Strong coupling from the static energy

Project lead by Viljami Leino
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Outline

- Introduction
- Static energy
- Improvements:
  - lattice ensembles
  - rotational symmetry
  - minimal renormalon subtraction scheme
- Some results
- Conclusion

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- Strong coupling α_s is a fundamental parameter of the Standard Model.
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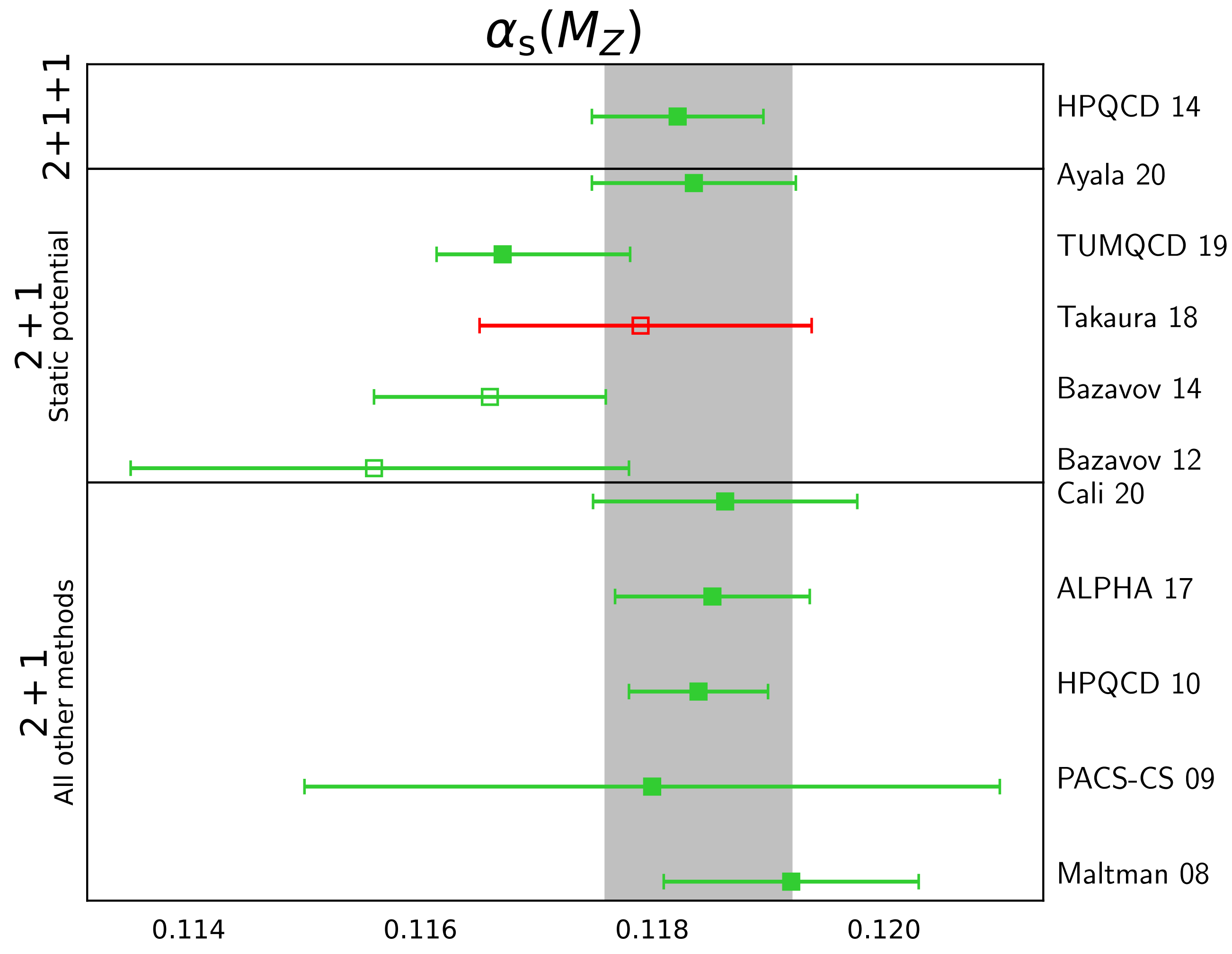

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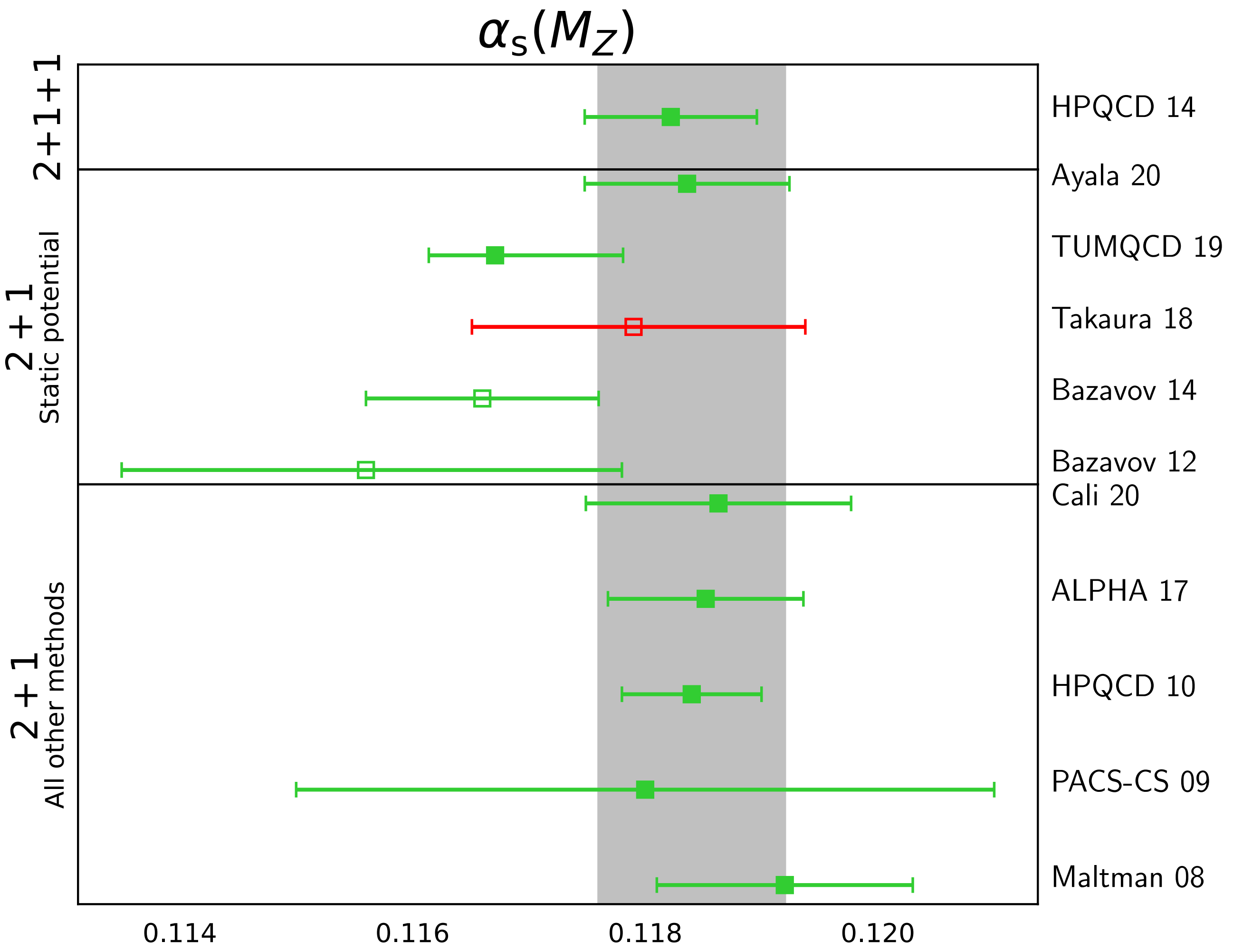
Work in progress:

