α_S determination from semi-inclusive regions in hadron and e^+e^- collisions

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Based on:

S. Camarda, G. F., M. Schott, EPJC 84 (2024), (2203.05394)
U. Aglietti, G. F. PRD 110 (2024) 11, (2403.04077)
U. Aglietti, G.F., W.-L. Ju, J. Miao PRL 134 (2025) 25, (2502.01570)

Workshop α_S-2025
Aussois – 17/12/2025

α_S from semi-inclusive, i.e. (real) radiation inhibited, regions.

Advantages:

- **higher sensitivity to α_S** w.r.t. *inclusive* observables;
- calculable at **higher theoretical accuracy** w.r.t. *exclusive* observables.

Challenges:

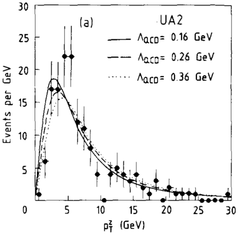
- sensitivity to **infrared (Sudakov) logs**;
- sensitivity **non perturbative QCD** effects.

Classical semi-inclusive obs. at colliders:

- **Drell–Yan production mechanism at small transverse-momentum (q_T).**
- **Shape variables in the back-to-back region.**

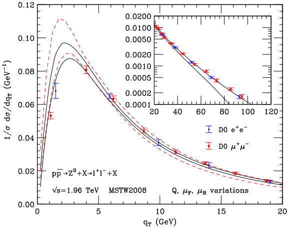
α_S from Z-boson q_T distribution

Sp$\bar{p}$S ($\sqrt{s} = 0.63$ TeV)



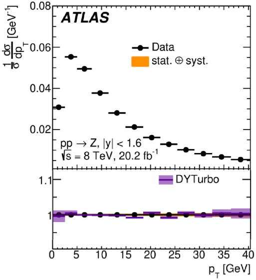
[UA2 Coll.('92)]
compared with
[Altarelli et al.('84)]

Tevatron ($\sqrt{s} = 1.96$ TeV)



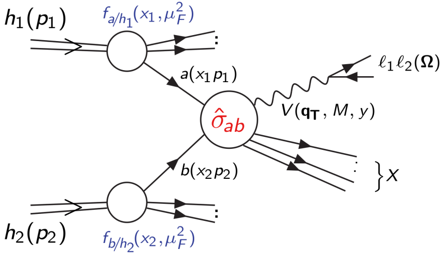
[D0 Coll.('08,'10)]
compared with
[Catani et al.('10)]

LHC ($\sqrt{s} = 7 - 8$ TeV)



[ATLAS Coll.('14)]
compared with
DYTurbo [Camarda et al.('10)]

Drell–Yan q_T distribution



$$h_1(p_1) + h_2(p_2) \rightarrow V + X \rightarrow \ell_1 + \ell_2 + X$$

QCD factorization formula:

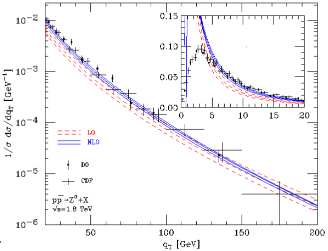
$$\frac{d\sigma}{dq_T^2} = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2\, f_{a/h_1}(x_1, \mu_F^2)\, f_{b/h_2}(x_2, \mu_F^2)\, \frac{d\hat{\sigma}_{ab}}{dq_T^2}(\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2).$$

Fixed-order perturbative expansion reliable only for $q_T \sim M$. When $q_T \ll M$:

$$\int_0^{q_T^2} d\bar{q}_T^2 \frac{d\hat{\sigma}_{q\bar{q}}}{d\bar{q}_T^2} \sim 1 + \alpha_S \left[c_{12} L_{q_T}^2 + c_{11} L_{q_T} + \cdots \right] + \alpha_S^2 \left[c_{24} L_{q_T}^4 + \cdots + c_{21} L_{q_T} + \cdots \right] + \mathcal{O}(\alpha_S^3)$$

with $\alpha_S^n L_{q_T}^m \equiv \alpha_S^n \log^m(M^2/q_T^2) \gtrsim 1$.

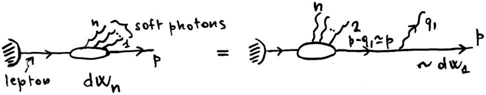
Resummation of logarithmic corrections mandatory.



Soft gluon exponentiation

Analytic resummation feasible if:
dynamics AND kinematics factorize ⇒ exponentiation.

- Dynamics factorization: general propriety of QCD for soft emissions [Gatheral('83)],[Frenkel,Taylor('84)], [Catani,Ciafaloni('84,'85)] analogous of eikonal approximation in QED [Yennie,Frautschi,Suura('61)]



$$dw_n(q_1,\ldots,q_n) \simeq \frac{1}{n!}\prod_{i=1}^{n} dw_1(q_i)$$

- Kinematics factorization: not valid in general. For q_T distribution it holds in the impact parameter space (Fourier transform)[Parisi,Petronzio('79)]

$$\int d^2\mathbf{q_T}\,\exp(-i\mathbf{b}\cdot\mathbf{q_T})\,\delta^{(2)}\Big(\mathbf{q_T}-\sum_{j=1}^{n}\mathbf{q_{T_j}}\Big) = \exp(-i\mathbf{b}\cdot\sum_{j=1}^{n}\mathbf{q_{T_j}}) = \prod_{j=1}^{n}\exp(-i\mathbf{b}\cdot\mathbf{q_{T_j}})\,.$$

- Exponentiation holds in the impact parameter space. Results have then to be transformed back to the physical space: $q_T \ll M \Leftrightarrow Mb \gg 1$, $\log M/q_T \gg 1 \Leftrightarrow \log Mb \gg 1$.

