



Complete 3 Loop unpolarized and polarized heavy flavor corrections to DIS for $Q^2 \gg m^2$

alphas-2025: CNRS Centre Paul Langevin, Aussois, F, December 14-19, 2025

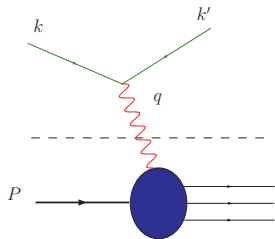
Johannes Blümlein | December 15, 2025

DESY AND TU DORTMUND

In collaboration with:

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Deep-Inelastic Scattering (DIS):



$$\longrightarrow L_{\mu\nu} \quad Q^2 := -q^2, \quad x := \frac{Q^2}{2P \cdot q} \quad \text{Bjorken-}x$$

$$\longrightarrow W_{\mu\nu} \quad \frac{d\sigma}{dQ^2 dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle =$$

$$\frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \\ + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda S^\sigma}{P \cdot q} g_1(x, Q^2) + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda (P \cdot q S^\sigma - S \cdot q P^\sigma)}{(P \cdot q)^2} g_2(x, Q^2).$$

The structure functions $F_{2,L}$ and $g_{1,2}$ contain light and heavy quark contributions. At 3-loop order also graphs with two heavy quarks of different mass contribute. \implies Single and 2-mass contributions: c and b quarks in one graph.

Factorization of the Structure Functions



At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{C_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{nonpert.}}$$

into (pert.) **Wilson coefficients** and (nonpert.) **parton distribution functions (PDFs)**.

\otimes denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \delta(x - yz) f(y) g(z).$$

Many of the subsequent calculations are performed in Mellin space, where \otimes reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx x^{N-1} f(x).$$

Wilson coefficients:

$$C_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) + H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right).$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i C_{i,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) A_{ij} \left(\frac{m^2}{\mu^2}, N \right)$$

[Buza, Matiounine, Smith, van Neerven 1996]

factorizes into the **light flavor Wilson coefficients** C and the **massive operator matrix elements (OMEs)** of local operators O_i between partonic states j

$$A_{ij} \left(\frac{m^2}{\mu^2}, N \right) = \langle j | O_i | j \rangle.$$

→ additional **Feynman rules with local operator insertions** for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are **known up to NNLO**

[Vermaseren, Moch, Vogt, 2005; JB, Marquard, Schneider, Schönwald, 2022].

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The main time-line for the 3-loop corrections



- 2005 F_L [no massive 3-loop OMEs needed]
- 2010 All unpolarized N_F terms and $A_{qg,Q}^{(3)}, A_{qq,Q}^{(3).PS}$
- 2014 unpolarized logarithmic 3-loop contributions and $A_{gq,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3).NS}, A_{Qq}^{(3).PS}$
- 2017 two-mass corrections $A_{gq,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3).NS}, A_{Qq}^{(3).PS}$
- 2018 two-mass corrections $A_{gg,Q}^{(3)}$
- 2019 2-loop correction: $(\Delta)A_{Qq}^{(2).PS}$ whole kinematic region and $\Delta A_{Qq}^{(3).PS}$
- 2019 two-mass corrections $\Delta A_{Qq}^{(3).PS}$
- 2020 two-mass corrections $\Delta A_{gg,Q}^{(3)}$
- 2021 polarized logarithmic 3-loop contributions and $\Delta A_{qg,Q}^{(3)}, \Delta A_{qq,Q}^{(3).PS}, \Delta A_{gq}^{(3)}$
- 2022 3-loop polarized massless Wilson coefficients [JB, Marquard, Schneider, Schönwald]
- 2022 corrected the polarized 2-loop contributions [Buza, van Neerven et al.]
- 2022 $(\Delta)A_{gg,Q}^{(3)}$
- 2023 $(\Delta)A_{Qg}^{(3)}$: 1st order factorizing parts
- 2024 $(\Delta)A_{Qg}^{(3)}$, [single-mass corrections]
- 2025 $(\Delta)A_{Qg}^{(3)}$, [two-mass corrections]

- [48 physics papers \(journals\)](#)
- [26 mathematical papers](#)
 - 1998 Harmonic sums [Vermaseren; JB]
 - 2000,2005 Analytic continuations of harmonic sums to $N \in \mathbb{C}$ [JB; JB, S. Moch]
 - 2003 Concrete shuffle algebras [JB]
 - 2009 Guessing large recurrences [JB, M. Kauers, S. Klein, C. Schneider]
 - 2009 Structural relations of harmonic sums [JB]
 - 2009 MZV Data mine [JB, D. Broadhurst, J. Vermaseren]
 - 2011 Cyclotomic harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]
 - 2013 Generalized harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]; 2001 [Moch, Uwer, Weinzierl]
 - 2014 Finite binomial sums and root-valued iterated integrals [Ablinger, JB, Raab, Schneider]
 - 2017 ${}_2F_1$ solutions (iterated non-iterative integrals) [J. Ablinger, JB, A. De Freitas, M. van Hoeij, E. Imamoglu, C. Raab, C.S. Radu, C. Schneider]
 - 2017 Methods of arbitrary high moments [JB, Schneider]
 - 2018 Automated solution of first-order factorizing differential equation systems in an arbitrary basis [J. Ablinger, JB, P. Marquard, N. Rana, C. Schneider]
 - 2023 Analytic continuation from t to x -space [JB, Behring, Schönwald]