- 2+1+1 flavor QCD,
- One-loop improvement of the static energy,
- Minimal renormalon subtraction (MRS) scheme.



Our computation strategy

- Compute the static energy $E_0(r)$ of quark and antiquark at separation r in lattice QCD from large-time behavior of the Wilson loops:

$$E(r) = -\lim_{T\to\infty} \frac{\ln\langle \mathrm{Tr}(W_{r\times T})\rangle}{T}, \quad W_{r\times T} = P\left\{\exp\left(i\oint_{r\times T} dz_\mu\, gA_\mu\right)\right\}.$$

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- Compute $\alpha_s(M_Z)$.

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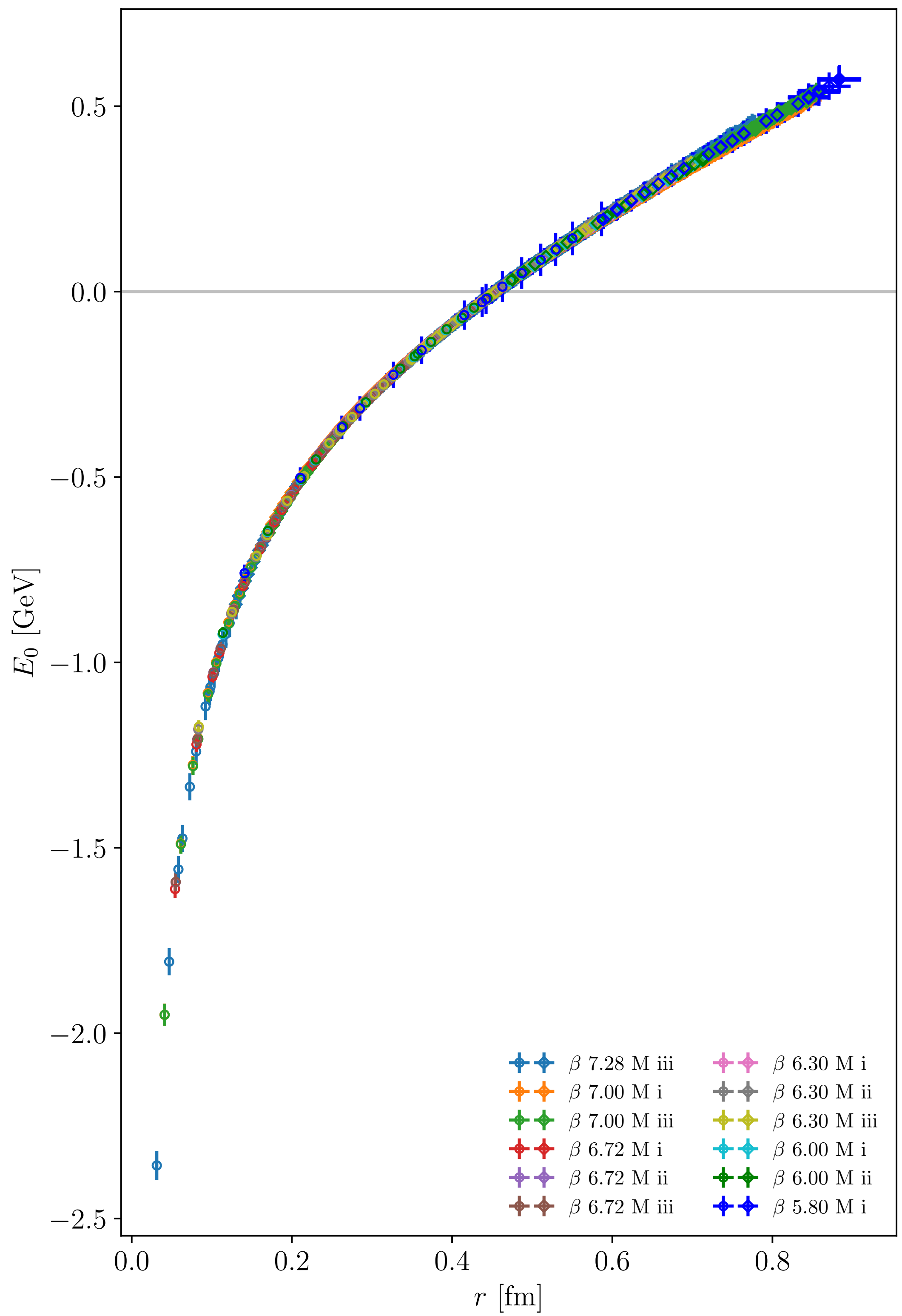
- Corrections at short distances to improve rotational symmetry.

