A precise α_s determination from the R-improved QCD Static Energy

José Manuel Mena Valle V.Mateu P.G.Ortega

Universidad de Salamanca

Instituto Universitario de Física Fundamental y Matemáticas

α_s 2025 Workshop

Full version of this work available in arXiv:2510.24846

Junta de Castilla y León Consejería de Educación

Ministerio de Trabajo y Economía Social

Fondos Europeos

Europa impulsa nuestro crecimiento

Cofinanciado por la Unión Europea

Introduction

- Our strategy $\to$ comparing the QCD Static Energy obtained in lattice simulations with highly accurate perturbative results.

- $V_{\mathrm{QCD}} \propto \alpha_s \to$ very sensitive.

- We improve this method building on previous analyses [A. Bazavov, N. Brambilla, X. Garcia i Tormo et al, 2012, A. Bazavov, N. Brambilla, X. Garcia i Tormo et al, 2014] in several ways:

  - Leading renormalon subtraction $\to$ short-distance scheme (MSR).

  - Resummation of associated large logs with R-evolution.

  - Profiles functions for the renormalization scales.

- We can fit (for the first time) lattice data up to $r \sim 0.5\,\mathrm{fm}$.

Static Energy and Static Potential

- The Static Energy is defined as the potential energy between an infinitely massive quark anti-quark pair at a distance r, corrected by ultra-soft effects. In pNRQCD

$$E_{\rm s}(r) = V_{\rm s}(r,\mu) + \delta_{\rm us}(r,\mu).$$

- The Static Potential is the basic object to understand the behavior of non-relativistic QCD:

$$V_{\rm s}(r,\mu) = V_{\rm s}^{\rm soft}(r,\mu) + V_{\rm s}^{\rm us}(r,\mu),$$

$$V_{\rm s}^{\rm soft}(r) = -C_F\,\frac{\alpha_s(\mu)}{r}\sum_{i=0}^{3}\left[\frac{\alpha_s(\mu)}{4\pi}\right]^i\sum_{j=0}^{i} a_{ij}\log^j\left(r\mu e^{\gamma_E}\right).$$

- Coefficients a_{i0} are known to three loops. $a_{ij\geq 0}$ obtained with RGE. $\alpha_s(\mu) = \alpha_s^{(n_\ell=3)}(\mu)$.
- At $\mathcal{O}(\alpha_s^4)$ ultra-soft contributions show up for the first time

$$V_{\rm s}^{\rm us}(r,\mu) = -\frac{C_A^3 C_F}{12\pi}\,\frac{\alpha_s^4(\mu)}{r}\log\left(\mu r e^{\gamma_E}\right).$$

Ultra-soft term and resummation

- μ_{us} appears both in the ultrasoft static potential V_s^{us} and in the matrix element δ_{us}

$$\delta_{\rm us}\left(\mu_s,\mu_{\rm us}\right)=-\frac{C_A^3C_F}{12\pi}\frac{\alpha_s^3\left(\mu_s\right)\alpha_s\left(\mu_{\rm us}\right)}{r}\log\left[\frac{C_A\alpha_s\left(\mu_s\right)e^{-5/6}}{\mu_{\rm us}r}\right]$$

$$V_{\rm s}^{\rm us}(r,\mu_s,\mu_{\rm us})=-\frac{C_A^3C_F}{12\pi}\frac{\alpha_s^3(\mu_s)\alpha_s(\mu_{\rm us})}{r}\log\left(\mu_{\rm us}re^{\gamma_E}\right)$$

Ultra-soft term and resummation

- μ_{us} appears both in the ultrasoft static potential V_s^{us} and in the matrix element δ_{us}

$$\delta_{us}(\mu_s,\mu_{us}) = -\frac{C_A^3 C_F}{12\pi}\frac{\alpha_s^3(\mu_s)\,\alpha_s(\mu_{us})}{r}\log\left[\frac{C_A\alpha_s(\mu_s)\,e^{-5/6}}{\mu_{us} r}\right]$$

$$V_s^{us}(r,\mu_s,\mu_{us}) = -\frac{C_A^3 C_F}{12\pi}\frac{\alpha_s^3(\mu_s)\alpha_s(\mu_{us})}{r}\log\left(\mu_{us} r e^{\gamma_E}\right)$$

- One can see the necessity of resummation with these two (incompatible) choices that minimize logs.
  - $\mu_{us} \sim \alpha_s/r$ [use this one: no large logs in δ_{us}]
  - $\mu_{us} \sim 1/r$ [discarded: sum up logs in V_s^{us}]

Ultra-soft term and resummation

- μ_{us} appears both in the ultrasoft static potential V_s^{us} and in the matrix element δ_{us}

$$\delta_{\rm us}\left(\mu_s, \mu_{\rm us}\right) = -\frac{C_A^3 C_F}{12\pi} \frac{\alpha_s^3\left(\mu_s\right)\alpha_s\left(\mu_{\rm us}\right)}{r} \log\left[\frac{C_A \alpha_s\left(\mu_s\right) e^{-5/6}}{\mu_{\rm us} r}\right]$$

$$V_{\rm s}^{\rm us}(r, \mu_s, \mu_{\rm us}) = -\frac{C_A^3 C_F}{12\pi} \frac{\alpha_s^3(\mu_s)\alpha_s(\mu_{\rm us})}{r} \log\left(\mu_{\rm us} r e^{\gamma_E}\right)$$

- One can see the necessity of resummation with these two (incompatible) choices that minimize logs.
  - $\mu_{us} \sim \alpha_s/r$ [use this one: no large logs in $\delta_{\rm us}$]
  - $\mu_{us} \sim 1/r$ [discarded: sum up logs in $V_s^{\rm us}$]

Ultra-soft term and resummation

- Resummation is known up to N^{3}LL ($\alpha_s^{4+n}\ln^n\alpha_s$) [N. Brambilla, A. Vairo, X. Garcia i Tormo and J. Soto, 2009].

Ultra-soft term and resummation

- Resummation is known up to N^{3}LL $(\alpha_s^{4+n}\ln^n\alpha_s)$ [N. Brambilla, A. Vairo, X. Garcia i Tormo and J. Soto, 2009].

- We perform resummation using pNRQCD RGE

$$\mu\frac{\mathrm{d}V_s(r,\mu_s,\mu)}{\mathrm{d}\mu} = -\frac{2C_F C_A^3}{24r}\frac{\alpha_s(\mu)}{\pi}\left[1+B\frac{\alpha_s(\mu)}{\pi}\right]\alpha_s^3(\mu_s)\left\{1+3\frac{\alpha_s(\mu_s)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_s e^{\gamma E}\right)\right]\right\}.$$

Ultra-soft term and resummation

- Resummation is known up to N^{3}LL $(\alpha_s^{4+n}\ln^n\alpha_s)$ [N. Brambilla, A. Vairo, X. Garcia i Tormo and J. Soto, 2009].

- We perform resummation using pNRQCD RGE

$$\mu\frac{\mathrm{d}V_s\left(r,\mu_s,\mu\right)}{\mathrm{d}\mu}=-\frac{2C_FC_A^3}{24r}\frac{\alpha_s(\mu)}{\pi}\left[1+B\frac{\alpha_s(\mu)}{\pi}\right]\alpha_s^3\left(\mu_s\right)\left\{1+3\frac{\alpha_s\left(\mu_s\right)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_se^{\gamma E}\right)\right]\right\}.$$

- Integrating from $\mu=\mu_s\sim 1/r$ to $\mu=\mu_{us}\sim\alpha_s/r$

$$\begin{aligned}V_s\left(r,\mu_s,\mu_{\mathrm{us}}\right)=&V_s\left(r,\mu_s\right)+U_{\mathrm{us}}\left(r,\mu_s,\mu_{\mathrm{us}}\right),\\U_{\mathrm{us}}\left(r,\mu_s,,\mu_{us}\right)=&\frac{C_A^3C_F}{6\beta_0r}\alpha_s^3\left(\mu_s\right)\left\{\left(1+3\frac{\alpha_s\left(\mu_s\right)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_se^{\gamma_E}\right)\right]\right)\log\left[\frac{\alpha_s\left(\mu_{\mathrm{us}}\right)}{\alpha_s\left(\mu_s\right)}\right]\right.\\&\left.+\left(B-\frac{\beta_1}{4\beta_0}\right)\left[\frac{\alpha_s\left(\mu_{\mathrm{us}}\right)}{\pi}-\frac{\alpha_s\left(\mu_s\right)}{\pi}\right]\right\}\end{aligned}$$

Ultra-soft term and resummation

- Resummation is known up to N^{3}LL $(\alpha_s^{4+n}\ln^n\alpha_s)$ [N. Brambilla, A. Vairo, X. Garcia i Tormo and J. Soto, 2009].

