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α_s from DIS data with large x resummation

OUTLINE

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1. Introduction

The search for new physics at the LHC accelerator requires a threshold resummation of the cross sections. (see, for example, the revent papers (Catani et al.: 2014, 2019), (Beneke et al.: 2019, 2020))

It would also be good to consider extracting from the deep-inelastic scattering (DIS) data (for the structure function (SF) $F_2(x,Q^2)$), taking into account resumming at large values of x ($=$ threshold resumming), where x is Bjorken's variable.

Usually the function $F_2(x,Q^2)$ is represented as a sum of the leading twist $F_2^{pQCD}(x,Q^2)$ and the twist four terms

$$F_2(x,Q^2) = F_2^{pQCD}(x,Q^2)\left(1+\frac{\tilde{h}_4(x)}{Q^2}\right) .$$

While analysing experimental data various corrections should be taken into account: nuclear effects, target mass corrections, heavy quark threshold corrections and higher twist terms are considered.

As is known there are two basic ways to perform QCD analysis over DIS data:
the first one (it is more popular now!) deals with Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) integro-differential equations and let the data be examined directly,
whereas the second one involves the SF moments and permits performing an analysis in analytic form as opposed to the former option. (but it is difficults to have experimental information!)

We take on the way in-between these two latter, i.e. analysis is carried out over the moments of SF $F_2^k(x, Q^2)$ defined as follows

$$M_n^{pQCD/\tau 2/\cdots}(Q^2) = \int_0^1 x^{n-2}\, F_2^{pQCD/\tau 2/\cdots}(x, Q^2)\, dx$$

and then reconstruct SF for each Q^2 by using Jacobi polynomial expansion method. (Parisi, Sourlas:1979), (Barnet et al.: 1981,1983,1984), (Krivokhizhin et al.: 1987,1990)

2. A brief theoretical input

The twist-two DIS SF can be represented as a sum of two terms:

$$F_2^{\tau 2}(x,Q^2)=\sum_q^f e_q^2\,\mathbf{f}_q(x,Q^2)=F_2^{NS}(x,Q^2)+F_2^S(x,Q^2)\,,$$

the singlet (S) and nonsinglet (NS) parts: $(< e_f^2 >= (1/f)\cdot\Sigma_q^f\, e_q^2)$

$$F_2^S(x,Q^2)=< e_f^2 > \sum_q^f \mathbf{f}_q(x,Q^2),\;\; F_2^{NS}(x,Q^2)=F_2^{\tau 2}(x,Q^2)-F_2^S(x,Q^2)\,.$$

Let's introduce PDFs, the gluon distribution function $\mathbf{f}_G(x,Q^2)$ and the singlet and nonsinglet quark distribution functions $\mathbf{f}_S(x,Q^2)$ and $\mathbf{f}_{NS}(x,Q^2)$ (hereafter PDFs are multiplied by x):

$$\begin{aligned}\mathbf{f}_S(x,Q^2) &\equiv \sum_q^f \mathbf{f}_q(x,Q^2)=V(x,Q^2)+S(x,Q^2)\,,\\ \mathbf{f}_{NS}(x,Q^2) &= \mathbf{u}_v(x,Q^2)-\mathbf{d}_v(x,Q^2)+\ldots\,,\end{aligned}$$

where f is the number of active quark flavors, strange,. . .), $V(x,Q^2)=\mathbf{u}_v(x,Q^2)+\mathbf{d}_v(x,Q^2)$ is the distribution of valence quarks and $S(x,Q^2)$ is a sum of sea parton distributions set equal to each other.

At large x values, no gluons and we can confine ourselves to considering only the NS (or valence) contribution (with the corresponding renormalization):

$$F_2^S(x,Q^2) \sim F_2^{NS}(x,Q^2) \quad \to \quad F_2^{\tau 2}(x,Q^2) \sim F_2^{NS}(x,Q^2),$$

There is a direct relation between SF moments $M(n,Q^2)$ and those of PDFs (I will omit the NS symbol here and below)

$$\mathbf{f}(n,Q^2) \;=\; \int_0^1 dx x^{n-2} \mathbf{f}(x,Q^2).$$

as

$$M(n,Q^2) = R(f) \times C(n, a_s(Q^2)) \times \mathbf{f}(n,Q^2)\,,$$

with

$$a_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}$$

and $C(n, a_s(Q^2))$ are the Wilson coefficient functions.

2.1 Strong coupling constant

The strong coupling constant is determined from the corresponding solution of the renormalization group equation.
At NLO level:

$$\frac{1}{a_1(Q^2)} - \frac{1}{a_1(M_Z^2)} + b_1 \ln\left[\frac{a_1(Q^2)}{a_1(M_Z^2)} \frac{(1+b_1 a_1(M_Z^2))}{(1+b_1 a_1(Q^2))}\right] = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right),$$

where hereafter

$$a_1(Q^2) = a_s^{NLO}(Q^2), \quad a_2(Q^2) = a_s^{NNLO}(Q^2)\,.$$

At NNLO level:

$$\frac{1}{a_2(Q^2)} - \frac{1}{a_2(M_Z^2)} + b_1 \ln\left[\frac{a_2(Q^2)}{a_2(M_Z^2)} \sqrt{\frac{1+b_1 a_2(M_Z^2) + b_2 a_2^2(M_Z^2)}{1+b_1 a_2(Q^2) + b_2 a_2 2(Q^2)}}\right]$$
$$+ \left(b_2 - \frac{b_1^2}{2}\right) \times (I(Q^2) - I((M_Z^2)) = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right).$$

The expression for I looks:

$$I(Q^2) = \begin{cases} \dfrac{2}{\sqrt{\Delta}} \arctan \dfrac{b_1 + 2b_2 a_2(Q^2)}{\sqrt{\Delta}} & \text{for } f = 3,4,5; \Delta > 0, \\ \dfrac{1}{\sqrt{-\Delta}} \ln \left[\dfrac{b_1 + 2b_2 a_2(Q^2) - \sqrt{-\Delta}}{b_1 + 2b_2 a_2(Q^2) + \sqrt{-\Delta}} \right] & \text{for } f = 6; \quad \Delta < 0, \end{cases}$$

where $\Delta = 4b_2 - b_1^2$ and $b_i = \frac{\beta_i}{\beta_0}$ are read off from the QCD β-function:

$$\beta(a_s) \; = \; -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 + \ldots$$

These equations allow us to eliminate QCD parameter Λ_{QCD} from the analysis.