Static energy on the lattice



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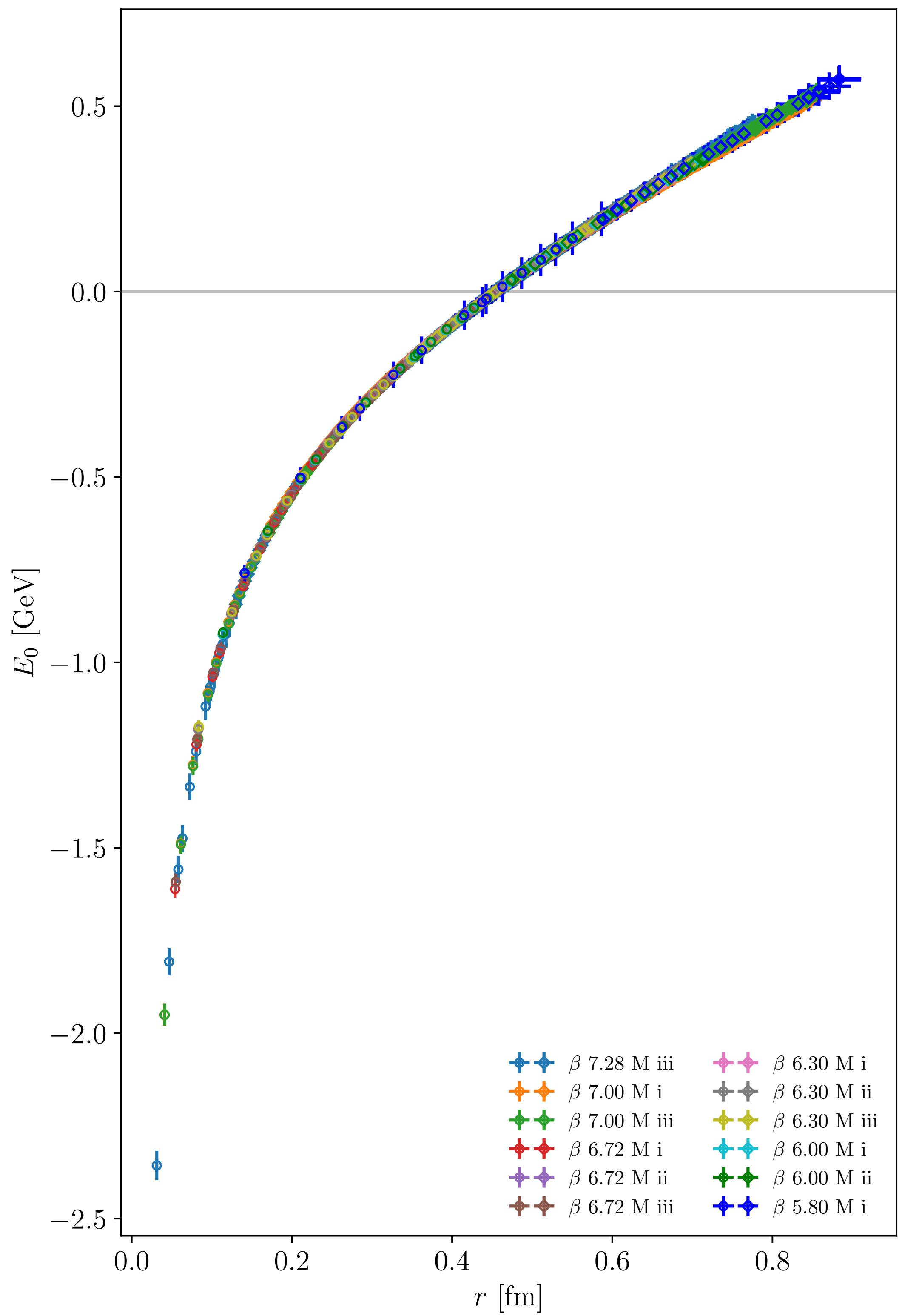

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- Simulations with the one-loop Symanzik-improved gauge action and the Highly Improved Staggered Quarks (HISQ) action.

  Lüscher, Weisz, Phys. Lett. B (1985)

  Follana et al. (HPQCD), Phys. Rev. D 75 (2007)

Connecting with perturbation theory

- Dimensional regularization: requires a renormalon subtraction.
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  - Minimal renormalon subtraction:
    relate the factorial growth of perturbative series to a power correction.

Komijani, JHEP08 (2017)

Brambilla et al., Phys. Rev. D 97 (2018)

Kronfeld, JHEP12 (2023)

Improvements: ensembles

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- Current work:

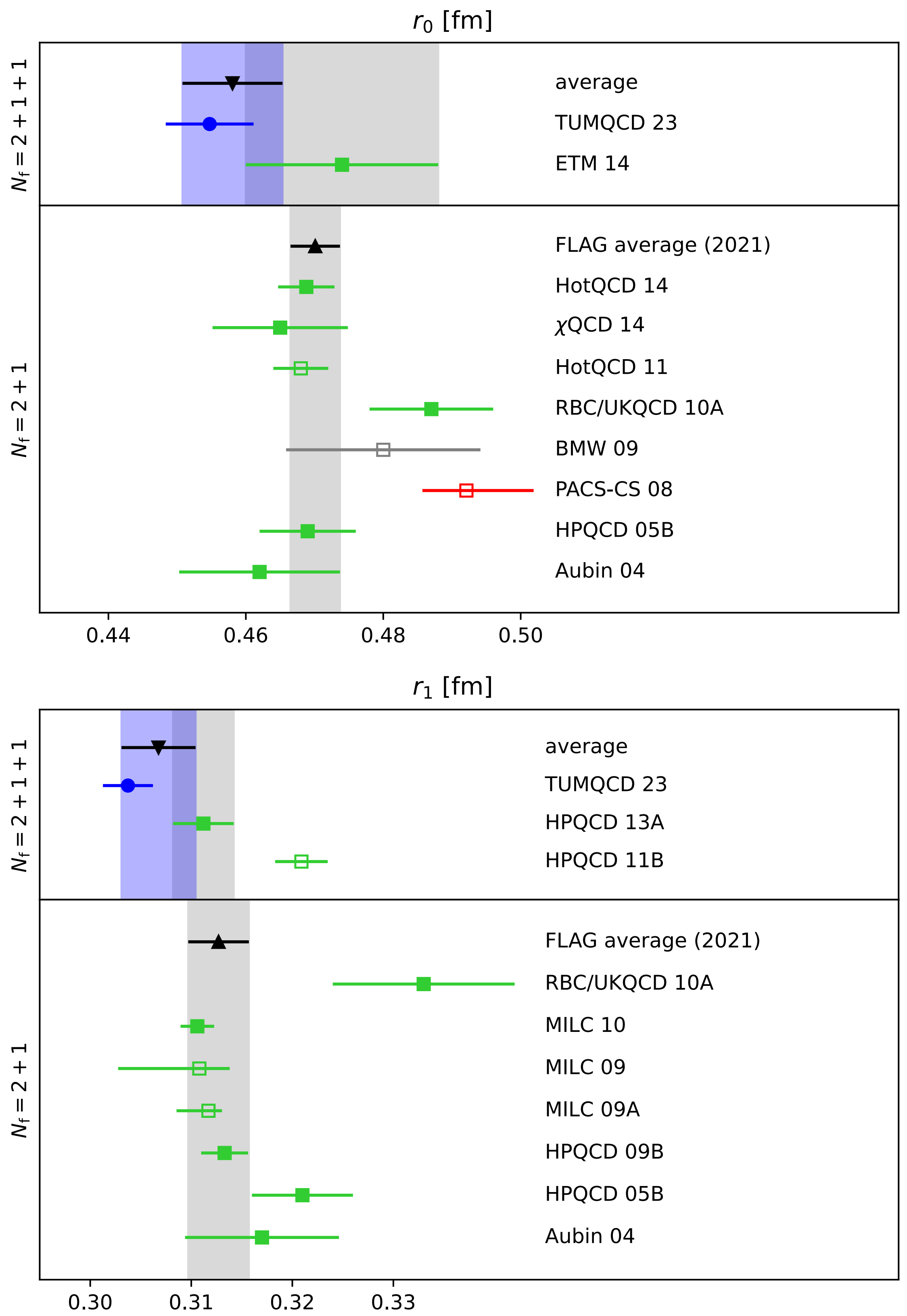
  2+1+1 flavor QCD, i.e. dynamical charm quark, HISQ action.