$\mathbf{q_T}$ resummation in QCD [Catani, de Florian, Grazzini('01)] [Bozzi, Catani, de Florian, Grazzini('03,'06)]

$$\frac{d\hat{\sigma}}{dq_T^2} = \frac{d\hat{\sigma}^{(res)}}{dq_T^2} + \frac{d\hat{\sigma}^{(fin)}}{dq_T^2};$$

In the impact parameter space: $q_T \ll M \Leftrightarrow Mb \gg 1, \quad \log M/q_T \gg 1 \Leftrightarrow \log Mb \gg 1$

$$\frac{d\hat{\sigma}^{(res)}}{dq_T^2} = \frac{M^2}{\hat{s}} \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q_T}} \mathcal{W}(b, M),$$

In the Mellin space (with respect to $z = M^2/\hat{s}$) we have:

$$\mathcal{W}_N(b,M) = \mathcal{H}_N(\alpha_S) \times \exp\left\{\mathcal{G}_N(\alpha_S, L)\right\}$$

with $\quad L \equiv \log(M^2 b^2) \quad$ and $\quad \alpha_S L \sim 1$

$$\mathcal{G}(\alpha_S, L) = L\, g^{(1)}(\alpha_S L) + g^{(2)}(\alpha_S L) + \frac{\alpha_S}{\pi} g^{(3)}(\alpha_S L) + \cdots \qquad \mathcal{H}(\alpha_S) = \hat{\sigma}^{(0)} \left(1 + \frac{\alpha_S}{\pi}\mathcal{H}^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 \mathcal{H}^{(2)} + \cdots\right)$$

LL ($\sim\alpha_S^n L^{n+1}$): $g^{(1)}$, ($\hat{\sigma}^{(0)}$); NLL ($\sim\alpha_S^n L^n$): $g^{(2)}$, $\mathcal{H}^{(1)}$; $\cdots$ N^{k}LL ($\sim\alpha_S^n L^{n+k-1}$): $g^{(k+1)}$, $\mathcal{H}^{(k)}$;

Resummed result at small q_T *matched* with corresponding fixed "finite" part at large q_T: *uniform accuracy* for $q_T \ll M$ and $q_T \sim M$.

- Resummed effects exponentiated in a universal of Sudakov form factor, process-dependence factorized in the hard-virtual factor $H_c^F(\alpha_S)$ via all-order formula [Catani,Cieri,de Florian,G.F.,Grazzini('14)].
- Resummation performed at partonic cross section level: (collinear) PDF evaluated at $\mu_F \sim M$, $f_N(b_0^2/b^2) = \exp\left\{ -\int_{b_0^2/b^2}^{\mu_F^2} \frac{dq^2}{q^2} \gamma_N(\alpha_S(q^2)) \right\} f_N(\mu_F^2)$: no PDF extrapolation in the non perturbative region, study of μ_R and μ_F dependence as in fixed-order calculations.
- No need for NP models: Landau singularity of α_S regularized using a *Minimal Prescription* without power-suppressed corrections [Laenen et al.('00)],[Catani et al.('96)].
- Introduction of resummation scale $Q \sim M$: variations give an estimate of the uncertainty from uncalculated logarithmic corrections.

$$\ln\left(M^2 b^2\right) \rightarrow \ln\left(Q^2 b^2\right) + \ln\left(M^2/Q^2\right)$$

- Perturbative unitarity constraint: recover *exactly* the total cross-section (upon integration on q_T)

$$\ln\left(Q^2 b^2\right) \rightarrow \widetilde{L} \equiv \ln\left(Q^2 b^2 + 1\right) \quad \Rightarrow \quad \exp\left\{\alpha_S^n \widetilde{L}^k\right\}\Big|_{b=0} = 1 \;\Rightarrow\; \int_0^\infty dq_T^2 \left(\frac{d\hat{\sigma}}{dq_T^2}\right) = \hat{\sigma}^{(tot)};$$

- General procedure to treat the q_T recoil [Catani,de Florian,G.F.,Grazzini('15)]:

$$\frac{d\hat{\sigma}^{(0)}}{d\mathbf{\Omega}} = \hat{\sigma}^{(0)}(M^2)\, F(\mathbf{q_T}; M^2, \mathbf{\Omega}) \;\; with \;\; F(\mathbf{q_T}; M^2, \mathbf{\Omega}) = F(\mathbf{0}; M^2, \mathbf{\Omega}) + \mathcal{O}(\mathbf{q_T^2}/M^2)$$

q_T resummation: perturbative accuracy

- Formalism implemented in numerically efficient and publicly available code:

  DYTurbo: computes resummed and fixed-order fiducial cross section and related distributions it retains full kinematics of the vector boson and of its leptonic decay products [Camarda,Boonekamp,Bozzi,Catani,Cieri,Cuth,G.F.,de Florian,Glazov, Grazzini,Vincter,Schott('20)]

  https://dyturbo.hepforge.org.

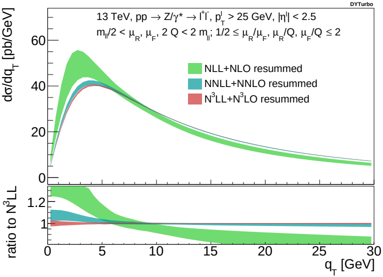
- We have explicitly included in DYTurbo up to:
  - N^{4}LL logarithmic contributions to all orders (i.e. up to $\exp(\sim\alpha_S^n L^{n-3})$);
  - Approximated N^{4}LO corrections (i.e. up to $\mathcal{O}(\alpha_S^4)$) at small q_T;
  - NLO corrections (i.e. up to $\mathcal{O}(\alpha_S^2)$) at large q_T;

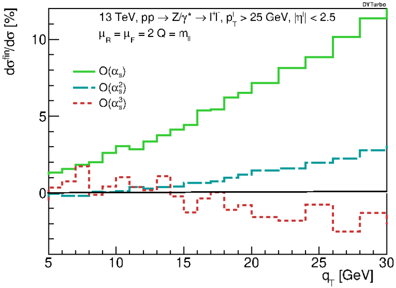
- Matching with NNLO corrections (i.e. up to $\mathcal{O}(\alpha_S^3)$) at large q_T from results in [Boughezal et al.('16)], [Gehrmann-De Ridder et al.('16)], [MCFM ('23)];

- Results up to N^{3}LO (i.e. up to $\mathcal{O}(\alpha_S^3)$) recovered for the total cross section (from unitarity).

