FLAG's 2024 report and future directions

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prepared for

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Outline

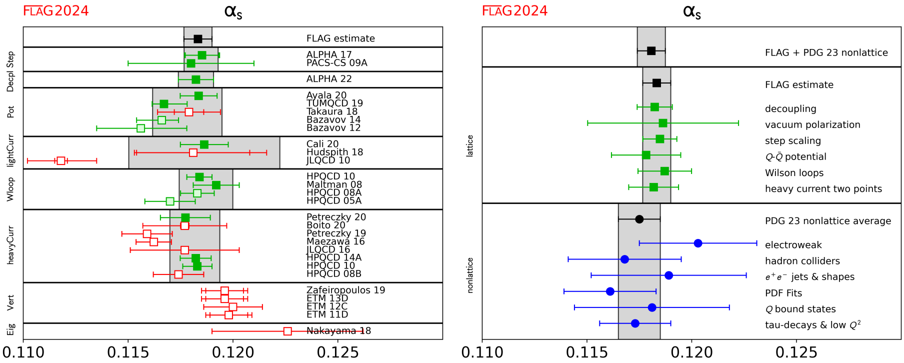
- The FLAG initiative
- FLAG 2024 estimate of α_s
- Reminders on determinations of $\alpha_s(m_Z)$
- Parametric uncertainties of Λ-parameter, perturbative behaviour and FLAG criteria
- New results & changes in FLAG 2024
- The decoupling strategy
- Situation for $N_{\rm f} = 0$
- Conclusions

FLAG = FLavour Lattice Averaging Group

- FLAG is an effort by the international lattice QCD community to provide the wider high energy physics community with lattice results for quantities of phenomenological interest, satisfying clearly defined quality criteria
- Original focus was on flavour physics, but now FLAG includes also sections on α_s, nucleon matrix elements and scale setting.
- FLAG website: `http://flag.itp.unibe.ch`
- Besides the quality criteria FLAG requires acceptance by or publication in a peer reviewed journal.
- **Cutoff date for FLAG 2024 was 30 April 2024**
- Status: The 2024 review will soon appear in Physical Review D.
- Time between successive reports 2-3 years, intermediate web updates by individual WG's are possible.

Nota Bene: FLAG requires that anyone using FLAG results cites the original references which enter the averages. A bibtex entry containing these references can be obtained from the FLAG website (go to the relevant plot and click on bib link next to it)

FLAG 2024 estimate of α_s



- FLAG 24: $\alpha_s(m_Z) = 0.1183(7) \quad [0.6\%]$

  FLAG 21: $\alpha_s(m_Z) = 0.1184(8) \quad [0.7\%]$
- Uncertainty: statistics dominated error of decoupling and step-scaling methods.
- Analogously when combined with PDG: $\alpha_s(m_Z) = 0.1181(7) \quad [0.6\%]$

Some reminders on determinations of $\alpha_s(m_Z)$

- FLAG 24 estimate: $\alpha_s(m_Z) = 0.1183(7) \quad [0.6\%]$
- All but one determinations: $N_\mathrm{f} = 2+1$, combined with 4-loop matching across charm and bottom thresholds
- A 1% error on α_s requires $\Delta\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f}=3)} < 5\%$

$\Rightarrow$ isospin breaking due to electromagnetism + mass differences is not yet relevant for α_s

Majority of determinations affected predominantly by systematics, in particular:

- Perturbative truncation errors: requires $\mu \gg \Lambda_{\overline{\mathrm{MS}}}$
- continuum limit: requires, $\mu \ll 1/a$

Note: given the very good quantitative perturbative description of decoupling across charm and bottom threshold, the determination of α_s is equivalent to a non-perturbative result for the Λ-parameter with $N_\mathrm{f} = 3, 4$

Reminder: parametric uncertainty of Λ-parameter & perturbative criteria

Λ-parameter in mass-independent renormalization scheme:

$$\Lambda_{\overline{\mathrm{MS}}} = \mu\varphi\left(\bar{g}(\mu)\right)$$

$$\varphi\left(\bar{g}\right) = \left[b_0\bar{g}^2\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2}} \exp\Big\{\underbrace{-\int_0^{\bar{g}} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]}_{=I[\bar{g};\beta]}\Big\}$$