- We perform resummation using pNRQCD RGE

$$\mu\frac{\mathrm{d}V_s(r,\mu_s,\mu)}{\mathrm{d}\mu}=-\frac{2C_FC_A^3}{24r}\frac{\alpha_s(\mu)}{\pi}\left[1+B\frac{\alpha_s(\mu)}{\pi}\right]\alpha_s^3(\mu_s)\left\{1+3\frac{\alpha_s(\mu_s)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_se^{\gamma E}\right)\right]\right\}.$$

- Replacing in the static energy $\rightarrow$ no large logs for $\mu_s\sim 1/r$ and $\mu_{us}\sim\alpha_s/r$

$$\begin{aligned}E_{\mathrm{s}}(r)=&V_{\mathrm{s}}^{\mathrm{soft}}(r,\mu_s)+\frac{C_A^3C_F}{12}\frac{\alpha_s^3(\mu_s)}{r}\left\{\frac{2}{\beta_0}\left(B-\frac{\beta_1}{4\beta_0}\right)\left[\frac{\alpha_s(\mu_{\mathrm{us}})}{\pi}-\frac{\alpha_s(\mu_s)}{\pi}\right]\right.\\&+\frac{2}{\beta_0}\left(1+3\frac{\alpha_s(\mu_s)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_se^{\gamma_E}\right)\right]\right)\log\left[\frac{\alpha_s(\mu_{\mathrm{us}})}{\alpha_s(\mu_s)}\right]\\&\left.-\frac{\alpha_s(\mu_{\mathrm{us}})}{\pi}\log\left[\frac{C_A\alpha_s(\mu_s)e^{-5/6}}{r\mu_{\mathrm{us}}}\right]-\frac{\alpha_s(\mu_s)}{\pi}\log(r\mu_se^{\gamma_E})\right\}.\end{aligned}$$

R-improvement, renormalon substractions

- Static energy suffers from an r-independant $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon ambiguity.

R-improvement, renormalon substractions

- Static energy suffers from an r-independant $\mathcal{O}(\Lambda_{\mathrm{QCD}})$ renormalon ambiguity.
- It is equals to $-2\,\times$ the ambiguity of the pole mass.

R-improvement, renormalon substractions

- Static energy suffers from an r-independant $\mathcal{O}(\Lambda_{\mathrm{QCD}})$ renormalon ambiguity.
- It is equals to $-2\times$ the ambiguity of the pole mass.

$$E_{\mathrm{Q\overline{Q}}}(r) = 2m_Q^{\mathrm{pole}} + E_s(r) \qquad \text{renormalon-free}$$

R-improvement, renormalon substractions

- Static energy suffers from an r-independant $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon ambiguity.
- It is equals to $-2 \times$ the ambiguity of the pole mass.

$$E_{Q\overline{Q}}(r) = 2m_Q^{\rm pole} + E_s(r) \qquad \text{renormalon-free}$$

- To cancel the renormalon we need a short-distance mass scheme $\rightarrow$ MSR mass [A. H. Hoang, A. Jain, C. Lepenik, V. Mateu, I. Scimemi and I. W. Stewart, 2018]

$$\delta m_Q^{\rm MSR}(\mu, R) \equiv m_Q^{\rm pole} - m_Q^{\rm MSR}(R) = R \sum_{n=1}^{\infty} \delta_n^R \left[\frac{\alpha_s(R)}{4\pi}\right]^n$$

$$= R \sum_{n=1}^{\infty} \left[\frac{\alpha_s(\mu)}{4\pi}\right]^n \sum_{j=0}^{n-1} \delta_{nj}^R \log^j\left(\frac{\mu}{R}\right).$$

R-improvement, renormalon substractions

- Static energy suffers from an r-independant $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon ambiguity.
- It is equals to $-2\times$ the ambiguity of the pole mass.

$$E_{Q\overline{Q}}(r) = 2m_Q^{\rm pole} + E_s(r) \qquad \text{renormalon-free}$$

- To cancel the renormalon we need a short-distance mass scheme $\rightarrow$ MSR mass [A. H. Hoang, A. Jain, C. Lepenik, V. Mateu, I. Scimemi and I. W. Stewart, 2018]

$$\delta m_Q^{\rm MSR}(\mu, R) \equiv m_Q^{\rm pole} - m_Q^{\rm MSR}(R) = R\sum_{n=1}^{\infty}\delta_n^R\left[\frac{\alpha_s(R)}{4\pi}\right]^n$$
$$= R\sum_{n=1}^{\infty}\left[\frac{\alpha_s(\mu)}{4\pi}\right]^n\sum_{j=0}^{n-1}\delta_{nj}^R \log^j\left(\frac{\mu}{R}\right).$$

- New type of large logs $\rightarrow$ resummation using R-evolution [A. H. Hoang, A. Jain, I. Scimemi and I. W. Stewart, 2008].

Previous works

- To see the advantages of R-evolution we go back to the first work that used the static energy to obtain α_s [A. Bazavov, N. Brambilla, X. Garcia i Tormo et al, 2012]

$$E_s(r) = V_s(r, \mu_s, \mu_{us}) + \delta_{US}(r, \mu_s, \mu_{us}) + RS(\rho),$$

  - $V_s(r, \mu_s, \mu_{us}) \to \ln(r\mu_{us})$
  - $RS(\rho) \to \ln(\rho/\mu_{us})$
- There is no right choice for $\mu_{us} \to$ fits only posible for small values of r.
- The force was used in [TUMQCD collaboration, 2019]

$$F_\mathrm{s}(r) = \frac{\mathrm{d}E_\mathrm{s}(r)}{\mathrm{d}r}$$

- Force avoids explicit substraction.
- Using fully canonical scales $\sim 1/r \to$ limits fit-range to $r \sim 0.076$ fm.

R-improvement, renormalon substractions

- The next goal is to obtain a renormalon-free potential.

$$E_{Q\overline{Q}}(r) = 2m_Q^{\mathrm{MSR}}(R_0) + 2\delta m_Q^{\mathrm{MSR}}(R_0,\mu) + E_s(r) \equiv 2m_Q^{\mathrm{MSR}}(R_0) + V_s^{\mathrm{MSR}}(r,\mu,R_0).$$

- We sum up large logs of μ/R_0.

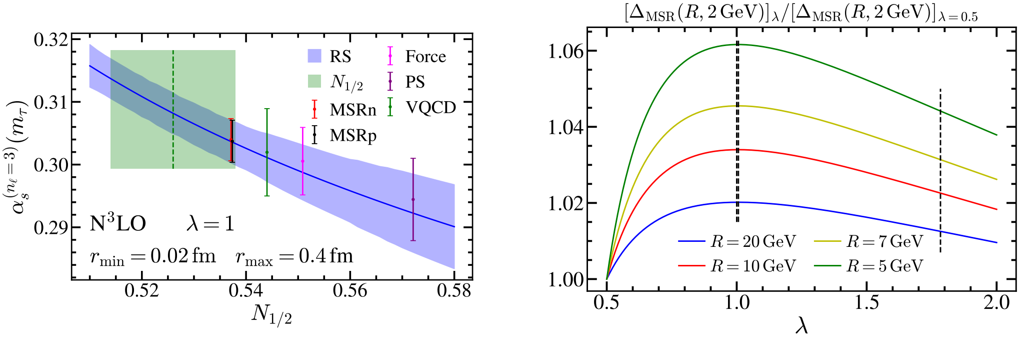
$$\begin{aligned}\delta m_Q^{\mathrm{MSR}}(R_0) &= \delta m_Q^{\mathrm{MSR}}(R_0) + \delta m_Q^{\mathrm{MSR}}(R) - \delta m_Q^{\mathrm{MSR}}(R)\\ &= \delta m_Q^{\mathrm{MSR}}(R) + m_Q^{\mathrm{MSR}}(R) - m_Q^{\mathrm{MSR}}(R_0)\\ &= \delta m_Q^{\mathrm{MSR}}(R) + \Delta^{\mathrm{MSR}}(R,R_0).\end{aligned}$$

- $\Delta^{\mathrm{MSR}}(R,R_0)$: solution to MSR mass R-RGE $\to$ sums up logs of R/R_0.
- We have $\delta m_Q^{\mathrm{MSR}}(R) \sim \log(\mu/R)$. By choosing $\mu \sim R \to$ no large logs.
- We define the R-improved static potential:

$$V_s^{\mathrm{MSR}}(r,\mu,R_0) = V_s(r,\mu) + 2\delta^{\mathrm{MSR}}(R,\mu) + 2\Delta^{\mathrm{MSR}}(R,R_0)\,.$$

Dependence on λ

- We don't know the R-evolution kernel up to infinite order $\rightarrow$ can estimate the truncation error with a dimensionless parameter λ, its variation account for higher-order remnantes (similar to renormalization scale variation).
- α_s is sensitive to $N_{1/2}$ (also deppends on λ), we have studied the R-evolution dependence with λ.