Z/γ^* production at N^{3}LL+N^{3}LO

[Camarda,Cieri,G.F.('21)]

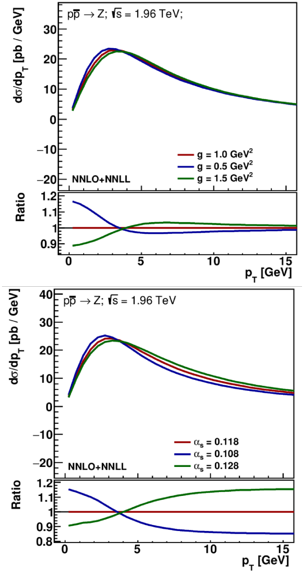




DYTurbo results. Left: Resummed NLL, NNLL and N^{3}LL results for Z/γ^* q_T spectrum.

Right: Impact of finite part at $\mathcal{O}(\alpha_S)$, $\mathcal{O}(\alpha_S^2)$ and $\mathcal{O}(\alpha_S^3)$

Non perturbative effects

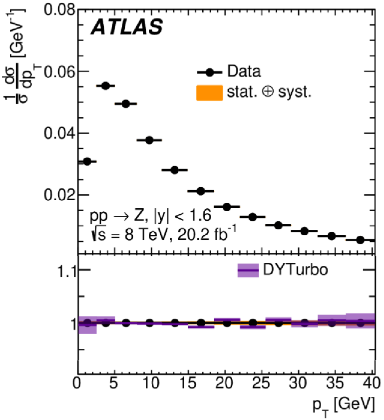


- Up to now discussed result in a complete perturbative framework (except for PDFs).
- Non perturbative *intrinsic* k_T effects parametrized by a NP form factor $S_{NP} = \exp\{-gb^2\}$ with $0 < g < 1.2\, GeV^2$:

$$\exp\{\mathcal{G}_N(\alpha_S, \widetilde{L})\} \quad \to \quad \exp\{\mathcal{G}_N(\alpha_S, \widetilde{L})\}\, S_{NP}$$

- NP effects increase the hardness of the q_T spectrum at small values of q_T. Non trivial interplay of perturbative and NP effects.
- However possible to disentangle the effects: scale of the NP effects is $\langle q_T \rangle \sim 1\, GeV$ ($g \sim 0.5 GeV^2$), scale of "soft gluon" recoil is $\langle q_T \rangle \sim 10\, GeV$.

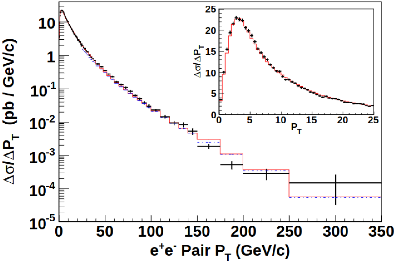
Z/γ^* production theory vs data



DYTurbo results at N^4LLa accuracy compared with data [ATLAS Coll.('23)].
Time performance of $\mathcal{O}$(*seconds*): (with exception of V+jet term with fiducial lepton cuts).

Z-boson q_T measurement at CDF

The CDF measurement of $Z/\gamma^* \to e^+e^-$ ($\sqrt{s} = 1.96\,TeV$ with $\int \mathcal{L} = 2.1 fb^{-1}$) [CDF Coll.('10)] is ideal for $\alpha_S(m_Z)$ determination.

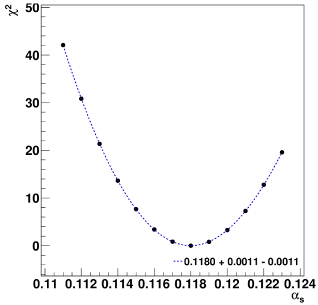


[CDF Coll.('10)]

- Measurement in full-lepton phase space with small extrapolation using angular coefficients method $\Rightarrow$ allows fast analytic predictions with DYTurbo.
- $p\bar{p}$ collisions: small contribution from heavy-flavour in initial state (0.4% $b\bar{b} \to Z$, 1.3% $c\bar{c} \to Z$). Quark mass effects negligible.
- Low pile-up and good electron resolution. Fine q_T bins ($0.5GeV$) with relatively small bin-to-bin correlations.

Determination of α_S from Z-boson q_T distribution

[Camarda,G.F.,Schott ('22)]]



- DYTurbo interfaced to xFitter. Fit region: $Z\ q_T < 30\,GeV$, predictions at $\mathrm{N}^3\mathrm{LL}+\mathcal{O}(\alpha_S^3)$ (i.e. $\mathrm{N}^3\mathrm{LL}+\mathrm{N}^3\mathrm{LO}$ at low q_T) with NNPDF4.0 PDF at NNLO.
- Defined χ^2 with experimental (β_{exp}) and PDFs (β_{th}) uncertainties (equivalent to including the new dataset in the PDF using profiling/reweighting).
- The non-perturbative form factor is $S_{NP} = \exp\{-gb^2\}$ with g left free in the fit.

$$\chi^2(\beta_{\rm exp}, \beta_{\rm th}) = \sum_{i=1}^{N_{\rm data}} \frac{\left(\sigma_i^{\rm exp} + \sum_j \Gamma_{ij}^{\rm exp}\beta_{j,\rm exp} - \sigma_i^{\rm th} - \sum_k \Gamma_{ik}^{\rm th}\beta_{k,\rm th}\right)^2}{\Delta_i^2} + \sum_j \beta_{j,\rm exp}^2 + \sum_k \beta_{k,\rm th}^2 \,.$$