Important Computer-Algebra Packages

C. Schneider: Sigma, EvaluateMultiSums, SumProduction, SolveCoupledSystem

J. Ablinger: HarmonicSums

The Wilson Coefficients at large Q^2



$$L_{q,(2,L)}^{\text{NS}}(N_F + 1) = a_s^2 [A_{qq,Q}^{(2),\text{NS}}(N_F + 1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(N_F)] + a_s^3 [A_{qq,Q}^{(3),\text{NS}}(N_F + 1)\delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1)C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F)]$$

$$L_{q,(2,L)}^{\text{PS}}(N_F + 1) = a_s^3 [A_{qq,Q}^{(3),\text{PS}}(N_F + 1)\delta_2 + N_F A_{qq,Q}^{(2),\text{NS}}(N_F) \tilde{C}_{g,(2,L)}^{(1),\text{NS}}(N_F + 1) + N_F \hat{C}_{q,(2,L)}^{(3),\text{PS}}(N_F)]$$

$$L_{g,(2,L)}^{\text{S}}(N_F + 1) = a_s^2 [A_{gg,Q}^{(1)}(N_F + 1)N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + a_s^3 [A_{gg,Q}^{(3)}(N_F + 1)\delta_2 + A_{gg,Q}^{(1)}(N_F + 1)N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1)N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1)N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) + N_F \hat{C}_{g,(2,L)}^{(3)}(N_F)]$$

$$H_{q,(2,L)}^{\text{PS}}(N_F + 1) = a_s^2 [A_{Qq}^{(2),\text{PS}}(N_F + 1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1)] + a_s^3 [A_{Qq}^{(3),\text{PS}}(N_F + 1)\delta_2 + A_{qq,Q}^{(2)}(N_F + 1)\tilde{C}_{g,(1,L)}^{(2)}(N_F + 1) + A_{Qq}^{(2),\text{PS}}(N_F + 1)\tilde{C}_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F + 1)]$$

$$H_{g,(2,L)}^{\text{S}}(N_F + 1) = a_s [A_{Qg}^{(1)}(N_F + 1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1)] + a_s^2 [A_{Qg}^{(2)}(N_F + 1)\delta_2 + A_{Qg}^{(1)}(N_F + 1)\tilde{C}_{q,(2,L)}^{(1)}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1)\tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1)] + a_s^3 [A_{Qg}^{(3)}(N_F + 1)\delta_2 + A_{Qg}^{(2)}(N_F + 1)\tilde{C}_{q,(2,L)}^{(1)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1)\tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1)\tilde{C}_{q,(2,L)}^{(2),\text{S}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1)\tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1)]$$

- The case for two different masses obeys an analogous representation.
- Note the contributions of the **massless Wilson coefficients**.

The variable flavor number scheme



- Matching conditions for parton distribution functions:

$$f_k(N_F + 2) + \bar{f}_k(N_F + 2) = A_{qq,Q}^{\text{NS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot [f_k(N_F) + \bar{f}_k(N_F)] + \frac{1}{N_F} A_{qq,Q}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) \\ + \frac{1}{N_F} A_{qg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F),$$

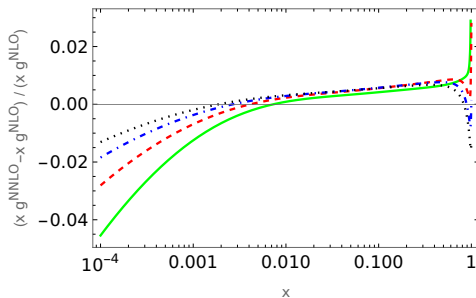
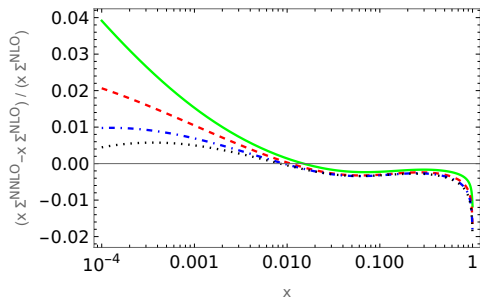
$$f_Q(N_F + 2) + \bar{f}_Q(N_F + 2) = A_{Qq}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) + A_{Qg} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F),$$

$$\Sigma(N_F + 2) = \left[A_{qq,Q}^{\text{NS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{qq,Q}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{Qq}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot \Sigma(N_F) \\ + \left[A_{qg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{Qg} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot G(N_F),$$

$$G(N_F + 2) = A_{gq,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) + A_{gg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F).$$

The charm and bottom quark masses are not that much different.

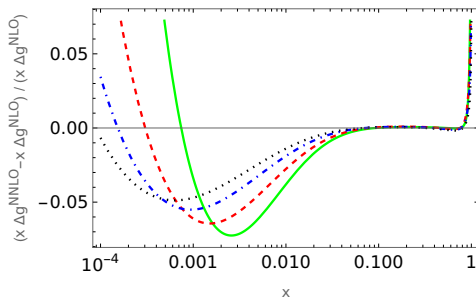
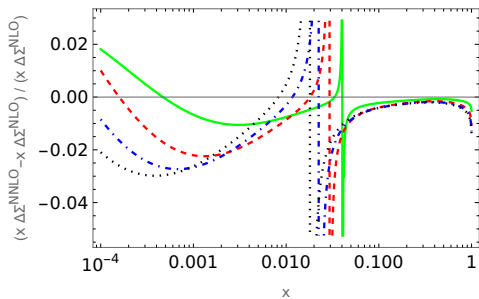
Relative effect in unpolarized NNLO evolution



$Q^2 = 10, 10^2, 10^3, 10^4 \text{ GeV}^2$ dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The unpolarized world deep-inelastic data have a precision of $O(1\%)$.