Improvements: scale setting

- Previous scale, 2+1 flavor:

  $r_1 = 0.3106(17)$ fm.

  Bazavov et al. (MILC), PoS LATTICE (2010)



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- Updates:

  2+1 flavor:

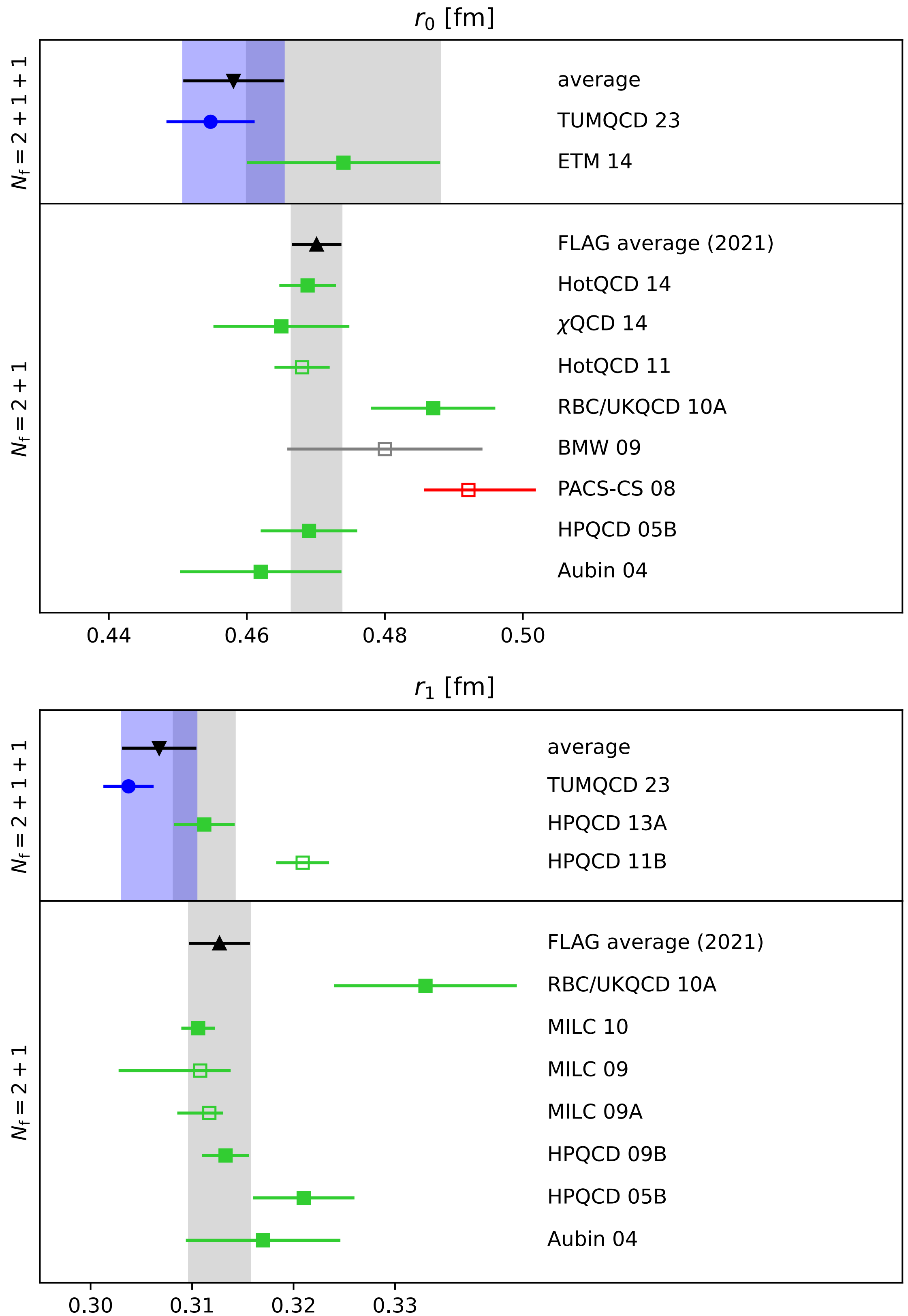
  $r_1 = 0.3072(22)$ fm.

  Larsen et al., 2502.08061

  2+1+1 flavor:

  $r_1 = 0.3037(25)$ fm

  Brambilla et al. (TUMQCD), Phys. Rev. D 107 (2023)



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$$E_0^{lat,tree}(\vec{r}) = -C_F g^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} D_{00}(k_0=0,\mathbf{k}) e^{i\mathbf{kr}} \equiv -C_F g^2 \frac{1}{4\pi r_I(\vec{r})},$$

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- One-loop improvement with HISQ has been recently computed by von Hippel, Leino, Steinbasser (in preparation).
- Calculation of one-loop diagrams is done with the HPsrc and HiPPy programs.