Theory uncertainties

	PDF fit	Hessian profiling
$\alpha_S(m_Z)$	0.1188 ± 0.0008	0.1184 ± 0.0006
g [GeV^2]	0.69 ± 0.05	0.71 ± 0.05
Dataset	χ^2/points	χ^2/points
NC DIS H1-ZEUS e^+p	955/905	
CC DIS H1-ZEUS e^+p	46/39	
NC DIS H1-ZEUS e^-p	219/159	
CC DIS H1-ZEUS e^-p	53/42	
H1-ZEUS correlated χ^2	91	
CDF Z p_T	41/55	40/55
Total	1405 / 1184	

- Bias from α_S-PDFs correlations [Forte,Kassabov('20)] $\rightarrow$ PDFs refitted.
- Other PDF sets considered. CT18, CT18Z, MSHT20, HERAPDF2.0, ABMP16 (NNLO) and MSHT20an3lo. The midpoint value is the nominal result and the PDF envelope as an additional uncertainty.
- Uncertainty from missing higher orders: $m_{ll}/2 < \{\mu_R, \mu_F, Q\} < 2m_{ll}$ with $0.5 < \{\mu_R/\mu_F, \mu_R/Q, \mu_F/Q\} < 2$.
- NP effects: b_*-pr. $b_* = b/\sqrt{1 + b^2/b_{lim}^2}$ ($b_{lim} = 2 - 3$ GeV^{-1}) and *minimal* pr. ($b_{lim} \rightarrow \infty$); quartic term $\exp(-qb^4)$ and different parametrization of S_{NP} [Collins,Rogers('15)].
- Uncertainty from finite component at $\mathcal{O}(\alpha_S^3)$.
- Check with D0 data and fit boundaries.

Theory uncertainties

	$\alpha_S(m_Z)$	g [GeV^2]	χ^2/dof
NNPDF4.0	0.1192 ± 0.0008	0.66 ± 0.05	41/53
CT18	0.1189 ± 0.0010	0.67 ± 0.05	40/53
CT18Z	0.1198 ± 0.0009	0.62 ± 0.05	41/53
MSHT20	0.1185 ± 0.0009	0.72 ± 0.05	40/53
HERAPDF2.0	0.1188 ± 0.0008	0.69 ± 0.05	40/53
ABMP16	0.1185 ± 0.0007	0.62 ± 0.05	42/53
MSHT20an3lo (N^{4}LL)	0.1184 ± 0.0009	0.73 ± 0.05	40/53
PDF fit	0.1184 ± 0.0006	0.71 ± 0.05	1405/1184

- Bias from α_S-PDFs correlations [Forte,Kassabov('20)] $\to$ PDFs refitted.
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$\mu_R/m_{\ell\ell}$	$\mu_F/m_{\ell\ell}$	$Q/m_{\ell\ell}$	$\alpha_S(m_Z)$	g [GeV^2]	χ^2/dof
1	1	1	0.1192 ± 0.0008	0.66 ± 0.05	41/53
1	1	2	0.1183 ± 0.0007	0.77 ± 0.05	40/53
1	1	0.5	0.1196 ± 0.0008	0.57 ± 0.05	42/53
1	2	1	0.1194 ± 0.0008	0.66 ± 0.05	41/53
1	2	2	0.1183 ± 0.0007	0.77 ± 0.05	41/53
1	0.5	1	0.1193 ± 0.0008	0.68 ± 0.05	42/53
1	0.5	0.5	0.1196 ± 0.0008	0.59 ± 0.05	42/53
2	1	1	0.1193 ± 0.0008	0.67 ± 0.05	42/53
2	1	2	0.1194 ± 0.0008	0.70 ± 0.05	41/53
2	2	1	0.1192 ± 0.0008	0.65 ± 0.05	42/53
2	2	2	0.1192 ± 0.0008	0.67 ± 0.05	41/53
0.5	1	1	0.1184 ± 0.0007	0.75 ± 0.05	42/53
0.5	1	0.5	0.1192 ± 0.0007	0.64 ± 0.05	41/53
0.5	0.5	1	0.1183 ± 0.0007	0.75 ± 0.05	42/53
0.5	0.5	0.5	0.1192 ± 0.0007	0.64 ± 0.05	42/53

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- Uncertainty from finite component at $\mathcal{O}(\alpha_S^3)$.
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Theory uncertainties

	$\alpha_S(m_Z)$	g [GeV^2]
$b_{\rm lim} = 2$ GeV^{-1}	0.1187 ± 0.0007	0.83 ± 0.05
$b_{\rm lim} \to \infty$	0.1199 ± 0.0008	0.42 ± 0.05
g_k	0.1186 ± 0.0008	0.65 ± 0.05
$q = 0.1$ GeV^4	0.1197 ± 0.0008	0.51 ± 0.05
VFN PDF evolution	0.1190 ± 0.0007	0.71 ± 0.05

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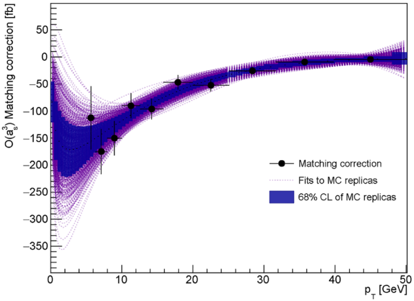

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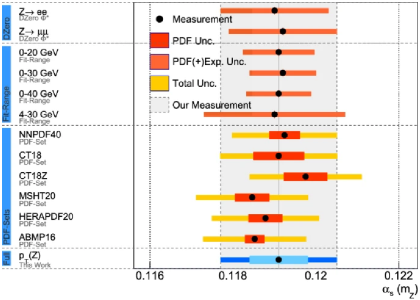

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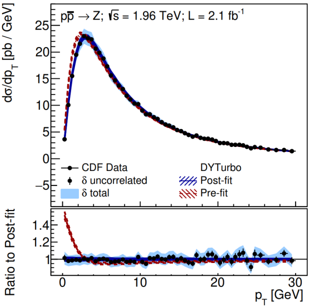
Fit results