Relative effect in polarized NNLO evolution



$Q^2 = 10, 10^2, 10^3, 10^4 \text{ GeV}^2$ dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The future polarized data at the **EIC** will reach a precision of $O(1\%)$.

Calculation methods



- Diagram generation: QGRAF [Nogueira, 1993]
- Lorentz and Dirac algebra: Form [Vermaseren, 2000]
- Color algebra: Color [van Ritbergen, Schellekens, Vermaseren, 1999]
- IBP reduction: Reduze 2 [von Manteuffel, Studerus 2009,2012]
- N space calculations:
 - Method of arbitrary large moments [JB, Schneider, 2017]
 - Summation theory and solving first-order factorizing recurrences: Sigma [Schneider, 2007,2013]
 - Reduce the results in the respective function spaces: HarmonicSums [Ablinger, 2009, 2012, etc.]
- x space calculations
 - solve 1st order factorizing differential equations
 - transform from $N \rightarrow t$ -space, solve the respective systems of differential equations (not necessarily factorizing to first order) [Behring, JB, Schönwald, 2023]
 - Reduce the results in the respective function spaces; iterated integrals over alphabets containing also higher transcendental letters [Ablinger et al. 2017]
 - The higher transcendental letters have to be known in analytic form for $z \in \mathbb{C}$.
- Both N and x space techniques are needed to solve the present problem. The recurrences for $A_{Qg}^{(3)}$ need far more than 15000 moments to be found & there are no technologies yet to solve non-first order factorizing recurrences analytically.
- Currently we have technologies to generate $O(25000)$ moments \implies massive 3-loop form factor and we guess differential equations based on $O(50000)$.

- massless and massive contributions to two-loops: **harmonic sums**
- all pole terms to three-loops: **harmonic sums**
- all massless Wilson coefficients to three-loops: **harmonic sums**

Single-mass OMEs

- all N_F of the massive OMEs three-loops: **harmonic sums**
- $(\Delta)A_{qq,Q}^{(3),NS}$, $(\Delta)A_{gq,Q}^{(3)}$, $(\Delta)A_{qg,Q}^{(3)}$, $(\Delta)A_{qq,Q}^{(3),PS}$ to three-loops: **harmonic sums**
- $(\Delta)A_{Qq}^{(3),PS}$ to three-loops: **generalized harmonic sums** and also $H_{\bar{a}}(1 - 2x)$
- $(\Delta)A_{gg,Q}^{(3)}$ to three-loops: **finite binomial sums** and square-root valued iterated integrals
- $(\Delta)A_{Qg}^{(3)}$ to three-loops:
 - first-order factorizing contributions: **finite binomial sums**; all iterated integrals in x -space can be rationalized
 - non-first-order factorizing contributions: **${}_2F_1$ letters** in iterated integrals in x -space

Inverse Mellin transform via analytic continuation: $a_{Qg}^{(3)}$



Resumming Mellin N into a continuous variable t , observing crossing relations. Ablinger et al. 2012

$$\sum_{k=0}^{\infty} t^k (\Delta \cdot p)^k \frac{1}{2} [1 \pm (-1)^k] = \frac{1}{2} \left[\frac{1}{1 - t\Delta \cdot p} \pm \frac{1}{1 + t\Delta \cdot p} \right]$$

$$\mathfrak{A} = \{f_1(t), \dots, f_m(t)\}, \quad G(b, \vec{a}; t) = \int_0^t dx_1 f_b(x_1) G(\vec{a}; x_1), \quad \left[\frac{d}{dt} \frac{1}{f_{a_{k-1}}(t)} \frac{d}{dt} \dots \frac{1}{f_{a_1}(t)} \frac{d}{dt} \right] G(\vec{a}; t) = f_{a_k}(t).$$

The $f_i(t)$ include higher transcendental letters. Regularization for $t \rightarrow 0$ needed.

$$F(N) = \int_0^1 dx x^{N-1} [f(x) + (-1)^{N-1} g(x)]$$

$$\tilde{F}(t) = \sum_{N=1}^{\infty} t^N F(N)$$

$$f(x) + (-1)^{N-1} g(x) = \frac{1}{2\pi i} \left[\text{Disc}_x \tilde{F} \left(\frac{1}{x} \right) + (-1)^{N-1} \text{Disc}_x \tilde{F} \left(-\frac{1}{x} \right) \right]. \quad (1)$$

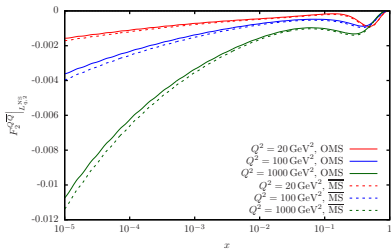
t -space is still Mellin space. One needs closed expressions to perform the analytic continuation (1). Analytic continuation is needed to calculate the **small x behaviour**. The **final expansion** maps the problem into a **very large number** of G -constants, including those with higher transcendental letters.

