \rangle = 0$$

Easiest case

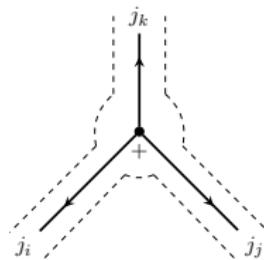
$$T_{in} = T_{out} = \left| + \begin{array}{c} j_k \\ \diagup \quad \diagdown \\ j_j \\ \diagdown \quad \diagup \\ j_i \end{array} - \right\rangle$$

Only need to consider

$$S = \left| \begin{array}{c} j_k \\ \diagup \quad \diagdown \\ j_j \\ \diagdown \quad \diagup \\ j_i \\ \hline a & m & b \end{array} \right\rangle$$

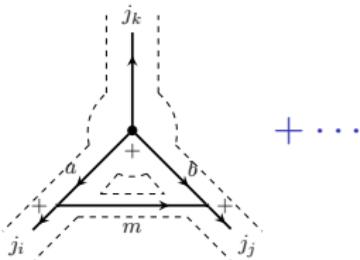
Trivalent Node I

Tri-valent node



$$\hat{H}_{\Delta}^m[N] := \frac{N(n)}{N_m^2} \epsilon^{ijk} \text{Tr} \left[\hat{h}_{\alpha_{ij}}^{(m)} \hat{h}_{s_k}^{(m)} \left[\hat{h}_{s_k}^{(m)-1}, \hat{V} \right] \right]$$

$$\sum_{a,b} A^{(m)}(j_i, a|j_j, b|j_k)$$



One-vertex Expansion I

We want to show:

$$\sum_S \langle T_{out} | Z[\kappa] | S \rangle \langle S | \hat{H}^{(m)} | T_{in} \rangle = 0$$

Easiest case

$$T_{in} = T_{out} = \left| + \begin{array}{c} j_k \\ \diagup \quad \diagdown \\ j_j \\ \diagdown \quad \diagup \\ j_i \end{array} - \right\rangle$$

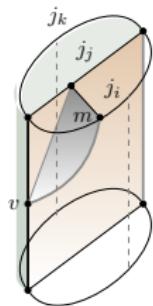
Only need to consider

$$S = \left| \begin{array}{c} j_k \\ \diagup \quad \diagdown \\ j_j \\ \diagdown \quad \diagup \\ j_i \\ \hline a & m & b \end{array} \right\rangle$$

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Two-complex κ

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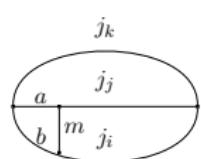
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One-vertex Expansion II

- Each node contributes a $9j$ symbol of the type



Boundary
graph Γ_v

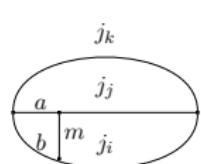
$$\sqrt{d_a d_b d_c} \begin{Bmatrix} a & b & c \\ a^+ & b^+ & c^+ \\ a^- & b^- & c^- \end{Bmatrix}$$

- The vertex trace gives:

$$\begin{Bmatrix} j_i^+ & j_j^+ & j_k^+ \\ b^+ & a^+ & m^+ \end{Bmatrix} \begin{Bmatrix} j_i^- & j_j^- & j_k^- \\ b^- & a^- & m^- \end{Bmatrix}$$

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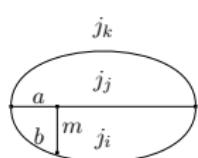
Complete spin-foam amplitude for $\gamma = 1$

$$(j^+, j^-) \xrightarrow{\gamma=1} (j, 0)$$

$$W_E(\kappa, \Theta, s)|_{\gamma=1} = (d_a d_b d_m)^{1/2} (-)^{j_i + j_j - j_k} \begin{Bmatrix} j_i & j_j & j_k \\ b & a & m \end{Bmatrix}$$

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Projection

$$\begin{aligned}
 & \sum_S W_E(\kappa, S, \Theta) |_{\gamma=1} \langle S | \hat{H}^{(m)} | \Theta \rangle \\
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 &\times \left[\sum_b d_b (-)^{2\beta} \begin{Bmatrix} j_i & \beta & j_j \\ j_k & c & m \\ j_j & m & b \end{Bmatrix} - \sum_a d_a (-)^{j_k + \beta} \begin{Bmatrix} j_j & j_k & j_i \\ \beta & c & m \\ j_i & m & a \end{Bmatrix} \right] \\
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 \end{aligned}$$

Plan of the Talk

1 Motivation

- Canonical LQG
- The Rigging Map
- Spin-Foams

2 Spin-Foam Projector

- Construction
- SF-Projector

3 Hamiltonian Constraint

- Quantization
- Action on trivalent nodes

4 One-vertex Expansion

5 Discussion

The Role of the Volume I

- Do not need to compute the volume, since SF- amplitude yields to $V_{j_k}{}^\beta(j_i, j_j, m, c) \delta_{\beta j_k}$
- Same behavior for four-valent and thus n -valent vertices

Main Result

The states $\sum_S W_E(\kappa, S, \Theta_n)|_{\gamma=1}\langle S|$ are weak solutions of the euclidean Hamiltonian constraint!

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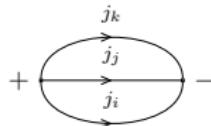
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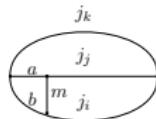
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The Role of the Volume II



- Each term associated to the triple (j_i, j_k, m) **vanish separately**

$\Downarrow \hat{H}$



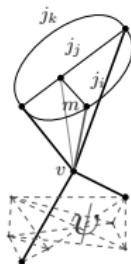
The Role of the Volume II

- Each term associated to the triple (j_i, j_k, m) **vanish separately**
- Volume is “**passive observer**”; Curvature F is **crucial**

Setting $\gamma = 1$ reduces EPRL-Amplitude
to SU(2)-BF theory

Generalization/Outlook

Arbitrary complex



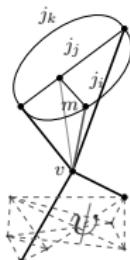
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- Counterexample: Trivial evolution
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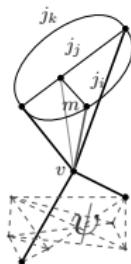
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Different \hat{H}_Δ

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Thank you for your attention!

◀ Contents

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