Statistical uncertainty		± 0.7	
Experimental systematic uncertainty		± 0.1	
PDF uncertainty (NNPDF4.0)		± 0.4	
PDF uncertainty (envelope of PDFs)		± 0.7	
Scale variations uncertainties	$+0.4$		-0.9
Matching at $\mathcal{O}(\alpha_S^3)$		± 0.1	
Non-perturbative model		± 0.7	
Flavour model	0		-0.3
QED ISR		$< \pm 0.1$	
Lower limit of fit range		± 0.2	
Total	$+1.3$		-1.6

Simultaneous fit of $\alpha_S(m_Z)$ and g at N^{3}LL+$\mathcal{O}(\alpha_S^3)$ (N^{3}LL+N^{3}LO):

$$\boldsymbol{\alpha_S(m_Z) = 0.1191^{+0.0013}_{-0.0016}}$$

$$g = 0.66 \pm 0.05 \; GeV^2$$

Energy-Energy Correlation (EEC) function

$$e^+ + e^- \;\to\; h_i + h_j + X$$

$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j=1}^{n} \int \frac{E_i}{Q}\frac{E_j}{Q}\,\delta(\cos\chi - \cos\theta_{ij})\, d\sigma_{e^+e^-\to h_i h_j + X}$$

where $Q=\sqrt{s}$ and θ_{ij} is the angle between momenta $\vec{p}_i$ and $\vec{p}_j$ [Basham,Brown, Ellis,Love('78)].

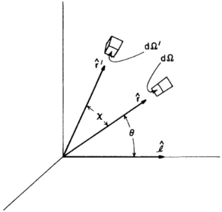


FIG. 2. Geometry for the experiment.

- EEC is IRC finite. While $d\sigma$ depends on parton fragmentation functions $D_{h,q}$, EEC does not: $\sum_h \int_0^1 dx\, x\, D_{h,q}(x,\mu_F^2) = 1$. EEC calculable in pure pQCD.
- Normalization gives

$$\int_{-1}^{+1} \frac{d\Sigma}{d\cos\chi}\, d\cos\chi = \int \left(\sum_{i=1}^{n}\frac{E_i}{Q}\right)^2 d\sigma = \sigma_{tot}.$$

- In the CoM frame at $\mathcal{O}(\alpha_S^0)$ we have a back-to-back $q\bar{q}$ pair:

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma}{d\cos\chi} = \frac{1}{2}\delta(1-\cos\chi) + \frac{1}{2}\delta(1+\cos\chi) + \mathcal{O}(\alpha_S).$$

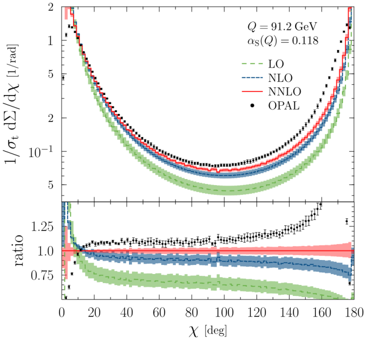
EEC in fixed-order pQCD

At higher orders in QCD we have (we use $z = (1 - \cos\chi)/2 = \sin^2(\chi/2)$):

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma}{dz} = \frac{1}{2}\left(\delta(1-z) + \delta(z)\right) + \frac{\alpha_S}{\pi}\mathcal{A}(z) + \left(\frac{\alpha_S}{\pi}\right)^2\mathcal{B}(z) + \left(\frac{\alpha_S}{\pi}\right)^3\mathcal{C}(z) + \mathcal{O}(\alpha_S^4)\,,$$

.

- The $\mathcal{O}(\alpha_S)$ function $\mathcal{A}(z)$ is known analytically from [Basham et al.('78)].
- At $\mathcal{O}(\alpha_S^2)$ function $\mathcal{B}(z)$ known analytically by [Dixon et al.('18)] (numerically by [Richards et al.('82,83)],[Kunszt, Nason('89)]).
- The $\mathcal{O}(\alpha_S^3)$ function $\mathcal{C}(z)$ known numerically by [Del Duca et al.('16)] from (fully differential) NNLO calculation of 3-jets cross-section in e^+e^- ann. using ColoRFulNNLO subtraction method [Somogyi et al.('05)].



EEC in the back-to-back limit

In the back-to-back limit $z \to 1$ ($\chi \to \pi$) we have

$$\mathcal{A}(z) = C_F \left\{ -\frac{1}{2}\left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{3}{4}\left[\frac{1}{1-z}\right]_+ - \left(\frac{\pi^2}{12} - \frac{11}{8}\right)\delta(1-z) + \cdots . \right\}$$

- In general at any order α_S^n large infrared (Sudakov) logarithms appears

$$\alpha_S^n \left[\frac{\ln^k(1-z)}{1-z}\right]_+ , \quad 0 \le k \le 2n-1$$

- Large logs spoils the convergence of fixed-order perturbative expansion. Reliable QCD predictions requires all order Sudakov resummation.
- In the back-to-back region the q_T between 2 hadrons is

$$q_T^2 \simeq Q^2 \cos^2(\chi/2) = Q^2(1-z) \to 0$$

  and EEC is closely related to Drell–Yan process at small-q_T.
- EEC function also contains large (single) logarithmic corrections of hard-collinear nature in the forward region $z \to 0$ (or $\chi \to 0$), $\ln^{n-1}(z)/z$, where hadrons have small angular separations [Dixon et al.('19)].