- Master integrals, solving differential equations not factorizing to 1st order
- ${}_2F_1$ solutions [Ablinger et al. \[2017\]](#)
- Mapping to complete elliptic integrals: **duplication** of the higher transcendental letters.
- Complete elliptic integrals, modular forms [Sabry, Broadhurst, Weinzierl, Remiddi, Tancredi, Duhr, Broedel et al. and many others.](#)
- Abel integrals
- K3 surfaces [Brown, Schnetz \[2012\]](#)
- Calabi-Yau motives [Klemm, Duhr, Weinzierl et al. \[2022\]](#)

Refer to as few as possible higher transcendental functions, the properties of which are known in full detail.

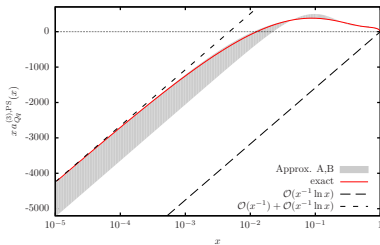
- $A_{Qg}^{(3)}$: effectively only one 3×3 system of this kind.
- The system is connected to that occurring in the case of ρ parameter. [Ablinger et al. \[2017\]](#), [JB et al. \[2018\]](#), [Abreu et al. \[2019\]](#)
- Most simple solution: **two** ${}_2F_1$ functions.

Non-singlet and pure singlet

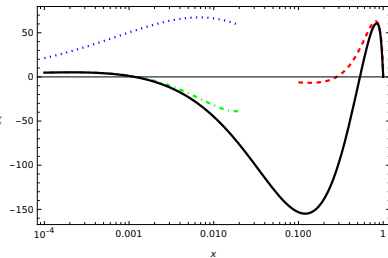


NS sHQ F_2

[J. Ablinger et al., 2014 a,b; 2020]

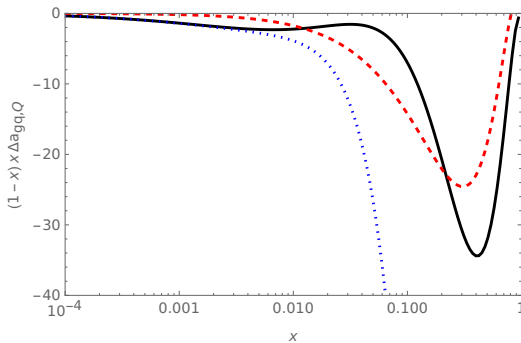
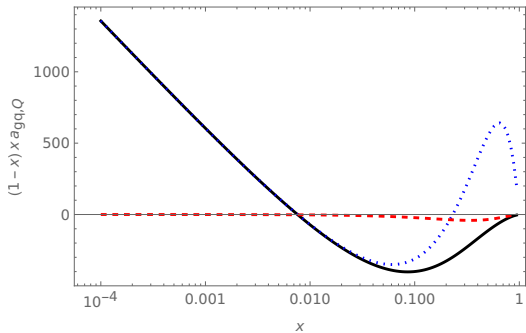


PS unpolarized



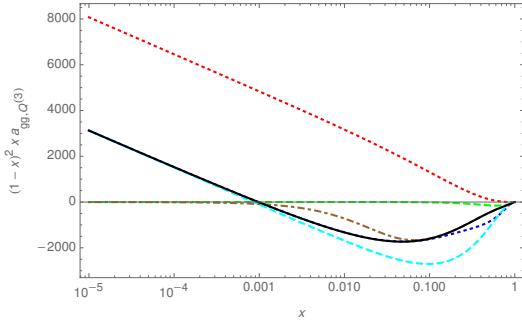
PS polarized

$a_{gq}^{(3)}$ and $\Delta a_{gq}^{(3)}$

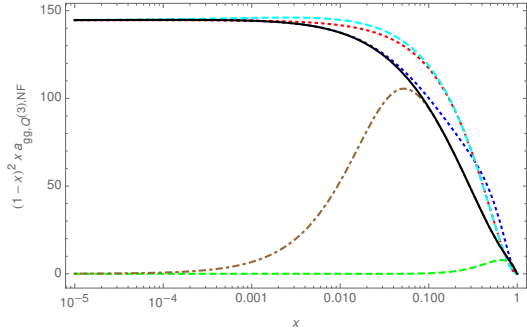


$N_F = 3$. Full line (black): complete result; dotted line (blue): small x expansion; dashed line (red): large x expansion.

[J. Ablinger et al., 2014; 2020]



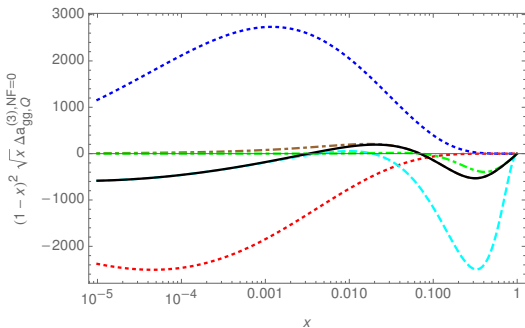
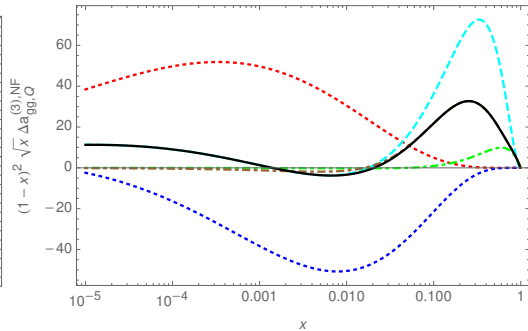
non N_F terms



N_F terms

Left panel: The non- N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x . Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution. Right panel: the same for the N_F contribution.