- The canonical value of $\lambda = 1$ is clearly biased.
- We pick $\lambda = 1.784$ and vary it from 1.5 to 2.1.

Profiles

- $\log(r\mu_s)$ and $\alpha_s(\mu_s)$ have to be small simultaneously.

Profiles

- $\log(r\mu_s)$ and $\alpha_s(\mu_s)$ have to be small simultaneously.
- Choosing $\mu_s \sim 1/r$ doesn't always work $\rightarrow \mu_s < \Lambda_{\rm QCD}$ for $r \gtrsim 0.2$ fm ~ 1 GeV.

Profiles

- $\log(r\mu_s)$ and $\alpha_s(\mu_s)$ have to be small simultaneously.
- Choosing $\mu_s \sim 1/r$ doesn't always work $\to \mu_s < \Lambda_{\rm QCD}$ for $r \gtrsim 0.2$ fm ~ 1 GeV.
- Solution: use profile functions that ensure series convergence:

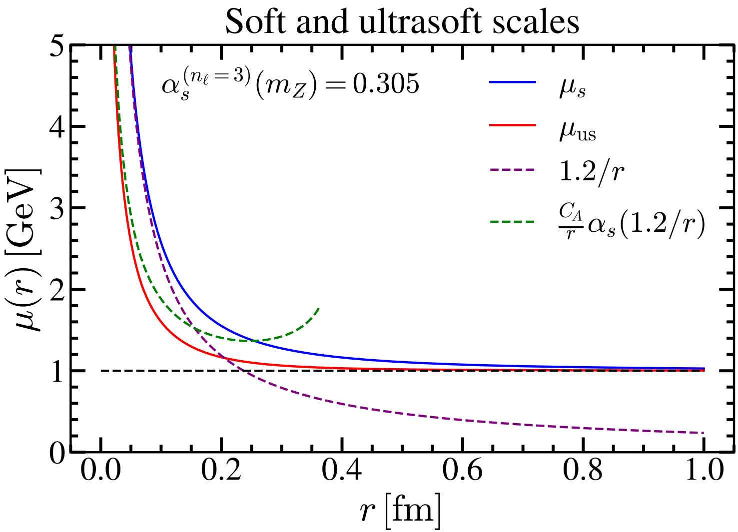
$$\mu_s(r,\xi,\mu_\infty,\Delta,b) = \sqrt{\left(\frac{\xi}{r}\right)^2 + \frac{b}{r} + (\mu_\infty - \Delta)^2} - \Delta = \begin{cases} \frac{\xi}{r} & \text{for } r \to 0 \\ \mu_\infty & \text{for } r \to \infty \end{cases}$$

$$R(r,\beta,R_\infty,\Delta,b) = \mu_s(r,\beta,R_\infty,\Delta,b)$$

$$\mu_{\rm us}(r,\xi,\kappa,\mu_\infty,\Delta,b) = C_A\Big\{\mu_s(r,\kappa\xi,\mu_\infty,\Delta,b)\alpha_s\big[\mu_s(r,\kappa\xi,\mu_\infty,\Delta,b)\big] - \mu_\infty\alpha_s(\mu_\infty)\Big\} + \mu_\infty$$

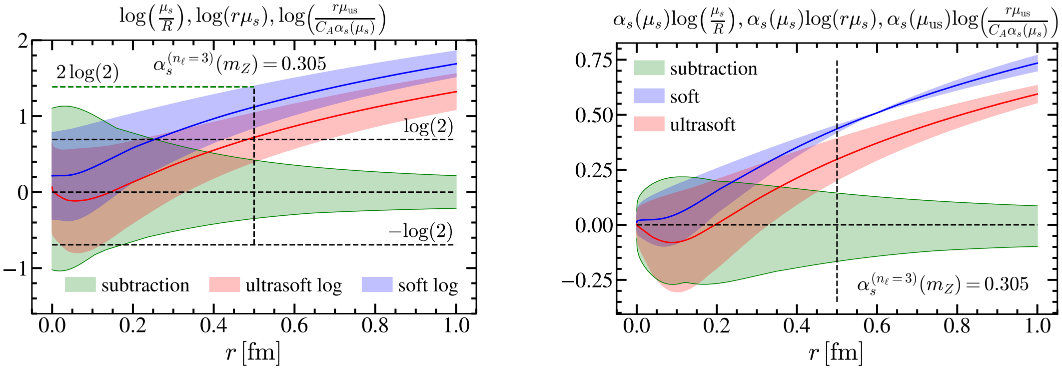
with $\xi = \mathcal{O}(1)$, $\mu_\infty \sim 1\,\text{GeV}$ and $\mu_s \gg \mu_{\rm us}$ for small distances. This makes sure the series is stable and convergent over the entire spectrum.

Profiles



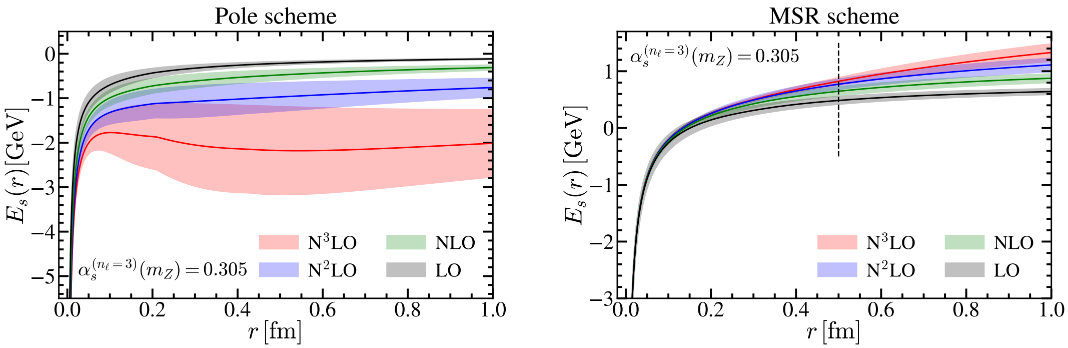
Soft and ultrasoft scales
$\alpha_s^{(n_\ell=3)}(m_Z) = 0.305$
μ_s
$\mu_{\rm us}$
$1.2/r$
$\frac{C_A}{r}\alpha_s(1.2/r)$
$\mu(r)$ [GeV]
r [fm]
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5

Profiles



- Ultrasoft logarithm smaller than the soft one and $\mathcal{O}(1)$ in the fit range.
- The product of $\alpha_s(\mu)\times\log(\mu)$ is smaller than 1.

Perturbative analysis



At $r = 0.3\,\text{fm}$ we have the following uncertainties for LO, NLO, N^{2}LO and N^{3}LO:

- Pole scheme $\rightarrow [0.16, 0.31, 0.71, 1.86]$ GeV.
- MSR scheme $\rightarrow [0.17, 0.15, 0.13, 0.07]$ GeV.