Sudakov resummation in pQCD

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma}{dz} = \frac{1}{\sigma_{tot}}\frac{d\Sigma_{res.}}{dz} + \frac{1}{\sigma_{tot}}\frac{d\Sigma_{fin.}}{dz}\ ;$$

In the impact parameter (b) space: $1-z \ll 1 \Leftrightarrow Qb \gg 1$, $\ln(1-z) \gg 1 \Leftrightarrow \ln(Qb) \gg 1$
[Parisi,Petronzio('79),Kodaira,Trentadue('81),Collins,Soper('83)]:

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma_{res.}}{dz} = \frac{1}{4}\,H(\alpha_S)\int_0^\infty db\, Q^2\, b\, J_0(\sqrt{1-z}\,Qb)\, S(Q,b),$$

$$S(Q,b) \;=\; L\, g^{(1)}(\alpha_S L) + g^{(2)}(\alpha_S L) + \frac{\alpha_S}{\pi} g^{(3)}(\alpha_S L) + \left(\frac{\alpha_S}{\pi}\right)^2 g^{(4)}(\alpha_S L) + \cdots$$

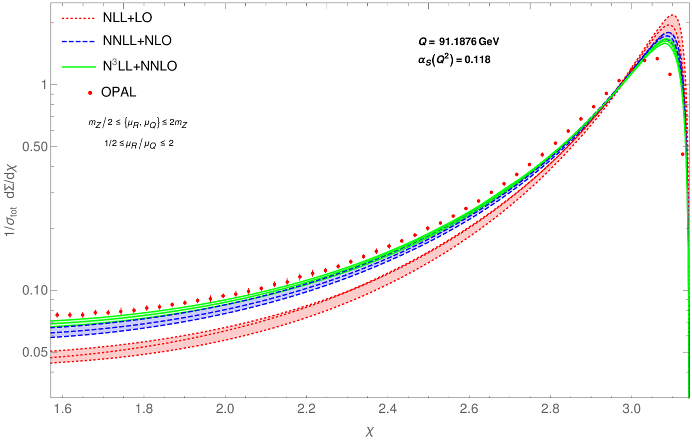
with $\quad L = \ln(Q^2 b^2), \quad \alpha_S L \sim 1.$

EEC resummation: perturbative ingredients

[Aglietti,G.F.('24)]

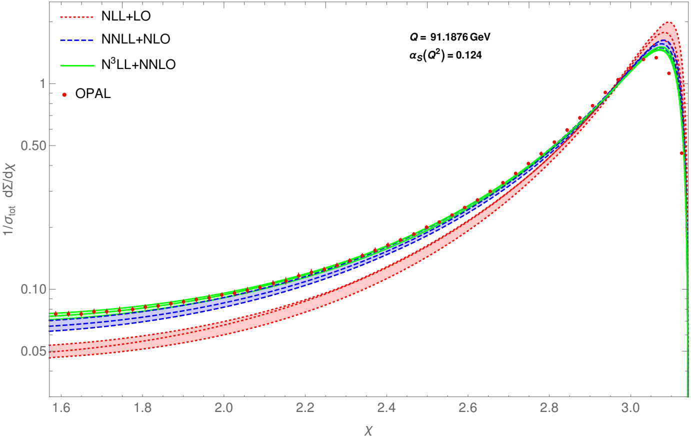
- NNLL coefficients already known [Basham at al.('78)],Kodaira,Trentadue('82),de Florian,Grazzini('04),Becher,Neubert('11). We have determined the N^{3}LL and NNLO coefficients in full QCD from results in SCET [Ebert,Mistlberger,Vita('20)].
- We thus performed all-order resummation up to N^{3}LL logarithmic accuracy all orders (i.e. up to $\exp(\sim \alpha_S^n L^{n-2})$) including hard-virtual contribution up to factor N^{3}LO.
- Matching with NNLO corrections (i.e. up to $\mathcal{O}(\alpha_S^3)$) from results in [Del Duca et al.('16)];
- Results up to N^{3}LO (i.e. up to $\mathcal{O}(\alpha_S^3)$) recovered for the total cross section (from unitarity).
- Full three-loop ($\mathcal{O}(\alpha_S^3)$) result also includes three-loop solution of the QCD coupling (β_0–β_3).

Numerical results: perturbative results



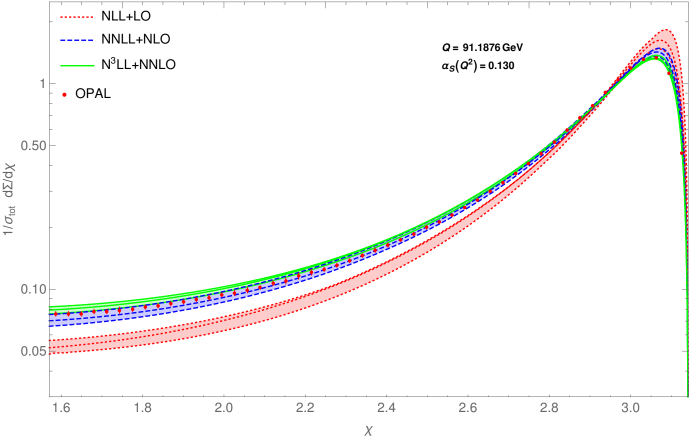
The resummed EEC spectrum at $\sqrt{s} = 91.1876$ GeV at various perturbative orders in QCD with $\alpha_S(m_Z^2) = 0.118$, compared with LEP data from [OPAL Coll.('92)]

Numerical results: perturbative results



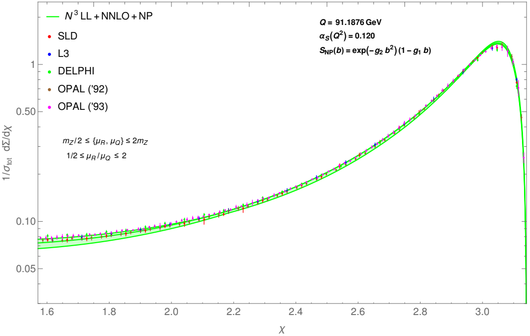
The resummed EEC spectrum at $\sqrt{s} = 91.1876$ GeV at various perturbative orders in QCD with $\alpha_S(m_Z^2) = 0.124$, compared with LEP data from [OPAL Coll.('92)]