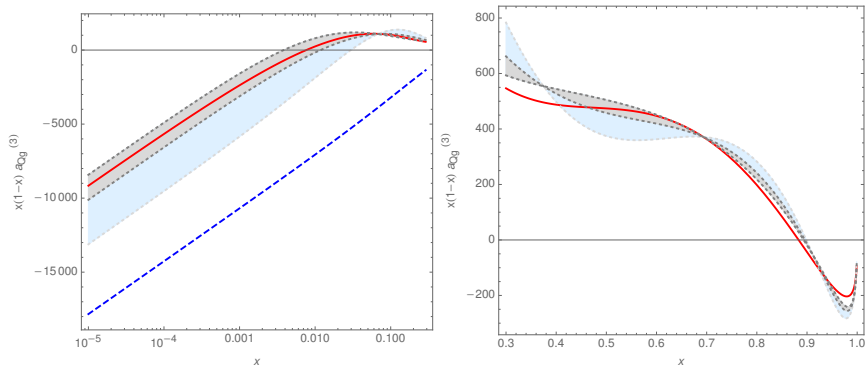
[J. Ablinger et al., 2022]


 non N_F terms

 N_F terms

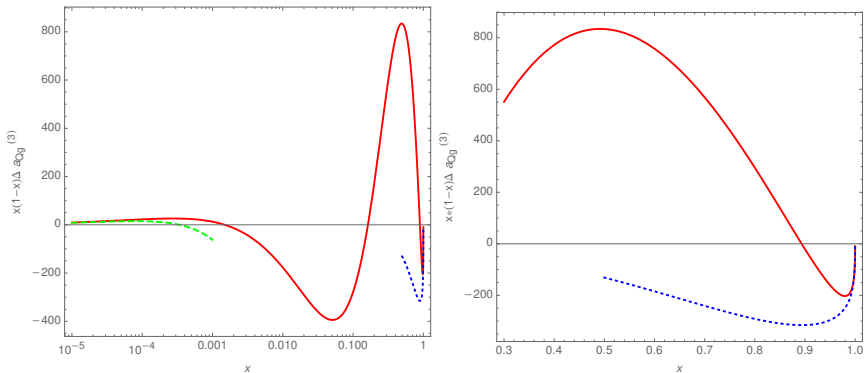
The non- N_F terms of $\Delta a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x . Full line (black): complete result; lower dotted line (red): term $\ln^5(x)$; upper dotted line (blue): small x terms $\propto \ln^5(x)$ and $\ln^4(x)$; upper dashed line (cyan): small x terms including all $\ln(x)$ terms up to the constant term; lower dash-dotted line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution. Right panel: the same for the N_F contribution.

[J. Ablinger et al., 2022]

1009 of the total 1233 Feynman diagrams have first-order factorizing contributions only and are given by G -functions up to root-values letters. The letters for all constants can be rationalized.



$a_{Qg}^{(3)}(x)$ as a function of x , rescaled by the factor $x(1-x)$. Left panel: smaller x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (blue): leading small- x term $\propto \ln(x)/x$ [Catani, Ciafaloni, Hautmann, 1990]; light blue region: estimates of [Kawamura et al., 2012]; gray region: estimates of [ABMP 2017]. Right panel: larger x region. Full line (red): $a_{Qg}^{(3)}(x)$; light blue region: estimates of [Kawamura et al., 2012] gray region: estimates of [ABMP 2017]. [J. Ablinger et al., 2022; 2023]

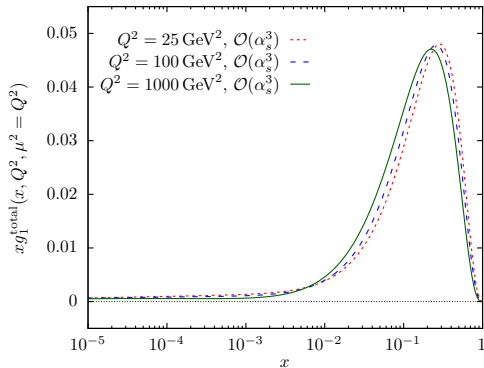
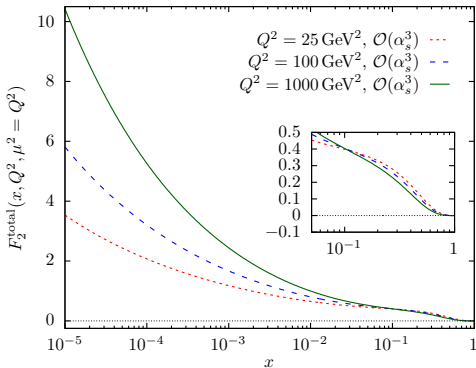


$\Delta a_{Qg}^{(3)}(x)$ as a function of x , rescaled by the factor $x(1-x)$. Left panel: full line (red): $\Delta a_{Qg}^{(3)}(x)$; dashed line (green): the small- x terms $\ln^k(x)$, $k \in \{1, \dots, 5\}$; dotted line (blue): the large- x terms $\ln^l(1-x)$, $l \in \{1, \dots, 4\}$. Right panel: larger x region. Full line (red): $\Delta a_{Qg}^{(3)}(x)$; dotted line (blue): the large- x terms $\ln^l(1-x)$, $l \in \{1, \dots, 4\}$.