Fits

- Lattice QCD data from HotQCD. $(2+1)$ flavor simulations. We have 9 lattices sets adding up to 2512 data points
- To change from lattice to physical units we use $r_1 = 0.3093(20)\,\mathrm{fm}$, average of [A. Bazavov et al, 2014] and [R. Larsen, S. Mukherjee, P. Petreczky et al, arXiv 2502.08061].
- Each dataset n has a different origin of the static energy $\rightarrow$ offset A_n, included in the χ^2 function:

$$\chi^2(\alpha_s, \{A_k\}) = \sum_{k=1}^{N_s} \chi_k^2(\alpha_s, A_k) \qquad \chi_k^2(\alpha_s, A_k) = \sum_{i=1}^{n^k_{\mathrm{data}}} \frac{\left[V_{ik}(\alpha_s) + A_k - V_{ik}^{\mathrm{exp}}\right]^2}{[\sigma_k^2]_i}$$

- We can marginalize analytically first with respect to the offsets.

$$\frac{\partial \chi^2}{\partial A_k} = 0 \Longrightarrow \tilde{A}_k(\alpha_s) = \frac{\sum_{i=1}^{n^k_{\mathrm{data}}} [\sigma_k^{-1}]_i^2 \left[V_{ik}^{\mathrm{exp}} - V_{ik}(\alpha_s)\right]}{\sum_{i=1}^{n^k_{\mathrm{data}}} [\sigma_k^{-1}]_i^2}\,,$$

$$\Longrightarrow \tilde{\chi}_k^2(\alpha_s) = \chi_k^2(\alpha_s, \tilde{A}_k(\alpha_s)) - \frac{\left\{\sum_{i=1}^{n^k_{\mathrm{data}}} [\sigma_k^{-1}]_i \left[V_{ik}(\alpha_s) - V_{ik}^{\mathrm{exp}}\right]\right\}^2}{\sum_{i=1}^{n^k_{\mathrm{data}}} [\sigma_k^{-1}]_i}.$$

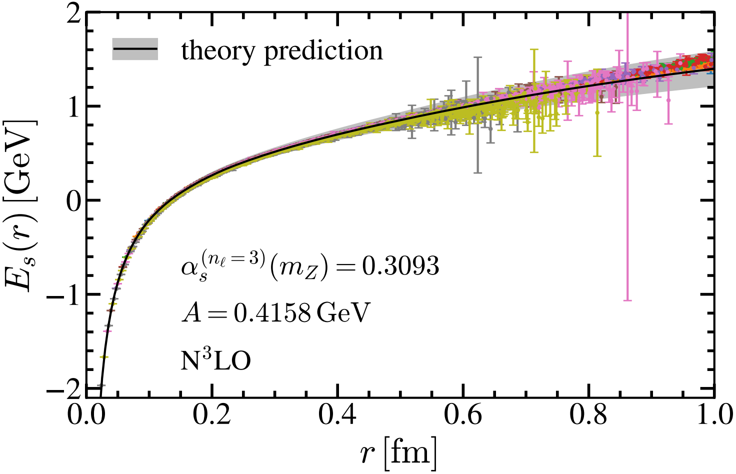
Fits

- Minimizing we obtain α_s^{BF} that verifies $\tilde{\chi}^2(\alpha_s^{\mathrm{BF}}) = \chi^2(\alpha_s^{\mathrm{BF}}, A^{\mathrm{BF}}) = \chi^2_{\mathrm{min}}$.
- The fit uncertainty is defined as $\tilde{\chi}^2(\alpha_s^{\mathrm{BF}} \pm \Delta_{\mathrm{fit}}\alpha_s) = \chi^2_{\mathrm{min}} + 1$.
- Theory uncertainties are estimated through a random scan.
- We perform the fit for 500 random profiles $\rightarrow$ α_s^{BF} and $\Delta_{\mathrm{fit}}\alpha_s$ for all of them.
- The theory (or perturbative) uncertainty is computed as

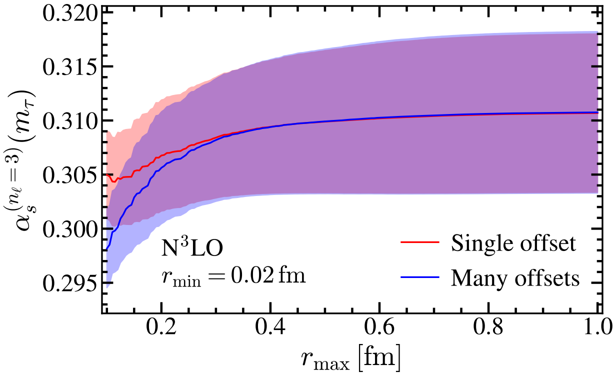
$$\Delta_{\mathrm{pert}}\alpha_s = (\alpha_{\mathrm{max}} - \alpha_{\mathrm{min}})/2$$

- The final result is the mean of the 500 α_s^{BF} and $\Delta_{\mathrm{fit}}\alpha_s$.
- We obtain $\mathcal{O}(10)\,\chi^2$ values $\rightarrow$ inflate lattice error by $\sqrt{\chi^2_{\mathrm{min}}/\mathrm{d.o.f.}}$.
- Fit for $r \in [r_{\mathrm{min}}, r_{\mathrm{max}}]$. $r_{\mathsf{data}} \geq 0.023$ fm.
- $r_{\mathrm{min}} \in [0.02, 0.045]$ fm and $r_{\mathrm{max}} \in [0.1, 1]$ fm with 0.05 fm spacing $\rightarrow$ 1086 different datasets.

Static Energy

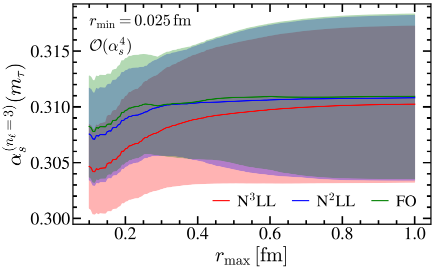


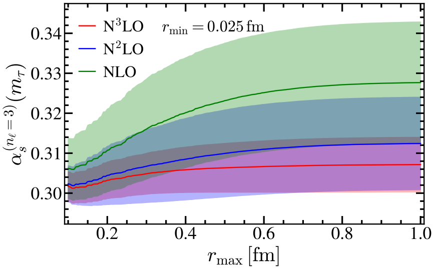
Offset approach

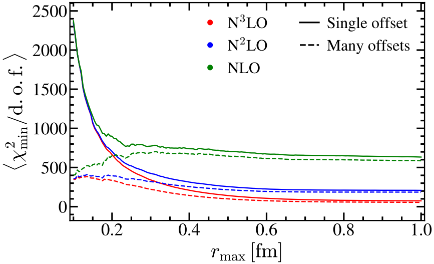


- We consider two offsets approaches:
  - Each ensemble has a different offset.
  - Common offset for all the ensembles.
- Both agree for $r_{\max} > 0.3$ fm.
- Overparametrization of χ^2 at small distances for the many offset case.
- We consider $r_{\max} \geq 0.35$ fm.

Order-by-order agreement

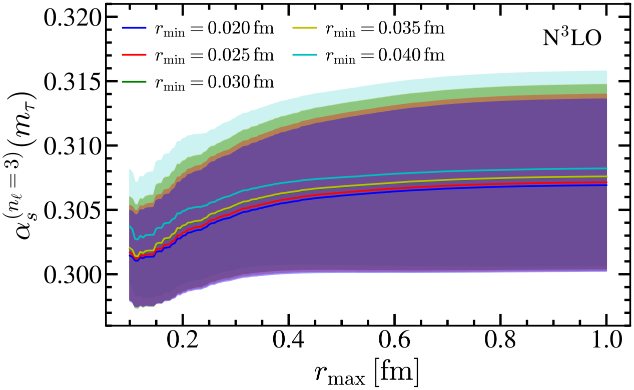






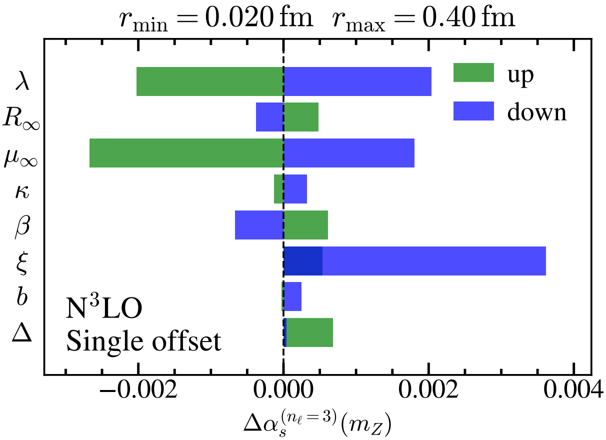
- Uncertainty bands nested for $r_{\rm max} < 0.5$ fm.
- We choose $r_{\rm max} \leq 0.45$ fm

Study of the dataset



- We consider the range $0.02 \leq r_{min} \leq 0.04$ fm .