Numerical results: perturbative results



The resummed EEC spectrum at $\sqrt{s} = 91.1876$ GeV at various perturbative orders in QCD with $\alpha_S(m_Z^2) = 0.130$, compared with LEP data from [OPAL Coll.('92)]

Numerical results: non perturbative effects



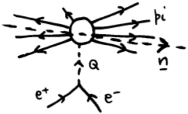
Comparison with data of the resummed EEC spectrum at N^3LL+NLO with non perturbative k_T *dependent* effects parameterized by a NP form factor $S_{NP} = \exp\{-g_2 b^2\}(1 - g_1 b)$ [Dokshitzer,Marchesini,Webber('99)].
Simultaneous fit of $\alpha_S(m_Z)$, g_1 and g_2 at N^{3}LL+$\mathcal{O}(\alpha_S^3)$ (N^{3}LL+N^{3}LO):

$$\alpha_S(m_Z) = 0.120 \pm 0.002$$

$$g_2 = 1.8 \pm 1.4\ GeV^2; \qquad g_1 = 0.3 \pm 0.1\ GeV;$$

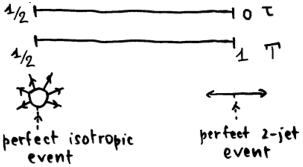
The Thrust in e^+e^- annihilation

$$T \equiv 1 - \tau = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p_i} \cdot \mathbf{n}|}{\sum_i |\mathbf{p_i}|}$$



- The sum is over all final state particles i with three-momentum $\mathbf{p_i}$.
- The maximum is taken with respect to the direction of the unit three-vector $\mathbf{n}$.
- T maximizes the longitudinal momentum along the vector $\mathbf{n}$.
- The vector which realizes the maximum is called thrust axis: $\mathbf{n}_T$.

The allowed kinematical range for T is: $1/2 \leq T \leq 1$ $(0 \leq \tau \leq 1/2)$



Upper limit for τ, $\tau_{max}^{(N)}$, depends on the number N of final-state particle. Only in the (formal) limit $N \to \infty$ it approaches $\tau_{max}^{(N\to\infty)} \to 1/2$.

Fixed-order QCD expansion

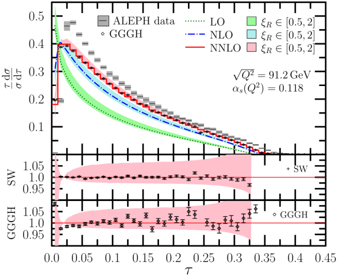
Thrust distribution can be calculated in at fixed-order in $\alpha_S = \alpha_S(\mu^2)$

$$\frac{1}{\sigma_{\rm tot}}\frac{d\sigma}{d\tau} = \delta(\tau) + \frac{\alpha_S}{\pi}\frac{d\mathcal{A}}{d\tau} + \left(\frac{\alpha_S}{\pi}\right)^2 \frac{d\mathcal{B}}{d\tau} + \left(\frac{\alpha_S}{\pi}\right)^3 \frac{d\mathcal{C}}{d\tau} + \mathcal{O}(\alpha_S^4)\,,$$

The LO function ($\tau > 0$) is

$$\frac{d\mathcal{A}}{d\tau} = 4 + 6\tau - \frac{2}{\tau} + \left(-4 + \frac{8}{3(1-\tau)\tau}\right)\ln\left(\frac{1-2\tau}{\tau}\right) \xrightarrow{\tau\to 0} -\frac{8}{3}\frac{\ln\tau}{\tau} - \frac{2}{\tau}\,.$$

QCD corrections up to NNLO known [Gehrmann-De Ridder et al. ('07)], [Weinzierl ('09)],[Del Duca et al. ('16)]. Calculations based on a numerical integration of the matrix elements. NNLO parton-level event generator public available EERAD3 [Gehrmann-De Ridder et al. ('14)].



Sudakov resummation

- Bulk of events in the two-jet limit $\tau \to 0$ (semi-inclusive region).
- In the fixed-order expansion large Sudakov logarithms appear due to incomplete cancellation between real radiation (constrained by kinematics) and virtual (unconstrained) emissions:

$$\frac{1}{\sigma_{tot}} \frac{d\sigma}{d\tau} \overset{\tau \to 0}{\sim} \sum_{n=1}^{\infty} \sum_{k=1}^{2n-1} \alpha_S^n \frac{1}{\tau} \ln^k \frac{1}{\tau} \,.$$

- *Cumulative* cross section (probability of having a *thrust* $< \tau$):

$$R_T(\tau) \equiv \frac{1}{\sigma_{\rm tot}} \int_0^{\tau} d\tau' \frac{d\sigma}{d\tau'} \qquad \text{thus} \qquad \frac{1}{\sigma_{\rm tot}} \frac{d\sigma}{d\tau} = \frac{dR_T(\tau)}{d\tau} \,.$$

$$R_T(\tau) \overset{\tau \to 0}{\sim} \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \alpha_S^n \ln^k \frac{1}{\tau} \,.$$

- Fixed-order expansion unreliable in the low τ region where $\alpha_S(Q) \ln^2 1/\tau \gtrsim 1$ ($\tau \lesssim 0.05$ for $Q = m_Z$).
- To obtain reliable predictions in the two-jet region, resummation of Sudakov logarithms is mandatory.

Sudakov resummation for Thrust

We follow the CTTW formalism [Catani,Trentadue,Turnock,Webber('91,'93)] Cumulative cross section can be written as:

$$R_T(\tau) = C(\alpha_S(Q^2))\,\Sigma(\tau, \alpha_S(Q^2)) + D(\tau, \alpha_S(Q^2))\,;$$

$C(\alpha_S)$ is a hard-virtual factor and $D(\alpha_S)$ is a *remainder* function vanishing at small τ:

$$C(\alpha_S) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n C_n\,, \qquad D(\tau, \alpha_S) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n D_n(\tau)\,.$$

$\Sigma\,(\tau, \alpha_s)$ is a long-distance form factor (resums the large Sudakov logarithms).