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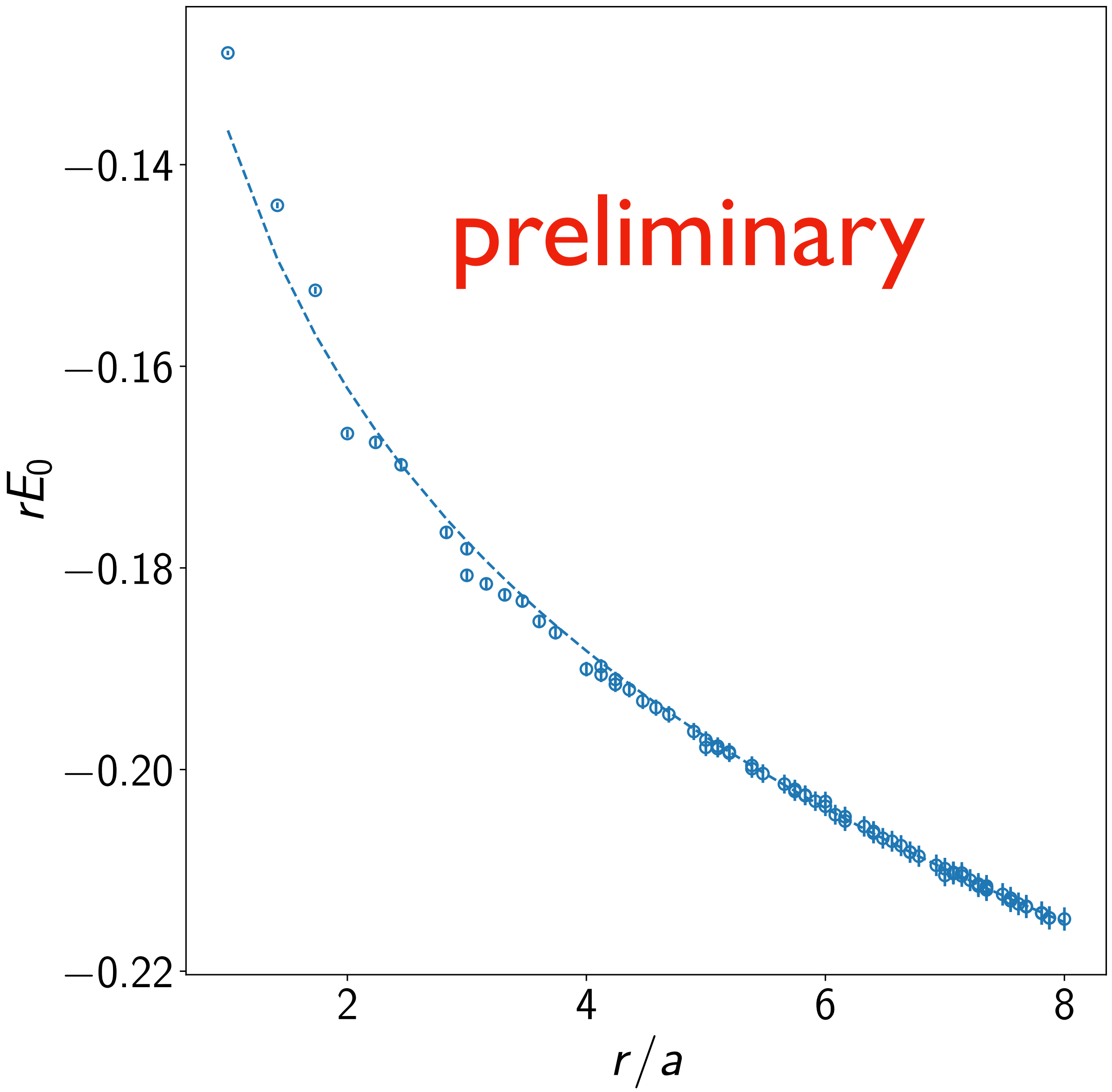

- Fit:

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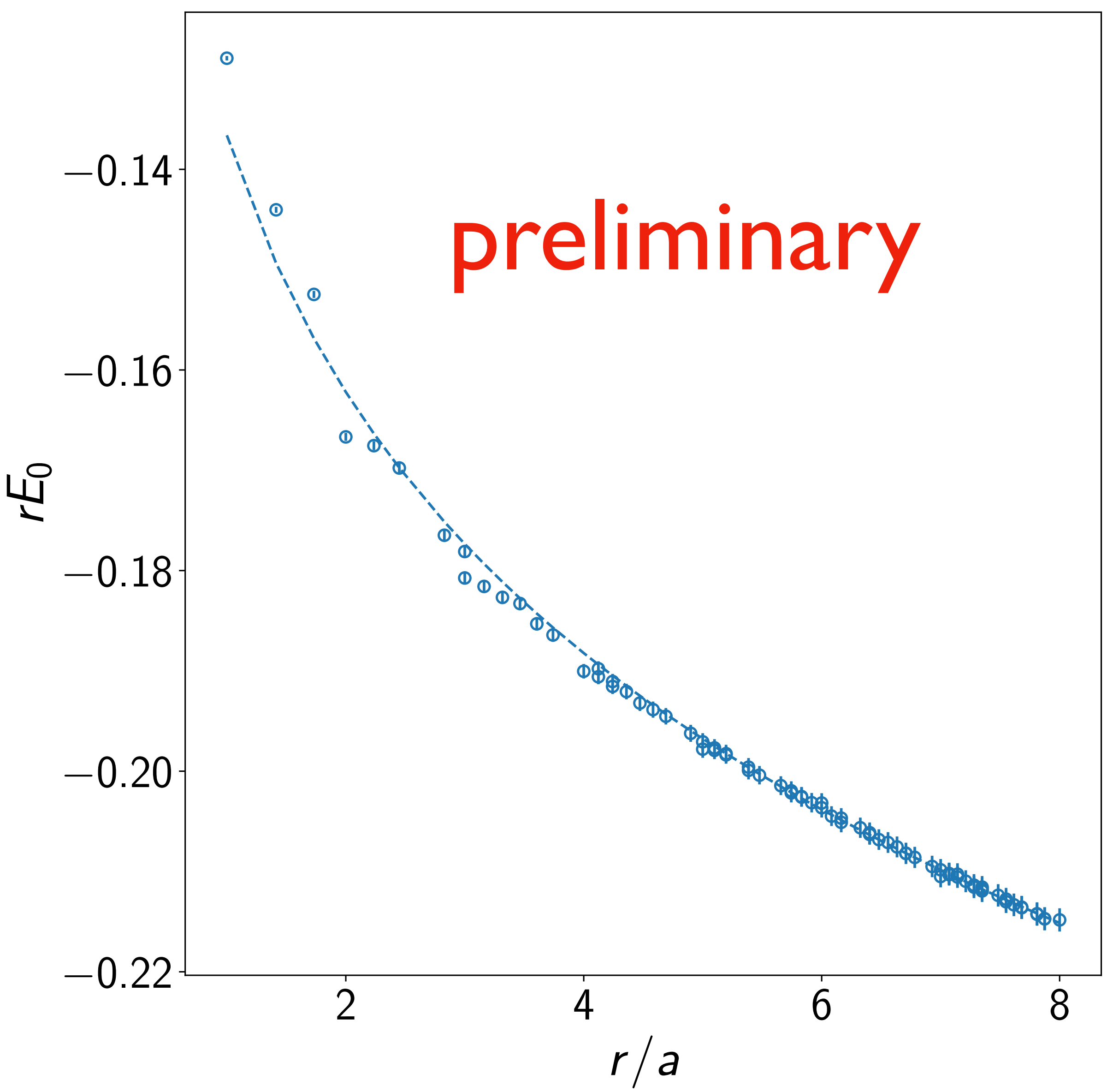