Variation of profiles parameters one at a time



Parameter	Default	Range
ξ	$1.2 \times \hbar c$	$[0.7, 2.2] \times \hbar c$
β	$1.2 \times \hbar c$	$[0.7, 2.2] \times \hbar c$
b	0	$[-0.3, 0.3] \times \hbar c$
Δ	0	$[-0.6, 0.6]$ GeV
μ_∞	1 GeV	$[0.9, 1.1]$ GeV
R_∞	1 GeV	$[0.9, 1.1]$ GeV
κ	1	$[0.8, 1.2]$
λ	1.8	$[1.5, 2.1]$

Final results

- Requirements: $r_{\min} \in [0.02, 0.04]$ fm and $r_{\max} \in [0.35, 0.45]$ fm $\rightarrow$ 105-element dataset.
- We perform a fit in all of them:

$$\alpha_s^{(n_f=3)}(m_\tau) = 0.3093 \pm 0.00001_{\text{lattice}} \pm 0.0061_{\text{th}} \pm 0.0011_{\text{set}} \pm 0.0011_{r_1}$$

$$\downarrow$$

$$\alpha_s^{(n_f=3)}(m_\tau) = 0.3093 \pm 0.0063$$

$$\alpha_s^{(n_f=5)}(m_Z) = 0.1170 \pm 0.0008_{\text{th}} \pm 0.0001_{\text{set}} \pm 0.0001_{r_1} \pm 0.0003_{\mu_c} \pm 0.0002_{\mu_b}$$

$$\downarrow$$

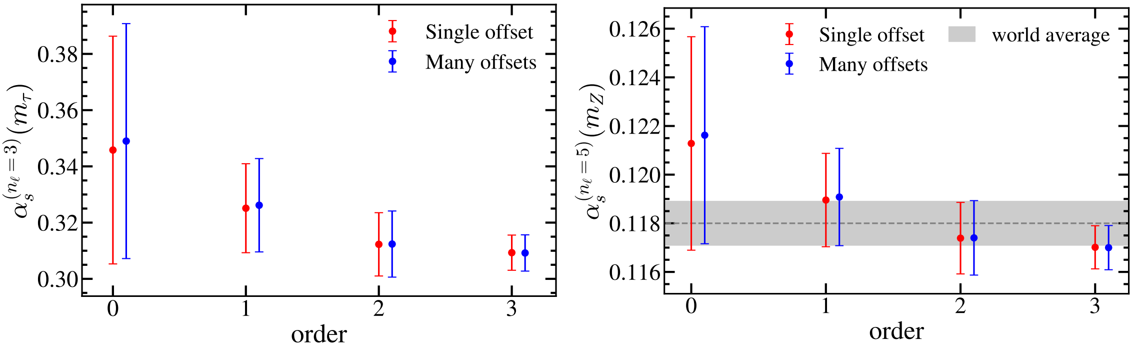
$$\alpha_s^{(n_f=5)}(m_Z) = 0.1170 \pm 0.0009$$

Competitive with the w.a. $\alpha_s^{(5)}(m_Z) = 0.1180 \pm 0.0009$ and compatible at 0.5-σ level.
Comparing with previous analyses:

$\alpha_s^{(n_f=5)}(m_Z) = 0.11660^{+0.00110}_{-0.00056} \rightarrow \alpha_s^{(n_f=5)}(m_Z) = 0.1162(8)$ [TUMQCD collaboration, 2019]

$\alpha_s^{(n_f=5)}(m_Z) = 0.1181 \pm 0.0009 \rightarrow \alpha_s^{(n_f=5)}(m_Z) = 0.1177(9)$ [C. Ayala, X. Lobregat, A. Pineda, 2020]

Final results



Conclusions

- Used the static energy to determine α_s, building (and improving) on previous analyses.
- Performed ultra-soft large-log resummation up to N^{3}LL.
- Employed the MSR mass and R-evolution to improve the static potential.
- Designed profile functions to increase the validity of the potential up to $r \sim 0.5\,\text{fm}$.
- Studied carefully the dependence on the dataset.
- Carried out fits to lattice data to obtain a very competitive result for α_s.

$$\alpha_s^{(n_f=5)}(m_Z) = 0.3093 \pm 0.0063 \qquad \alpha_s^{(n_f=5)}(m_Z) = 0.1170 \pm 0.0009$$

BACKUP

Force-type subtractions

- Integrating the Force is equal to perform R-evolution.
- The renormalon doesn't depend on r $\rightarrow$ we can subtract the potential at r_0.

$$E_{\rm s}^{F}\left(r,r_0\right)\equiv E_{\rm s}(r)-E_{\rm s}\left(r_0\right)=E_{\rm s}(r)-E_{\rm s}\left(r_1\right)+\left[E_{\rm s}\left(r_1\right)-E_{\rm s}\left(r_0\right)\right]$$

$$=E_{\rm s}(r)-E_{\rm s}\left(r_1\right)+\int_{r_0}^{r_1}\mathrm{d}r'F_{\rm s}\left(r'\right)\equiv E_{\rm s}(r)-E_{\rm s}\left(r_1\right)+\Delta_F\left(r_0,r_1\right)$$

- [A. Bazavov, N. Brambilla, X. Garcia i Tormo et al, 2014] chooses $r_1=r\rightarrow$ only Δ_F left.
- Connecting with R-evolution, the substraction term is (choosing $R=1/r_1$ and $\mu=R$):

$$E_s(r_1)\equiv\delta^F_{\rm soft}(R)=\frac{1}{2}V_{\rm s}^{\rm soft}\left(\frac{1}{R}\right)=-2\pi C_F R\sum_{i=1}\left[\frac{\alpha_s(R)}{4\pi}\right]^i\sum_{j=0}^{i-1}a_{i-1,j}\gamma_E^j\equiv R\sum_{i=1}\left[\frac{\alpha_s(R)}{4\pi}\right]^i\delta_i^F.$$

- We can express Δ_F as an R-evolution integral:

$$\gamma^F_{\rm soft}(R)=-\frac{1}{2}\left[r^2F_{\rm s}^{\rm soft}(r)\right]_{r=1/R},$$

$$\Delta_F^{\rm soft}\left(r_0,r_1\right)=\int_{r_0}^{r_1}\frac{\mathrm{d}r'}{\left(r'\right)^2}\left[\left(r'\right)^2F_{\rm s}^{\rm soft}\left(r'\right)\right]=\frac{1}{2}\int_{1/r_0}^{1/r_1}\mathrm{d}R'\gamma^F_{\rm soft}\left(R'\right).$$

- It inherits the infrared sensitivity of the Static Potential.