A non-perturbatively defined coupling $\bar{g}^2(\mu)$ implies a non-perturbative β-function:

$$\beta(\bar{g}) \stackrel{\mathrm{def}}{=} \mu\frac{\partial\bar{g}(\mu)}{\partial\mu}, \qquad \beta(g) = -b_0 g^3 - b_1 g^5 + \ldots$$

with universal 1- and 2-loop coefficients b_0, b_1:

$$b_0 = (11 - \tfrac{2}{3}N_\mathrm{f})/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}N_\mathrm{f})/(4\pi)^4\,.$$

At large μ, use perturbative knowledge $\leq n_l + 1$-loops for $\beta \to \beta^{(n_l+1)}$ ($\alpha = g^2/(4\pi)$)

$$I[g;\beta] \quad \stackrel{g\to 0}{\simeq} \quad I[g,\beta^{(n_l+1)}] + O\left(g^{2n_l}\right)$$

$\Rightarrow$ For large μ expect remaining μ-dependence $\Lambda_{\overline{\mathrm{MS}}}^{\text{estimated}}/\Lambda_{\overline{\mathrm{MS}}} = 1 + O\left(\alpha^{n_l}(\mu)\right)$

Reminder: definition of effective coupling, α_{eff}

Starting point for all α_s determinations: Euclidean short distance quantity Q, that

- can be measured in a lattice simulation
- has a perturbative expansion, $Q = c_0 + c_1\alpha + c_2\alpha^2 + \ldots$.

We associate an effective coupling to Q, by normalizing

$$\alpha_{\text{eff}} = (Q - c_0)/c_1$$

- Advantage: no need to refer to a particular scale, α_{eff} is measured, possibly after chiral and continuum extrapolations (exception: couplings at $\mu = 1/a$, e.g. from small Wilson loops).
- Loop counting: Relate to the $\overline{\mathrm{MS}}$ scheme:

$$\alpha_{\text{eff}} = \alpha_{\overline{\mathrm{MS}}} + d_1\alpha_{\overline{\mathrm{MS}}}^2 + d_2\alpha_{\overline{\mathrm{MS}}}^3 + d_3\alpha_{\overline{\mathrm{MS}}}^4 + \ldots$$

  If d_k are known up to $k = n_l$ the loop order is n_l. Currently best cases have $n_l = 3$ (plus partial information on $n_l = 4$ for static potential/force)

Reminder FLAG 24 criteria (unchanged since FLAG 19!)

Renormalization scale

- ★ all points in the analysis have $\alpha_{\text{eff}} < 0.2$
- ○ all points have $\alpha_{\text{eff}} < 0.4$ and at least 1 with $\alpha_{\text{eff}} < 0.25$
- ■ otherwise

Perturbative behaviour

- ★ verified over a range of a factor 4 change in $\alpha_{\text{eff}}^{n_l}$ without power corrections or alternatively $\alpha_{\text{eff}}^{n_l} < \frac{1}{2}\Delta\alpha_{\text{eff}}/(8\pi b_0 \alpha_{\text{eff}}^2)$ is reached.
- ○ verified over a range of a factor $(3/2)^2$ change in $\alpha_{\text{eff}}^{n_l}$ possibly fitting with power corrections or alternatively $\alpha_{\text{eff}}^{n_l} < \Delta\alpha_{\text{eff}}/(8\pi b_0 \alpha_{\text{eff}}^2)$ is reached.
- ■ otherwise

Continuum limit: at a reference point of $\alpha_{\text{eff}} = 0.3$ (or less) require

- ★ three lattice spacings with $\mu a < 1/2$ and full O(a) improvement, or three lattice spacings with $\mu a \leq 1/4$ and 2-loop O(a) improvement, or $\mu a \leq 1/8$ and 1-loop O(a) improvement
- ○ three lattice spacings with $\mu a < 3/2$ reaching down to $\mu a = 1$ and full O(a) improvement, or three lattice spacings with $\mu a \leq 1/4$ and 1-loop O(a) improvement
- ■ otherwise

plus convention for μ in different quantities (e.g. $\mu = q$ in momentum space observables, or $\mu = 1/L$ for step-scaling)

FLAG 24, new results and changes:

- Petreczky and Weber '22: update on moments of heavy quark 2-point functions: $\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 332(17)(2)_{\text{scale}}$ MeV (discussed in FLAG 21 review but only published after FLAG 21 deadline)
- ALPHA '22 Decoupling strategy, relates $N_\mathrm{f} = 3$ to $N_\mathrm{f} = 0$

$\Rightarrow$ Decoupling strategy triggers renewed activity for $N_\mathrm{f} = 0$.