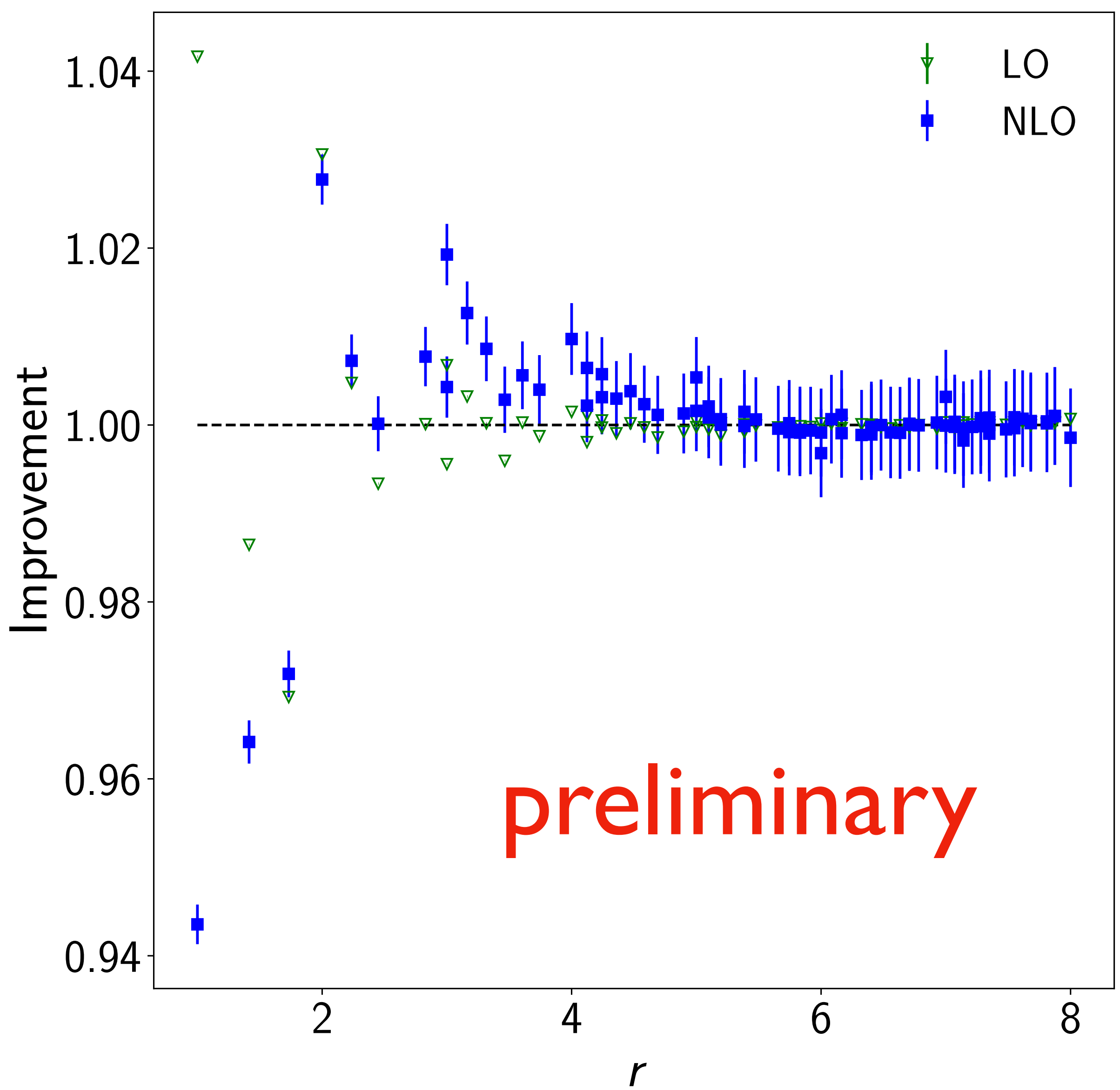

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- Known information is x_0, x_1 (new one-loop) and v_0, v_1, v_2, v_3 ($\overline{\rm MS}$).

Improvements: MRS scheme

- Static energy in Potential NRQCD:

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  - evaluate $\sum_{l=0}^{\infty} v_l \alpha_s^{l+1}$, i.e., truncate on f_l, not v_l.

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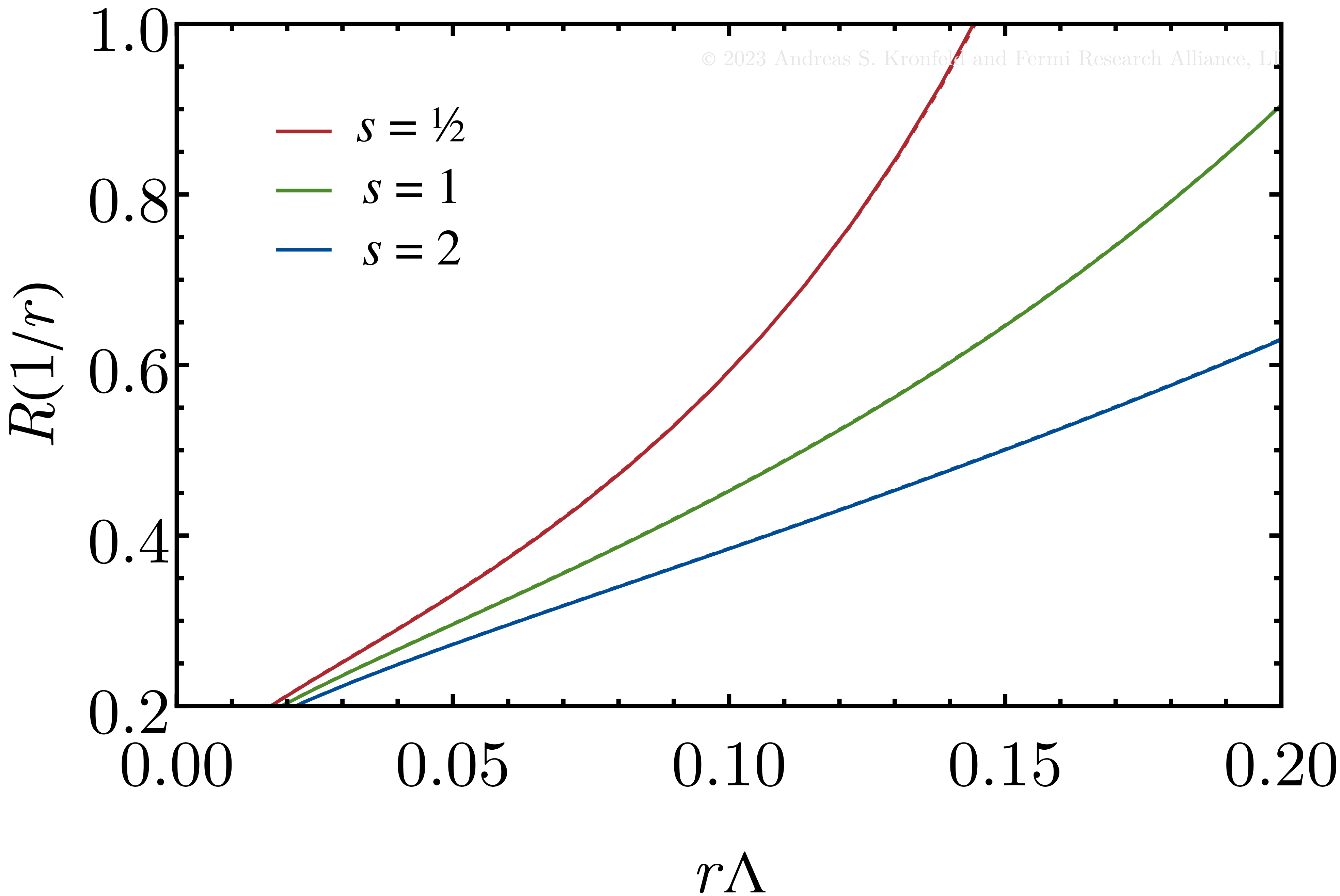