Potential-type subtractions

- If we choose $\mu = R$ and $R = \frac{e^{-\gamma_E}}{r}$

$$\delta^V_{\text{soft}}(R) \equiv V_s^{\text{soft}}\left(\frac{e^{-\gamma_E}}{R}\right) = -2\pi e^{\gamma_E} C_F R \sum_{i=1} \left[\frac{\alpha_s(R)}{4\pi}\right]^i a_{i-1,0} \equiv R \sum_{i=1} \left[\frac{\alpha_s(R)}{4\pi}\right]^i \delta^V_i .$$

- We can obtain their anomalous dimensions

$$\gamma^V_{\text{soft}}(R) = -\frac{1}{2}\left[r^2 V_s^{\text{soft}}(r)\right]_{r=1/R}$$

$$\Delta^{\text{soft}}_V(r_0, r_1) = \int_{r_0}^{r_1} \frac{\mathrm{d}r'}{(r')^2}\left[(r')^2 V_s^{\text{soft}}(r')\right] = \frac{1}{2}\int_{1/r_0}^{1/r_1} \mathrm{d}R' \gamma^V_{\text{soft}}(R')$$

Subtraction schemes: PS scheme

- Defined from its relation to the pole mass: $m_p - m^{\mathrm{PS}}(R) \equiv \delta_{\mathrm{PS}}(R)$

$$\delta^{\mathrm{PS}}_{\mathrm{soft}}(R) = -\frac{1}{2}\int_{|\vec{q}|<R} \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\tilde{V}^{\mathrm{soft}}_{\mathrm{s}}(q,R) \equiv C_F R \frac{\alpha_s(R)}{\pi}\sum_{i=0}\left[\frac{\alpha_s(R)}{4\pi}\right]^i c_i,$$

$$c_i = \sum_{j=0}^{i} a_{i,j} h_j, \qquad \text{with} \qquad h_j = j!\sum_{\ell=0}^{j}\sum_{k=0}^{\text{floor}\left[\frac{\ell}{2}\right]} \kappa_{\ell-2k}\left(\frac{\pi}{2}\right)^{2k}\frac{(-1)^k}{(2k)!}.$$

Subtraction schemes: ultrasoft terms

- Since all these three schemes come from the potential, they have ultrasoft terms in their R-evoluition.

$$\gamma^{F}(R) = \gamma^{F}_{\text{soft}}(R) - \frac{C_A^3 C_F}{24\pi}[\alpha_s(R)]^4 \log\left[C_A \alpha_s(R) e^{\gamma_E - 5/6}\right],$$

$$\gamma^{V}(R) = \gamma^{V}_{\text{soft}}(R) - \frac{C_A^3 C_F}{24\pi} e^{\gamma_E}[\alpha_s(R)]^4 \log\left[C_A \alpha_s(R) e^{\gamma_E - 5/6}\right],$$

$$\gamma^{\mathrm{PS}}(R) = \gamma^{\mathrm{PS}}_{\mathrm{soft}}(R) - \frac{C_F C_A^3}{12\pi^2}[\alpha_s(R)]^4 \log\left[C_A \alpha_s(R) e^{\gamma E - 5/6}\right].$$

- They inherit the infrared problem of the static potential.

RS scheme

- It is defined from the pole mass by subtracting its leading asymptotic behavior [Antonio Pineda, 2001].

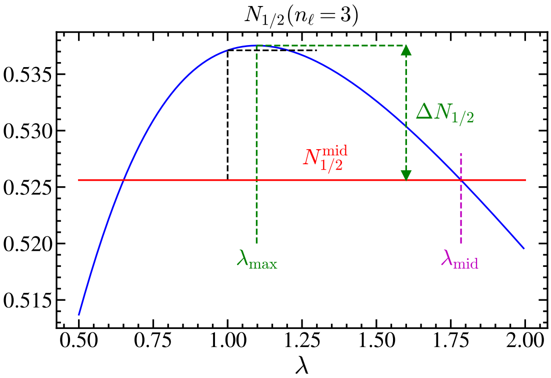
$$m_Q^{\rm pole} - m_Q^{\rm RS}(R) = \frac{2\pi}{\beta_0} R N_{1/2} \sum_{n=1}^{\infty} \left[\frac{\beta_0 \alpha_s(R)}{2\pi}\right]^n \sum_{\ell=0}^{\infty} g_\ell \left(1+\hat{b}_1\right)_{n-1-\ell},$$

- $N_{1/2}$ is the normalization of the leading renormalon

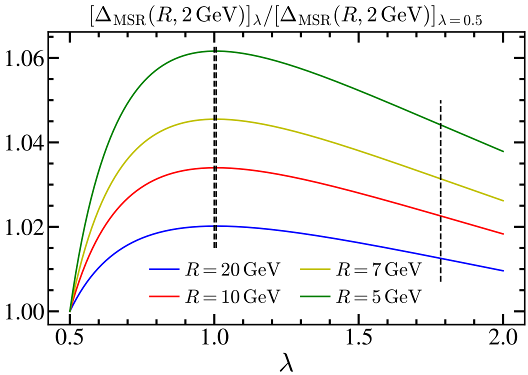
$$N_{1/2}^{(n)} = \frac{\beta_0}{2\pi} \sum_{k=0}^{n} \frac{S_k}{\left(1+\hat{b}_1\right)_k} \qquad S_k = \sum_{k=0}^{j} \tilde{\gamma}_k^R \sum_{i=0}^{j-k} (-1)^i \tilde{b}_i^N \tilde{g}_{j-i-k}^N,$$

- $N_{1/2}$ depends on λ, it reshuffles higher perturbative orders. We vary it to estimate $N_{1/2}$ (similar to scale variation).
- λ is also used to estimate R-evolution uncertainty.

$N_{1/2}$

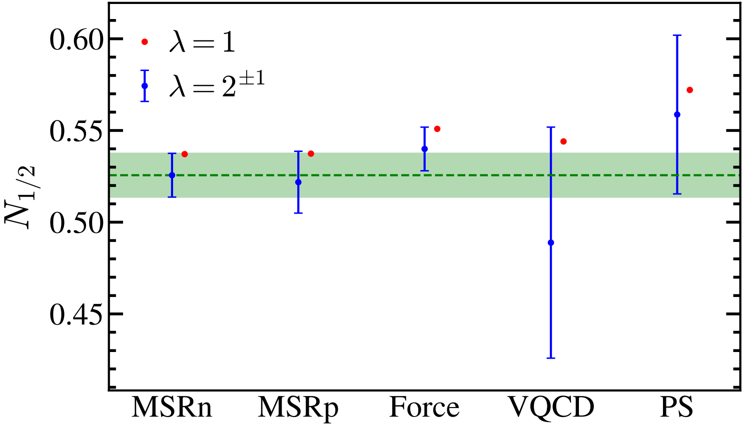


$N_{1/2}(n_\ell = 3)$
0.535
0.530
0.525
0.520
0.515
0.50
0.75
1.00
1.25
1.50
1.75
2.00
λ
$\Delta N_{1/2}$
$N_{1/2}^{\mathrm{mid}}$
λ_{max}
λ_{mid}

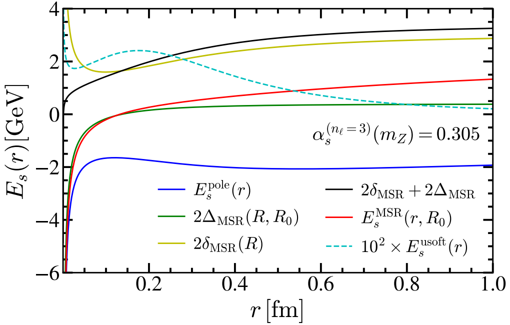


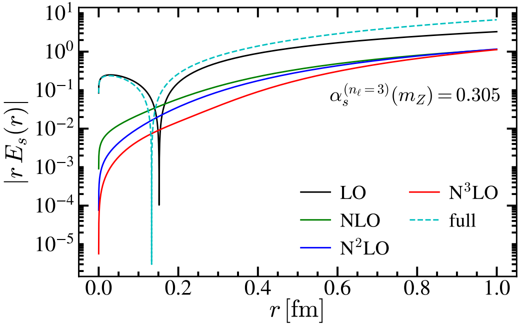
$[\Delta_{\mathrm{MSR}}(R, 2\,\mathrm{GeV})]_\lambda / [\Delta_{\mathrm{MSR}}(R, 2\,\mathrm{GeV})]_{\lambda=0.5}$
1.06
1.04
1.02
1.00
0.5
1.0
1.5
2.0
λ
$R = 20\,\mathrm{GeV}$
$R = 7\,\mathrm{GeV}$
$R = 10\,\mathrm{GeV}$
$R = 5\,\mathrm{GeV}$