$N_\mathrm{f} = 0$

- Bribian et al. (2021, published): step-scaling for gradient flow coupling with twisted periodic boundary conditions
- Hasenfratz et al. (2023, published) and Wong et al (2023, Lattice '22 proceedings): use flow time as renormalization scale (infinite volume), 2-loop conversion to $\overline{\mathrm{MS}}$ by Harlander & Neumann (2016)
- Chimirri (2022, Lattice '22 proceedings) study of moments of heavy quark currents
- TUMQCD 23: static force with insertion of chromo-electric field. and gradient flow time as intermediate regulator.

FLAG 24, new results and changes:

- A scale variation analysis was included following the procedure proposed by Del Debbio & Ramos 2021

$\Rightarrow$ The results found to be in line with the quoted ranges.

- In view of the decoupling strategy we have reviewed $N_f = 0$ results and introduced categories and estimated corresponding ranges.

$\Rightarrow$ Conceptual shift: In the past, $N_f = 0$ was considered a test bed to cheaply test methods for full QCD; Now $N_f = 0$ results acquire physical significance!

$\Rightarrow$ renewed activity with $N_f = 0$ following the FLAG 21 review;

- Decoupling strategy may be generalized, e.g. to quark masses;
- Some FLAG WG's may have to reverse the phasing out of $N_f < 3$ results.

α_s from the decoupling strategy

[ALPHA 2019-2025]:

- Decoupling well described by PT (available up to 4 loops) $\Rightarrow$ use it as a tool!
- Connection between QCD Λ-parameters for $N_\mathrm{f} = 3$ and $N_\mathrm{f} = 0$, by decoupling triplet of heavy mass degenerate quarks with RGI mass M:

$$\bar{g}_s^{(3)}(\mu/\Lambda_s^{(3)}, M) = \bar{g}_s^{(0)}(\mu/\Lambda^{(0)}) + \mathrm{O}(\mu^2/M^2)\,, \qquad (1)$$

- in PT this leads to

$$[\bar{g}_{\overline{\mathrm{MS}}}^{(0)}(\mu)]^2 = C\left(\bar{g}_{\overline{\mathrm{MS}}}^{(3)}(m_\star)\right)[\bar{g}_{\overline{\mathrm{MS}}}^{(3)}(m_\star)]^2, \qquad m_\star = \overline{m}_{\overline{\mathrm{MS}}}(m_\star),$$

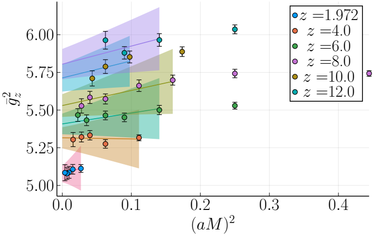
and for $\mu = m_\star$ one finds $C(x) = 1 + c_2 x^4 + c_3 x^6 + c_4 x^8 + \ldots$.

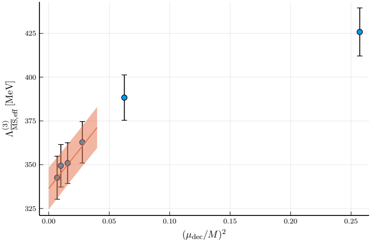
- Reformulation with $P = \varphi_{\overline{\mathrm{MS}}}^{(0)}\left(g_\star\sqrt{C(g_\star)}\right)/\varphi_{\overline{\mathrm{MS}}}^{(3)}(g_\star)$, $g_\star = g_{\overline{\mathrm{MS}}}^{(3)}(m_\star)$:

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_\mathrm{dec}} = \frac{\Lambda_{\overline{\mathrm{MS}}}^{(0)}}{\Lambda_s^{(0)}} \times \lim_{M/\mu_\mathrm{dec}\to\infty}\left[\frac{\varphi_s^{(0)}\left(\bar{g}_s^{(3)}(\mu_\mathrm{dec}, M)\right)}{P\left(\frac{M}{\mu_\mathrm{dec}}\Big/\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_\mathrm{dec}}\right)}\right]$$

- Corrections $\mathrm{O}(1/M^2)$ and $\mathrm{O}(\alpha^4(m_\star))$, requires extrapolation

Continuum and large mass extrapolations





- Data for $z = M/\mu_{\mathrm{dec}} \in \{1.972, 4, 6, 8, 10, 12\}$, extrapolated to $a = 0$ using global fits with 2 cuts in $(aM)^2 < 0.16, 0.25$; fixed $\hat{\Gamma} \in [-1, 1]$ and $\hat{\Gamma}' \in [-1/9, 1]$.

$$\bar{g}^2(z_i, a) = C_i + p_1[\alpha_{\overline{\mathrm{MS}}}(a^{-1})]^{\hat{\Gamma}}(a\mu_{\mathrm{dec}})^2 + p_2[\alpha_{\overline{\mathrm{MS}}}(a^{-1})]^{\hat{\Gamma}'}(aM_i)^2 \,.$$

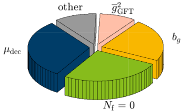
- $1/z^2$ extrapolation: solve equation for target ρ,

$$\rho \times \underbrace{P\,(z/\rho)}_{\text{PT} + \mathrm{O}\left(\alpha^4_{\overline{\mathrm{MS}}}(m_\star)\right)} = \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}}, \qquad \rho = \frac{\Lambda^{(3)}_{\overline{\mathrm{MS}},\mathrm{eff}}}{\mu_{\mathrm{dec}}} = \frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} + \mathrm{O}(1/z^2)$$

Combination with ALPHA 17 and prospects of further error reduction

$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} = 336(10)(6)_{b_\mathrm{g}}(3)_{\hat{\Gamma}_m}\,\mathrm{MeV} = 336(12)\mathrm{MeV} \quad \Rightarrow \quad \alpha_s(m_Z) = 0.1182(8)$$

- Total error is of the same size as in ALPHA '17 ($341(12)\mathrm{MeV}$, $\alpha_s(m_Z) = 0.1185(8)$)



contributions to error^2

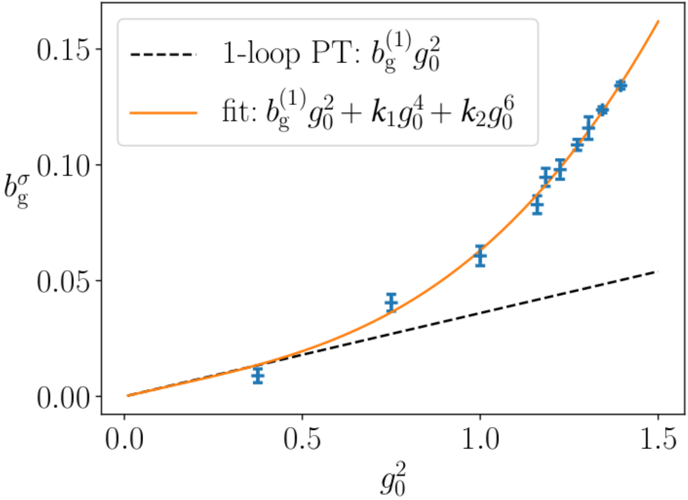
$\Rightarrow$ common (squared) error with ALPHA '17 only 28%! (from common scale)

- Combine published results ALPHA 17 and ALPHA 22:

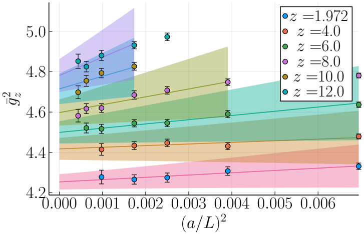
$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} = 339.5(9.6) \qquad \Rightarrow \qquad \alpha_s(m_Z) = 0.1184(7)$$

- ALPHA 25: improved precision and control of systematics (cf. A. Ramos)
  - Improved physical scale setting for from hadronic observables (via $\sqrt{t_0}$);
  - Complete non-perturbative elimination of O(aM) effects (improvement coefficient b_g)
  - Quantify decoupling of charm and bottom quarks both perturbatively and non-perturbatively.
  - Improve analysis of continuum limit for step-scaling functions.