- Fixed-order perturbation theory: $R(1/r) = -\,rE_0(r)/C_F, \quad \mu = s/r.$

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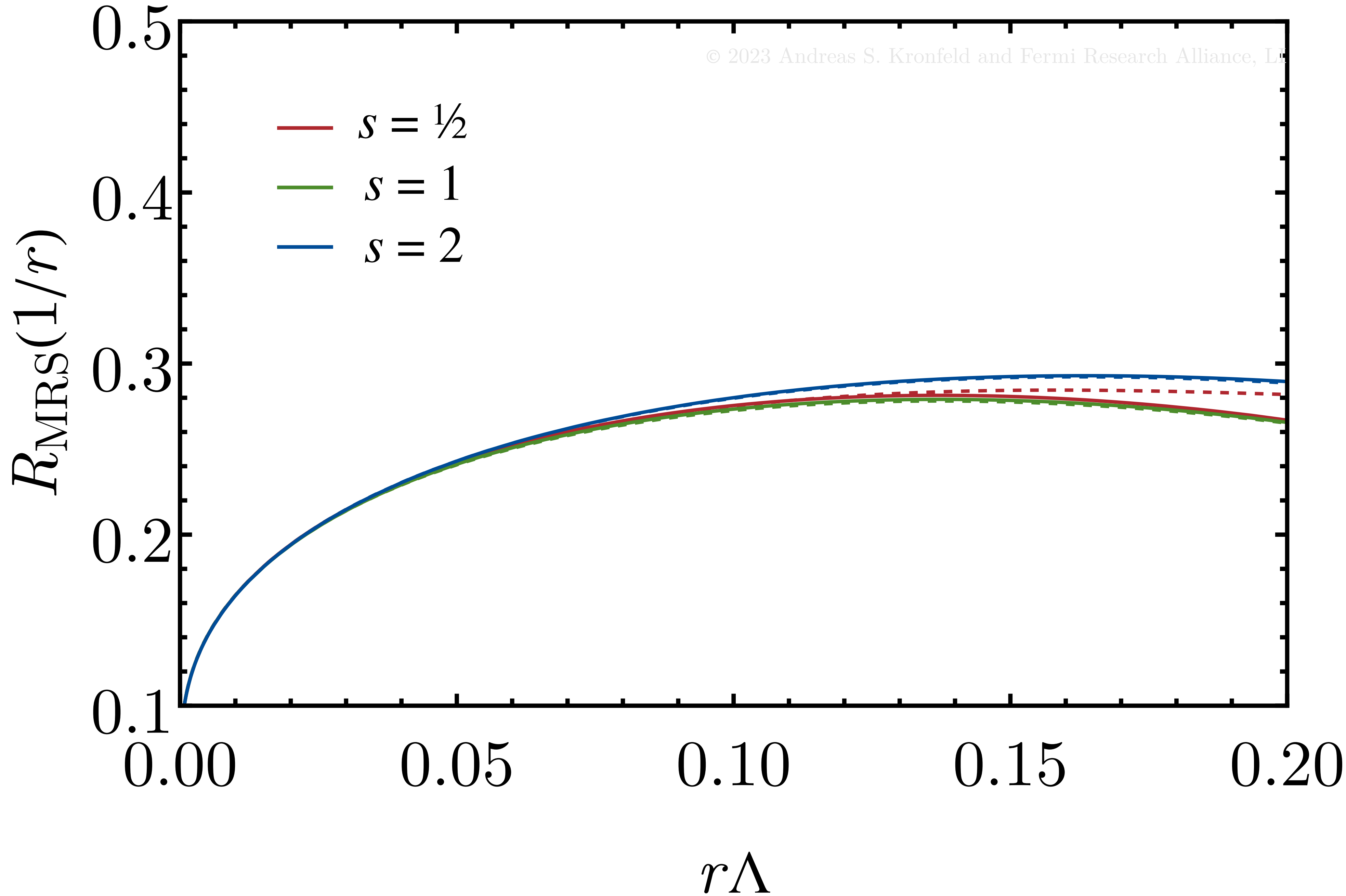
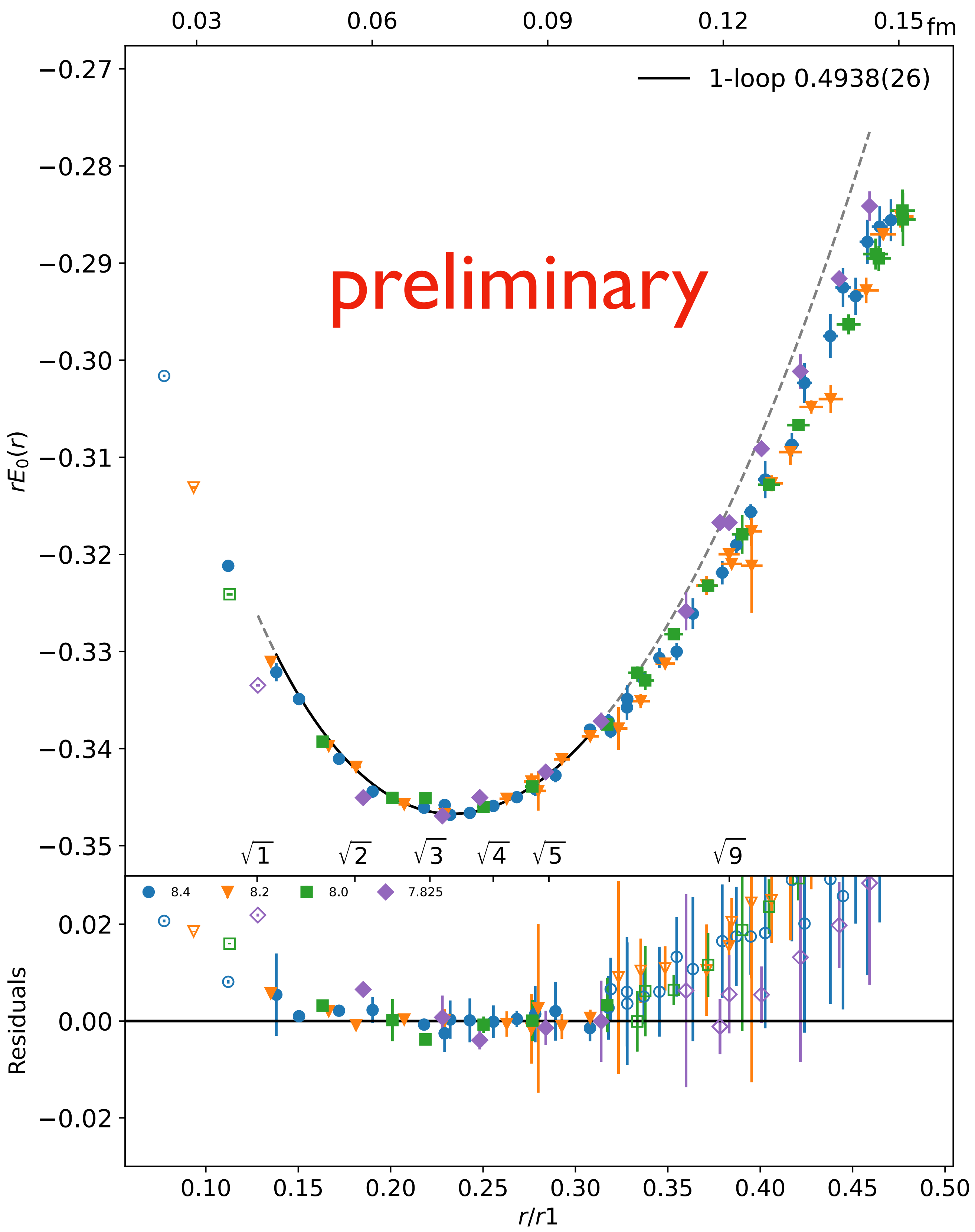