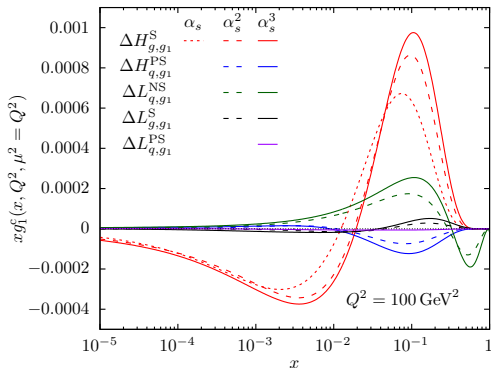
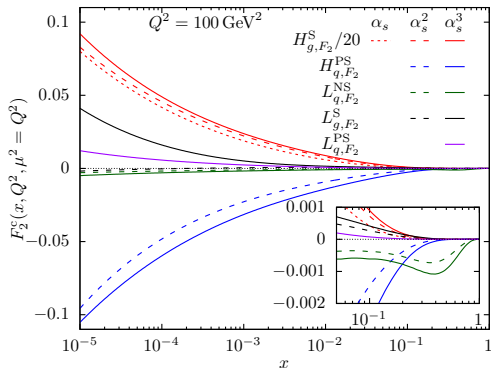
[J. Ablinger et al., 2022; 2023]

Precise and fast numerical representations of all single mass VFNS OMEs are given in [arXiv:2510.02175].

F_2 and xg_1 : massless and single-mass HQ contributions

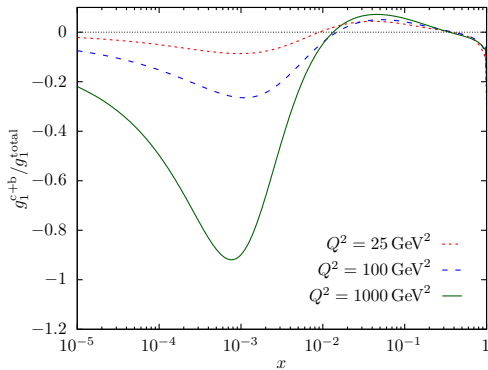
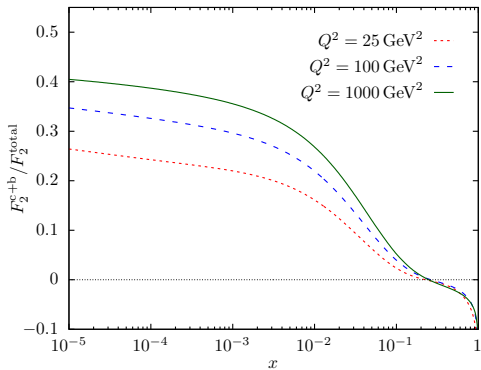


Contributions of the different Wilson coefficients to $F_2^{c,b}$ and $xg_1^{c,b}$ for the different orders in α_s

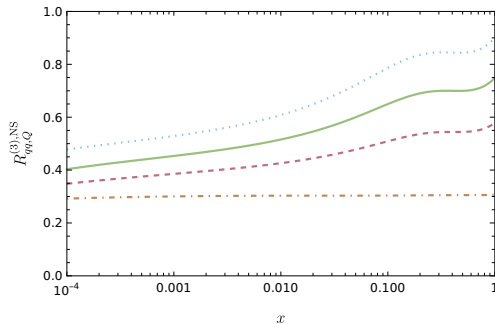


charm contributions at $Q^2 = 100 \text{ GeV}^2$: cumulative in α_s

Single mass heavy flavor ratios for F_2 and xg_1



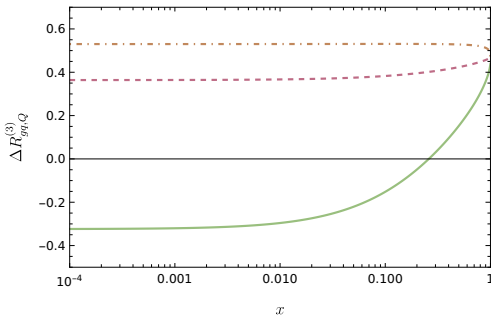
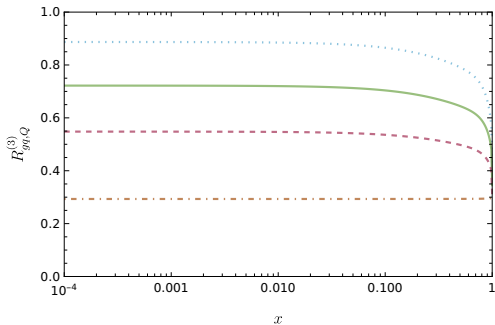
Two-mass Corrections: $(\Delta)A_{qq,Q}^{NS,2m}$



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Ablinger, JB, De Freitas, Hasselhuhn, Schneider, Wißbrock '17]