Renormalon substraction

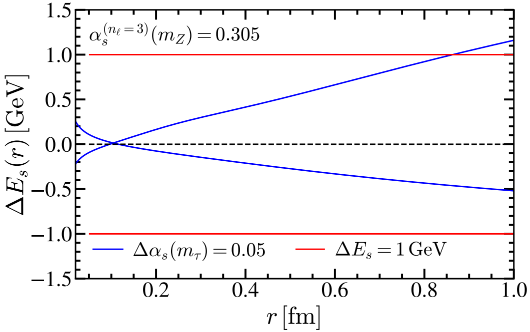


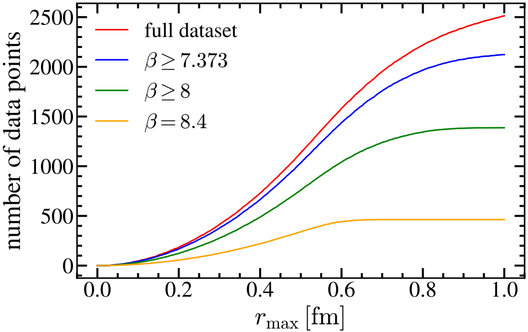
Perturbative analysis



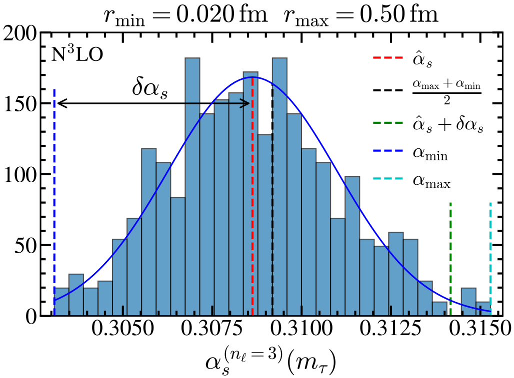


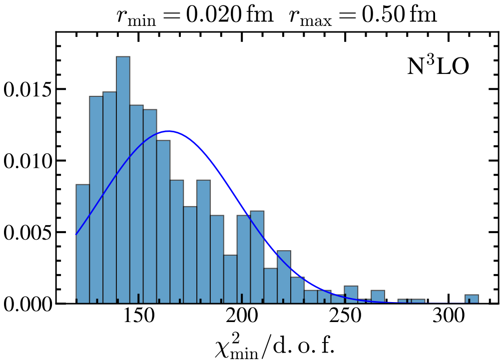
Auxiliar figures



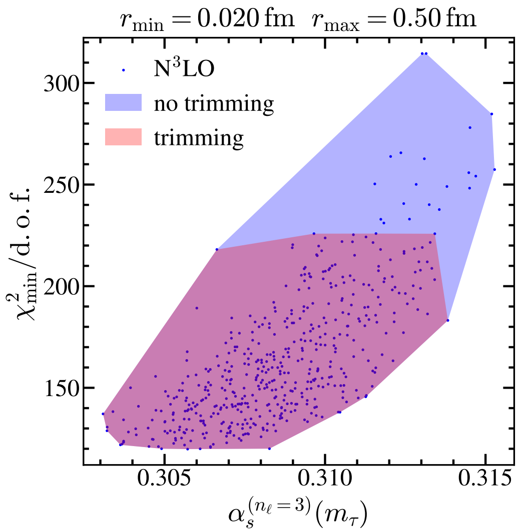


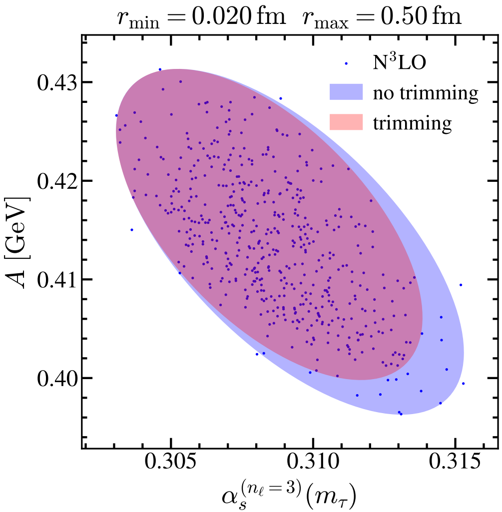
Random scan histograms



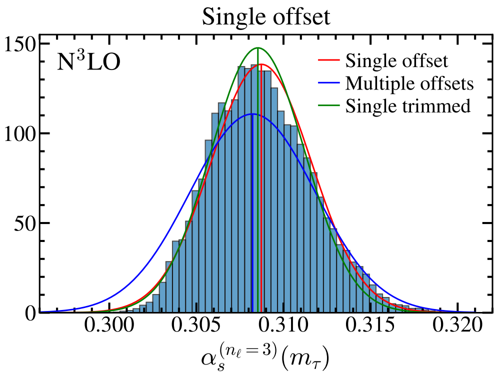


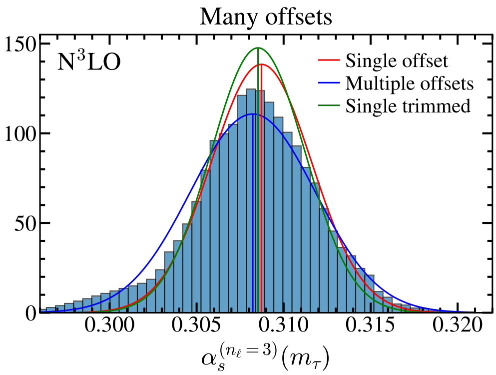
χ^2 trimming



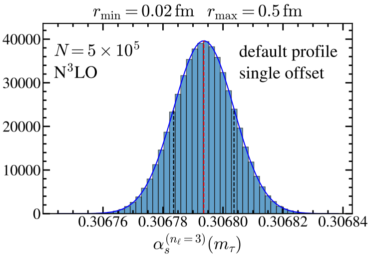


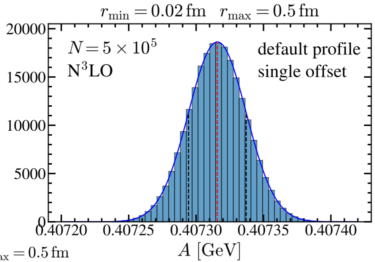
Fits histograms

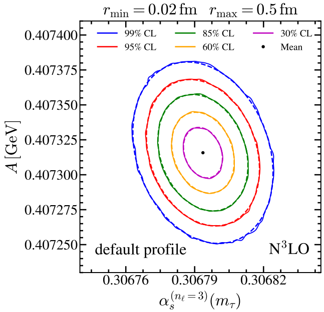




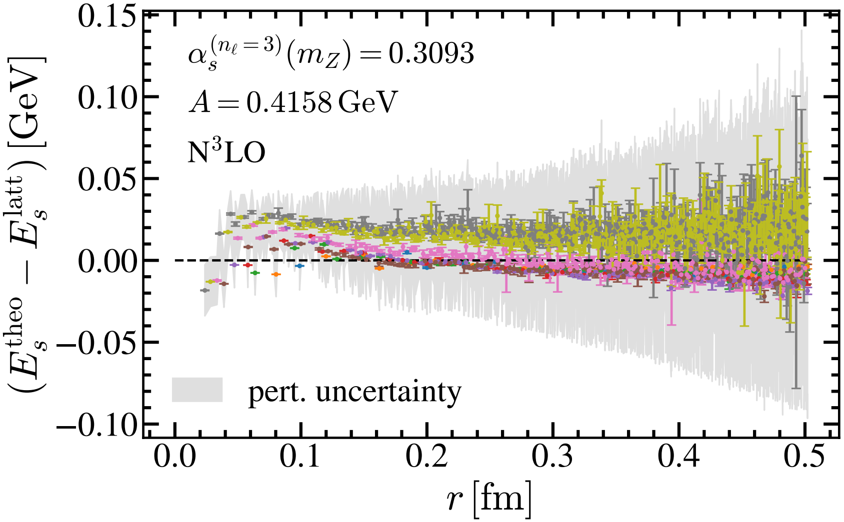
Replica method



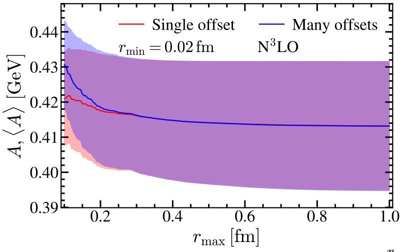


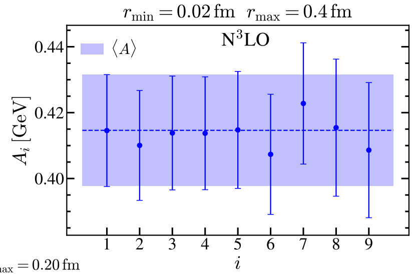


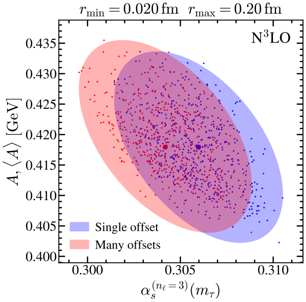
Lattice data



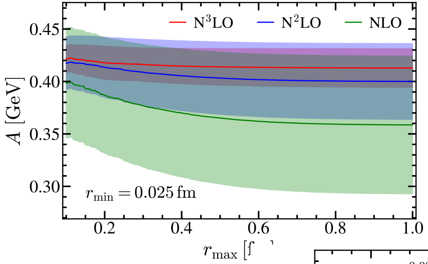
Offset analysis

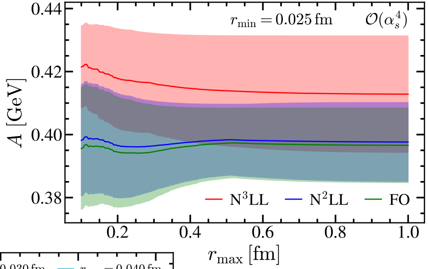