Nonperturbative result for b_g, ALPHA '24

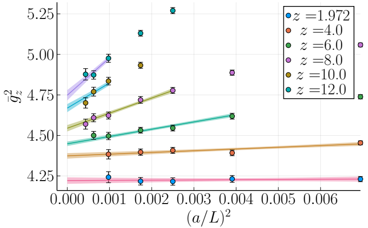


Improvement: The continuum extrapolation of massive couplings



Previous determination, ALPHA '22

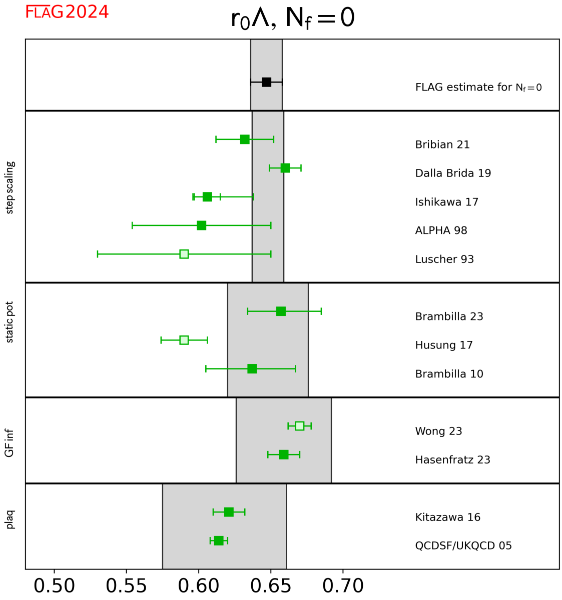
- Most of error from estimate of $b_g - b_g^{1-\mathrm{loop}}$
- This is a systematic!
- But error in $\bar{g}^2(\mu, M)$ subdominant (assuming 100 percent error on 1-loop b_g)



NP determination of b_g (ALPHA '24)

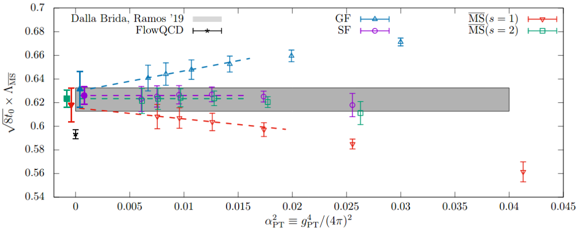
- Much more precise continuum values
- Completely removes largest systematic effect in α_s

Situation for $N_f = 0$



Situation for $N_f = 0$

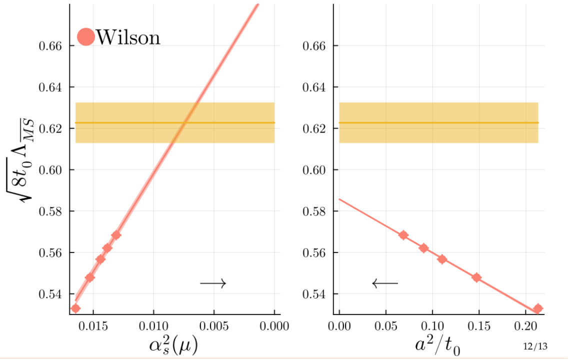
- Dalla Brida and Ramos '19, Nada and Ramos '21
  $\sqrt{8t_0}\Lambda^{(0)}_{\overline{\rm MS}} = 0.6227(98)$ $\Leftarrow$ enters the ALPHA decoupling result



- Result enters $N_f = 3$ decoupling result! Seemed rather large; but carefully cross checked by Nada & Ramos '21
- Now supported by several new $N_f = 0$ results