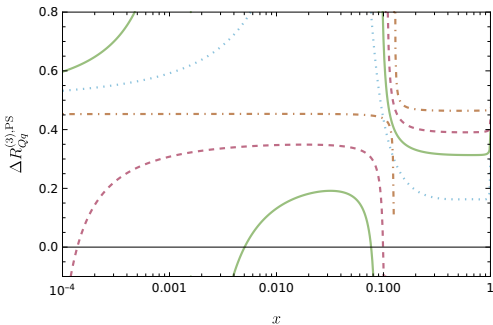
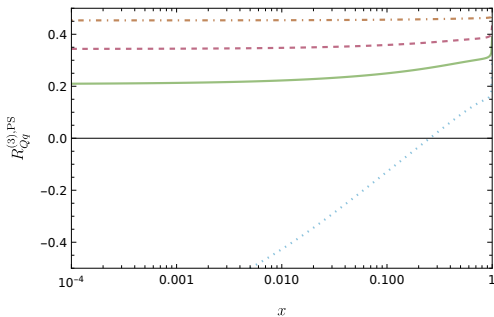
$$R_{ij} = \frac{A_{ij}^{2m,irred}}{A_{ij,tot}^{T_F^2}}$$



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Abinger, JB, De Freitas, Hasselhuhn, Schneider, Wißbrock '17]

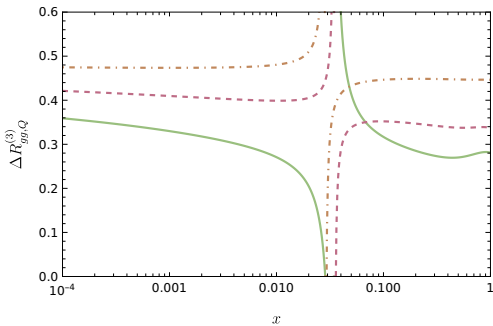
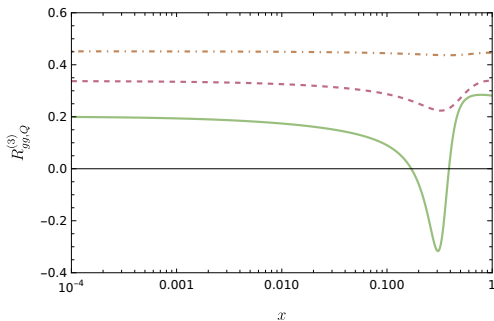
[Behring, JB, De Freitas, von Manteuffel, Schönwald, Schneider '21]



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Ablinger, JN, De Freitas, Schneider, Schönwald '17]

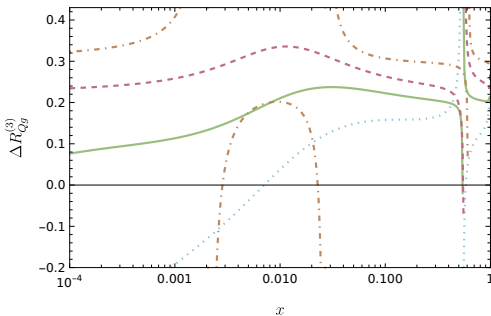
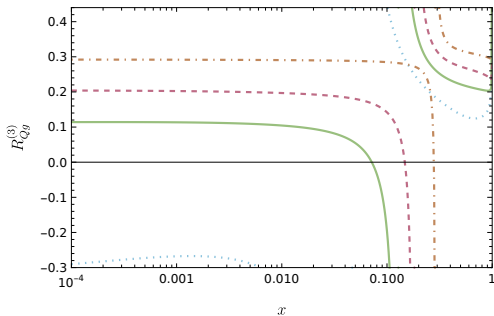
[Ablinger, JB, De Freitas, Saragnese, Schneider, Schönwald '19]



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Ablinger, JB, De Freitas, Schneider, Schönwald '18]

[Ablinger, JB, Goedicke, De Freitas, Saragnese, Schneider, Schönwald '20]

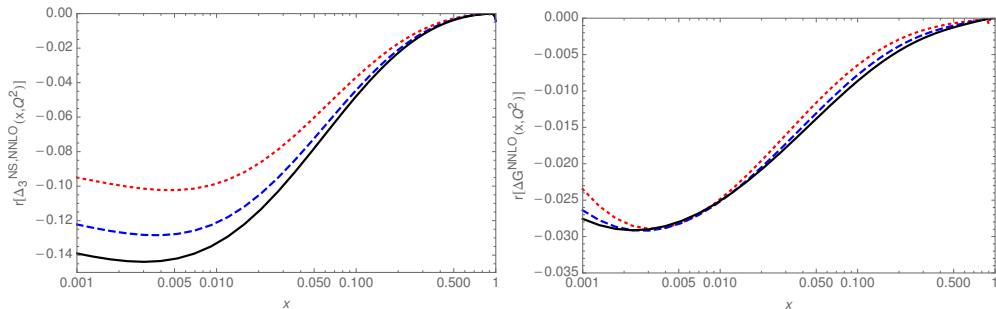


[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[J.Ablinger et al. 2025]

In summary: the 3-loop 2-mass OMEs form $\sim 50 \%$ of the $T_F^2 C_i$ contributions in all cases.