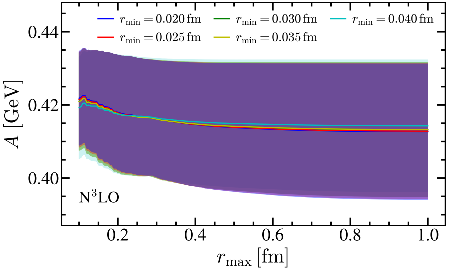




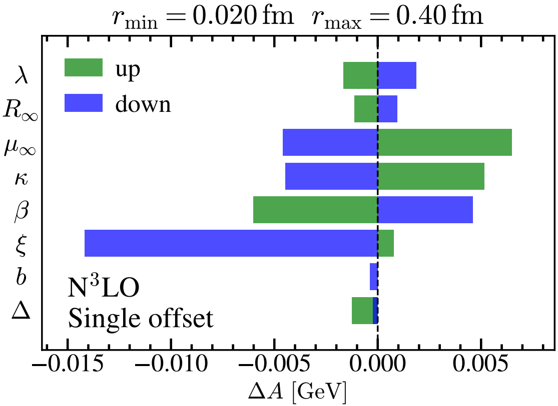
Offset analysis

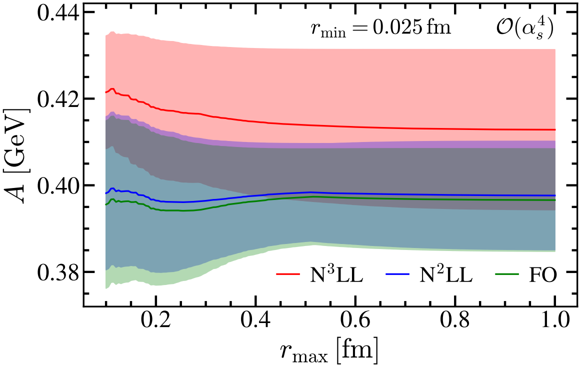




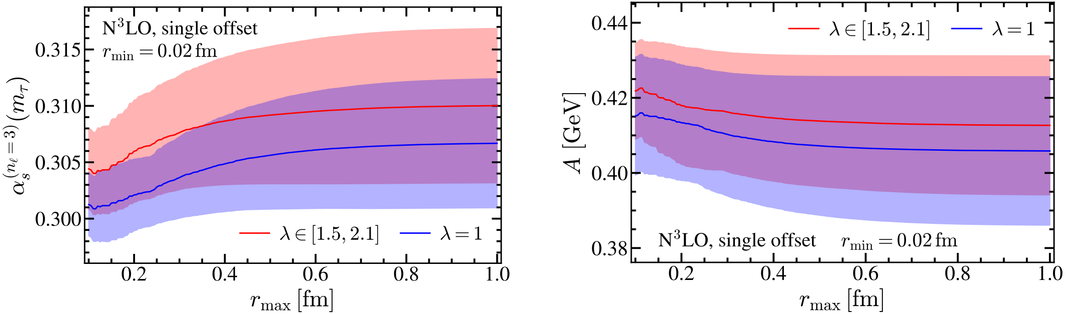


Offset analysis

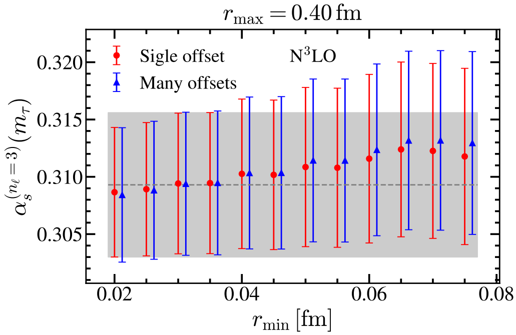


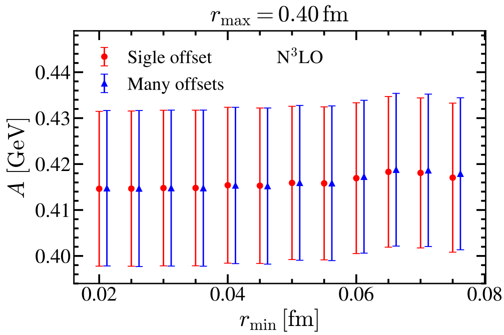


λ variation

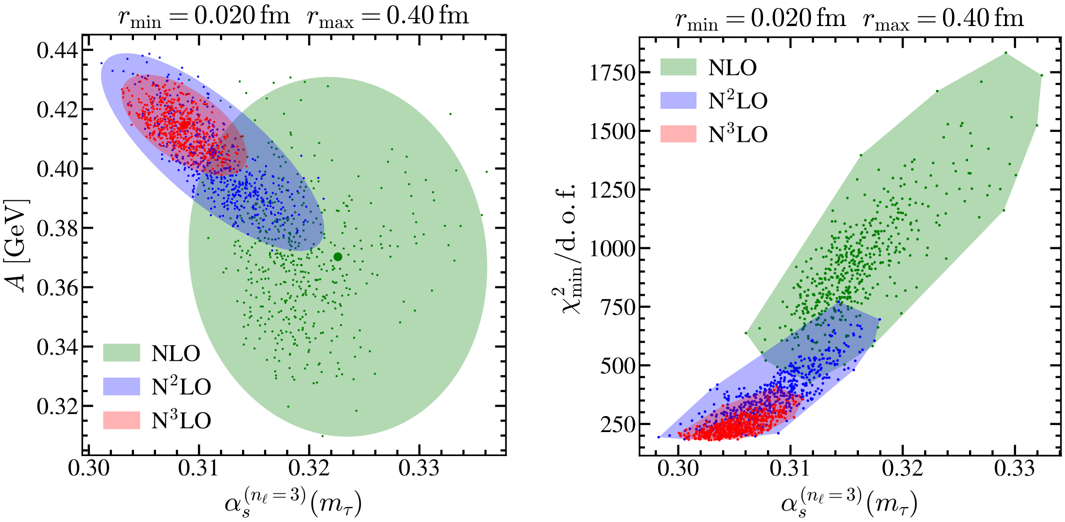


Further dependence check on r_{min}





Scattered fit points



Aditional fits

	full	$\beta \geq 7.737$	$\beta \geq 8$	$\beta = 8.4$
$\alpha_s^{(n_\ell=3)}(m_\tau)$	0.3093 ± 0.0063	0.3089 ± 0.0061	0.3089 ± 0.0070	0.3080 ± 0.0072
$\alpha_s^{(n_\ell=5)}(m_Z)$	0.1170 ± 0.0009	0.1170 ± 0.0009	0.1170 ± 0.0010	0.1169 ± 0.0010

	$\mu_c = \overline{m}_c$	$\lambda = 1$	FO	many offsets
$\alpha_s^{(n_\ell=3)}(m_\tau)$	0.3093 ± 0.0063	0.3055 ± 0.0049	0.3108 ± 0.0057	0.3092 ± 0.0064
$\alpha_s^{(n_\ell=5)}(m_Z)$	0.1174 ± 0.0009	0.1165 ± 0.0007	0.1172 ± 0.0008	0.1170 ± 0.0009