Power vs. perturbative corrections, study by Catumba, Lang & Ramos, Lattice '24

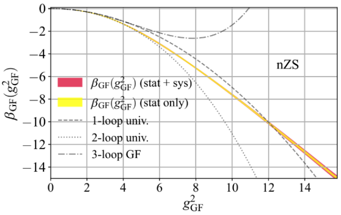
- Bare coupling $\alpha_0 = \alpha_{\text{lat}}(1/a)$ is defined at the lattice cutoff scale $1/a$
- $$\sqrt{8t_0}\Lambda_{\overline{\text{MS}}} = \left(\frac{\sqrt{8t_0}}{a}\right) \times (a\Lambda_{\overline{\text{MS}}}) \quad + \quad \mathcal{O}(a^2) + O(\alpha(1/a)^p)$$

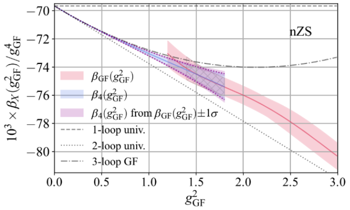


Hasenfratz et al. (2023, published) and Wong et al. (proceedings)

- renormalized coupling in infinite volume gradient flow scheme
- β-function known to 3-loops (Harlander and Neumann 2016)
- β-function from differentiating w.r.t. flow time
- Quote final result $\sqrt{8t_0}\Lambda^{(0)}_{\overline{\mathrm{MS}}} = 0.622(10)$ (perfectly agrees with Dalla Brida 19).

HOWEVER: Perturbative behaviour:





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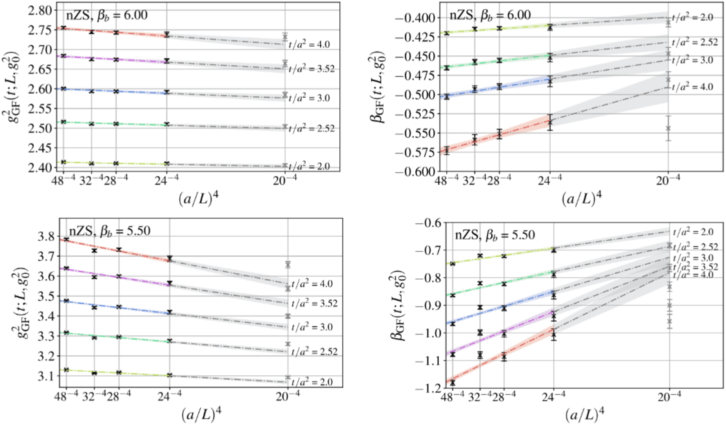
Infinite volume extrapolations:



Conclusions I

- Lattice QCD completely by-passes problems with quark confinement (hadronisation, quark-hadron duality etc.)
- But α_s poses a hard problem for lattice QCD due to large scale difference between perturbative and hadronic regimes!
- Complete solution exists in terms of step-scaling procedure; requires dedicated effort and resources;

$\Rightarrow$ finite volume essential, ie. most high order QCD PT results cannot be used directly!

- Potential game changer: Numerical Stochastic Perturbation Theory (NSPT), e.g 3-loop β-function for finite volume GF scheme (Dalla Brida & Lüscher 2018)
- Still, most calculations do not use step-scaling: trade higher PT order for lower energy scale:

$\Rightarrow$ systematics from truncation of perturbative series and non-perturbative effects;

- Other possible compromise: extrapolation to infinite volume; BUT: systematics at high/intermediate/low energy scales?

Conclusions II

- Decoupling strategy: $N_\mathrm{f} = 0$ results are physically relevant; uptake in community effort since FLAG 2021 report
- New methods come with new systematics: extrapolations to
  - decoupling limit $M \to \infty$
  - infinite volume limit (Hasenfratz et al, Wong et al.)
  - $\alpha \to 0$ for parametric uncertainty in Λ-parameter desirable ($N_\mathrm{f} = 0$); requires a wide range for α^{n_l}!
  - zero gradient flow time limit (e.g. TUMQCD 23)

⇒ use case-by-case assessments, possibly data driven (i.e. error in relation to distance covered by extrapolation)

- Reference scales: switch from r_0 to $\sqrt{8t_0}$: Need ratios of r_0, r_1, w_0 (also for $N_\mathrm{f} = 0$!)
- Scale variations could become a complementary criterion for perturbative truncation errors.
- Set-up of FLAG criteria still seems adequate in structure; numerical limits on α_eff, $a\mu$ should be tightened eventually.