Polarized PDF evolution in the Larin Scheme



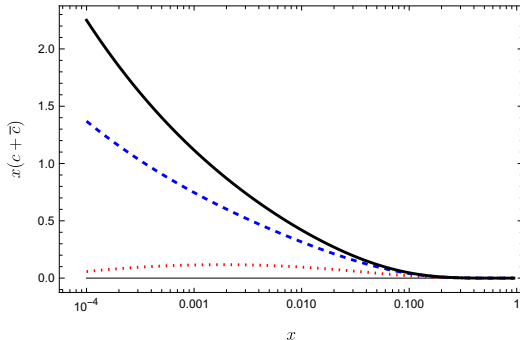
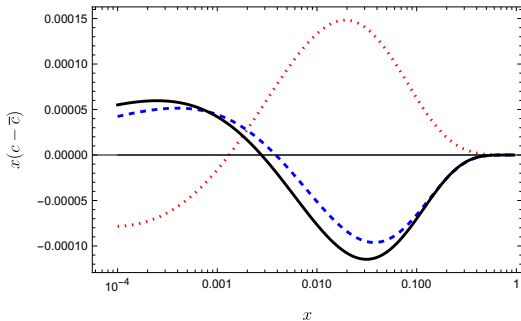
[Dotted line: $Q^2 = 100 \text{ GeV}^2$, dashed line: $Q^2 = 1000 \text{ GeV}^2$, full line: $Q^2 = 10000 \text{ GeV}^2$]

$$r(x, Q^2) = \frac{f^L(x, Q^2)}{f^M(x, Q^2)} - 1$$

The pdfs are necessary to match HO Larin-scheme calculations.

[JB and M. Saragnese, 2024]

The difference and the sum of heavy flavor pdfs in the VFNS



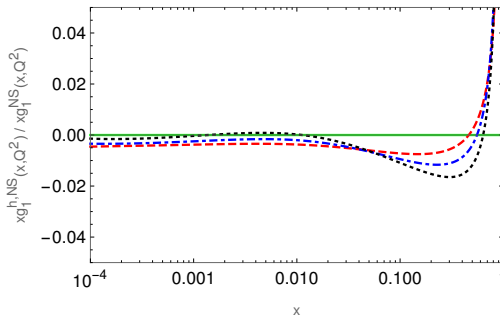
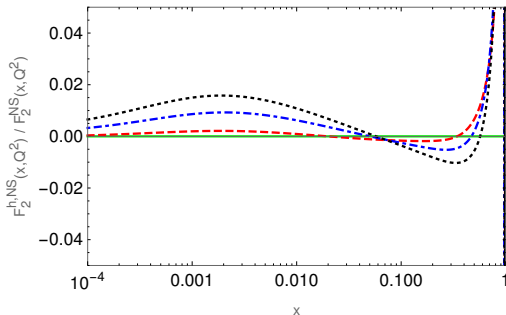
The unpolarized distributions $x[c(x, Q^2) - \bar{c}(x, Q^2)]$ (left panel) and $x[c(x, Q^2) + \bar{c}(x, Q^2)]$ (right panel). Dotted lines: $Q^2 = 4 \text{ GeV}^2$. Dashed lines: $Q^2 = 30 \text{ GeV}^2$. Full lines: $Q^2 = 100 \text{ GeV}^2$. [A. Behring, JB, A. De Freitas, A. von Manteuffel, C. Schneider, K. Schönwald, DESY 25–185].

$$\Delta\gamma_{qq}^{\text{NS},s,(2)} = 4 \frac{d_{abc} d^{abc}}{N_c} N_F \frac{1}{2} [1 + (-1)^{N_f}] \left\{ \frac{S_1 Q_4}{N^4 (1+N)^4} + \left[-\frac{2(1+N+N^2)(2+N+N^2)}{N^3 (1+N)^3} - \frac{4(N-1)(2+N)}{N^2 (1+N)^2} S_1 \right] S_{-2} - \frac{(2+N+N^2)}{N^2 (1+N)^2} [S_3 - 2S_{-3} + 4S_{-2,1}] \right\}.$$

The relative contribution of HQ to non-singlet structure functions at N³LO



Scheme-invariant evolution: the way to high-precision $\alpha_s(M_Z^2)$



The relative contribution of the heavy flavor contributions due to c and b quarks to the structure function F_2^{NS} and xg_1^{NS} ; dashed lines: 100 GeV²; dashed-dotted lines: 1000 GeV²; dotted lines: 10000 GeV². massless: N³LO; single+two-mass massive contributions: $O(\alpha_s^3)$. [JB, M. Saragnese, 2021].

Conclusions



- All unpolarized and polarized **single-mass** and **two-mass** OMEs and the associated massive Wilson coefficients for $Q^2 \gg m_Q^2$ have been calculated. The unpolarized and **polarized** massless three-loop Wilson coefficients were calculated and contribute to the present results.
- **Analytic all Q^2 results** are available in the unpolarized and polarized NS- and PS-cases at **2 loop order**.
- Various new **mathematical and technological** methods were developed during the present project. They are available for use in further single- and two-mass calculations in other QFT projects.
- Both the single- and two-mass terms of the **VFNS at 3-loop order** are finished and the single-mass contributions were made available in form of numerical codes.
- Very soon new precision analyses of the world DIS-data to measure $\alpha_s(M_Z)$ and m_c at higher precision can be carried out.
- The results in the **polarized case** prepare the analysis of the precision data, which will be taken at the **EIC** starting at the end of this decade.
- For all sub-processes it turned out that the small x **BFKL approaches fail** to present the physical result due to quite a series of missing subleading terms, which substantially correct the LO behaviour. The correct description requires the full calculation.