Thrust kinematics factorize in Laplace space

$$\Theta(\tau - \sum_{j=1}^{n} \frac{k_i^2}{Q^2}) = \frac{1}{2\pi i}\int_C \frac{dN}{N} e^{N\tau} \prod_{j=1}^{n} e^{-Nk^2/Q^2}$$

Exponentiation holds in Laplace space (results then transformed into physical space):

$\tau \ll 1 \quad \Leftrightarrow \quad N \gg 1\,, \qquad \ln 1/\tau \gg 1 \quad \Leftrightarrow \quad \ln N \gg 1$. In the Laplace space:

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i}\int_C \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} = \frac{1}{2\pi i}\int_C \frac{dN}{N} e^{N\tau} e^{L\, f_1(\lambda) + f_2(\lambda) + \sum_{n=3}^{\infty}\left(\frac{\alpha_S}{\pi}\right)^{n-2} f_n(\lambda)}$$

where the contour C runs parallel to the imaginary axis and lies to the right of all singularities.

Inversion of Laplace transform

Formal inversion from N space to τ space is:

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} \,,$$

- This formula involves (formally non-integrable) Landau singularity of α_S
- Exact analytic Laplace inversion cannot be computed.

CTTW['93] : Taylor expand $\mathcal{F}(\alpha_S, L)$ around the point (not the saddle point)

$$\ln N = \ln(1/\tau) \equiv \ell \,,$$

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_C \frac{dN}{N} e^{N\tau} \exp\left[\sum_{k=0}^{\infty} \frac{\partial^k \mathcal{F}(\alpha_S, \ell)}{\partial \ell^k} \frac{\ln^k(\tau N)}{k!}\right] \,,$$

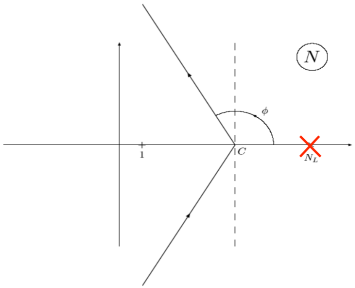
not possible to evaluate the series exactly: a new hierarchy is defined in $\tau-$space.
N^{n}LL in τ space defined by keeping the terms $\alpha_S^{n-1}(\alpha_S \ell)^k$, (for all k).
Crucial point: the correspondence $\ln N$ to $\ln(1/\tau)$ is not exact. Kinematics factorization and exponentiation are valid only in N space.

Exact numerical inversion (Minimal Prescription)

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_{C_{MP}} \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} ,$$

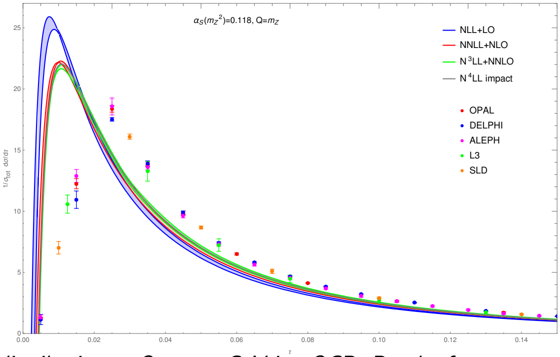
where the contour C runs parallel to the imaginary axis and lies to the right of all singularities of the integrand.

Exact numerical inversion can be performed with a prescription to avoid the Landau Pole. Minimal Prescription [Catani,Mangano,Nason,Trentadue('96)]: the contour of integration C_{MP} lies to the right of all physical singularities but to the left of the (unphysical) Landau pole. The results obtained by using this prescription converge asymptotically to the perturbative series and do not include any power correction.



Numerical results: perturbative effects

[Aglietti,G.F.,W.-L. Ju,J.Miao('25)]

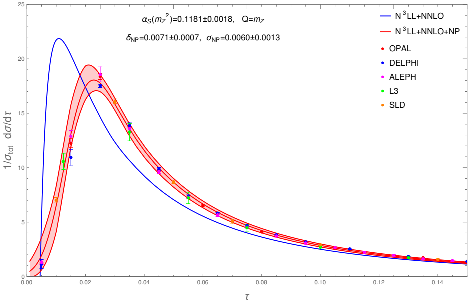


Thrust distribution at $Q = m_Z$ GeV in pQCD. Results from resummation in Laplace-conjugated space, including renormalization scale variations $Q/2 \leq \mu_R \leq 2Q$.

Numerical results: non perturbative effects

NP effects included using an analytic model based on a correlation [Catani at al.('91)] or shape function [Korchemsky,Sterman('99)] $f_{NP}(\tau_h,\tau)$ depending on 2 parameters. Simultaneous fit of $\alpha_S(m_Z)$ and NP parameters at N^{3}LL+N^{3}LO.

$$\frac{d\sigma_h}{d\tau_h} = \int d\tau \frac{d\sigma}{d\tau}\, f_{NP}(\tau,\tau_h)\,, \qquad f_{NP}(\tau_h,\tau) = \frac{1}{\sqrt{2\pi}\sigma_{NP}} \exp\left[-\frac{(\tau_h-\tau-\delta_{NP})^2}{2\,\sigma_{NP}^2}\right],$$



The thrust distribution at $Q = 91.1876$ GeV at N^{3}LL+NNLO in QCD without (blue solid line) and with (red band) the inclusion of NP effects and fit uncertainties.

Conclusions

- Semi-inclusive processes important to test pQCD predictions, extract information on NP QCD and determine the value of α_S.
- Precise theoretical predictions based on based on Sudakov resummation perturbative QCD at N^3LL+N^3LO (and beyond).
- Resummation performed in conjugated space. Inversion into momentum space performed exactly in numeric way.
- Determination of $\alpha_S(m_Z)$ from comparison with experimental data in semi-inclusive regions.
- Result in agreement with the world average. Uncertainty comparable to other determinations.