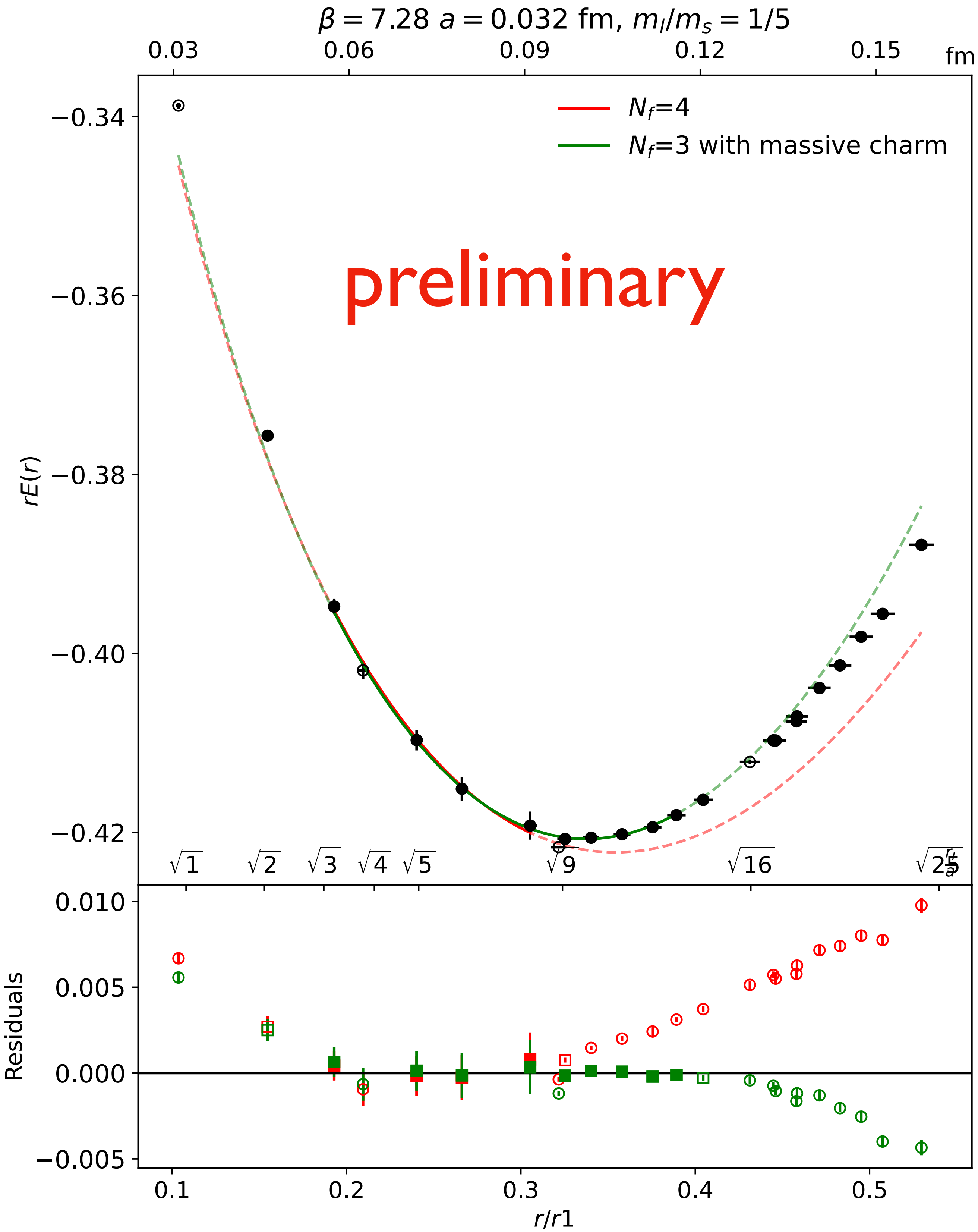
- Factorially resummed with MRS scheme.

Some results



- Revisiting previous work.
- Global fit to 2+1 flavor HISQ data, 4 lattice spacings.
- One-loop improvement.
- MRS scheme.

Some results



- Fit to the fines 2+1+1 flavor HISQ ensemble.
- One-loop improvement.
- MRS scheme.
- Including massive charm vs four massless flavors.

Conclusion

- Previous TUMQCD α_s extraction was done in 2+1 flavor QCD with the integrated force approach.
- The ongoing work focuses on several improvements:
  - 2+1+1 flavor QCD,
  - one-loop improvement of the static energy to better handle the discretization effects,
  - minimal renormalon subtraction (MRS) scheme for the static energy directly, rather than integrated force.
  - revisiting the scale setting for 2+1 and 2+1+1 flavors.