2.2 Q^2-dependence of SF moments

The coefficient functions $C(n, a_s(Q^2))$ is further expressed through the functions $B_j(n)$ $(j = 1, 2)$which are known exactly

$$C(n, a_s(Q^2)) = 1 + a_s(Q^2)B_1(n) + a_s^2(Q^2)B_2(n) + \mathcal{O}(a_s^3(Q^2))\,.$$

The Q^2-evolution of the PDF moments can be calculated within the framework of perturbative QCD:

$$\frac{\mathbf{f}(n, Q^2)}{\mathbf{f}(n, Q_0^2)} = \left[\frac{a_s(Q^2)}{a_s(Q_0^2)}\right]^{\frac{\gamma_0(n)}{2\beta_0}} \times \frac{h(n, Q^2)}{h(n, Q_0^2)}\,,$$

$$h^{NS}(n, Q^2) = 1 + a_s(Q^2)Z_1(n) + a_s^2(Q^2)Z_2(n) + \mathcal{O}\left(a_s^3(Q^2)\right)\,,$$

with

$$\begin{aligned} Z_1(n) &= \frac{1}{2\beta_0}[\gamma_1(n) - \gamma_0(n)\, b_1]\,, \\ Z_2(n) &= \frac{1}{4\beta_0}\left[\gamma_2(n) - \gamma_1(n)b_1 + \gamma_0(n)(b_1^2 - b_2)\right] + \frac{1}{2}Z_1^2(n)\,. \end{aligned}$$

Here $\gamma_k(n)$ $(k = 0, 1, 2)$ are so-called anomalous dimensions of Wilson operators.

2.3 Factorization μ_F and renormalization μ_R scales

It is good to consider the dependence of results on the factorization μ_F and renormalization μ_R scales, caused by the truncation of a perturbative series while doing the calculus.
A modification is achieved by replacing $a_s(Q^2)$ by $a_s(\mu_F^2)$ in the coefficient function C and $a_s(\mu_R^2)$ in the PDF evolution (i.e. in h), where $\mu_F^2 = k_F Q^2, \quad \mu_R^2 = k_R \mu_F^2 = k_R k_F Q^2$.

Then,

$$M_(n, Q^2) = R(f) \times \hat{C}(n, a_s(k_F Q^2)) \times \mathbf{f}(n, k_F Q^2),$$

and

$$\frac{\mathbf{f}(n, k_F Q^2)}{\mathbf{f}(n, k_F Q_0^2)} = [\frac{a_s(k_F k_R Q^2)}{a_s(k_F k_R Q_0^2)}]^{\gamma_0(n)/2\beta_0} \times \frac{\hat{h}(n, k_F k_R Q^2)}{\hat{h}(n, k_F k_R Q_0^2)}.$$

The function $\hat{C}$ can be obtained from C as follows:

$$
\begin{aligned}
a_s(Q^2) \ &\rightarrow \ a_s(k_F Q^2)\,, \quad B_1(n) \rightarrow B_1(n) + \frac{1}{2}\gamma_0(n) \ln k_F\,, \\
B_2(n) \ &\rightarrow \ B_2(n) + \frac{1}{2}\gamma_1(n) \ln k_F + \left(\frac{1}{2}\gamma_0 + \beta_0\right) B_1 \ln k_F \\
&+ \ + \frac{1}{8}\gamma_0\left(\gamma_0 + 2\beta_0\right) \ln^2 k_F\,,
\end{aligned}
$$

and function $\hat{h}$ can be obtained from h as follows:

$$
\begin{aligned}
a_s(Q^2) \ &\rightarrow \ a_s(k_F k_R Q^2), \quad a_s(Q_0^2) \ \rightarrow \ a_s(k_F k_R Q_0^2)\,, \\
Z_1(n) \ &\rightarrow \ Z_1(n) + \frac{1}{2}\gamma_0(n) \ln k_R \\
Z_2(n) \ &\rightarrow \ Z_2(n) + \frac{1}{2}\gamma_1(n) \ln k_R + \frac{1}{2}\gamma_0(n) Z_1 \ln k_R
\end{aligned}
$$

Below we will perform firstly the standart analyses:
the first analysis with $k_F = k_R = 1$ (as a basic one) to extract $a_s(M_z)$ and the high-twist (HT) corrections, i.e. $\tilde{h}(x)$;
several analyses with $k_R = 1/2$, $k_R = 2$, $k_F = 1/2$, $k_F = 2$ to estimate the contribution of higher-orders of perturbation theory, i.e. the values of the theoretic error for $a_s(M_z)$.

Additionaly, we will perform the analyses (1) with CT threshold resummation and also with (2) $k_R = 1$ and n-dependent values of k_F. At large n values, $\gamma_i(n) \sim \ln n$ and $B_i(n) \sim \ln^{2k} n$.
So, $B_i(n)$ rise strongly at large n values. We would like to resume large terms (by using n-dependent values of k_F) and study an influence of the resummation on $\alpha_S(M_Z)$ and HT terms.

To study it, we will use the DIS scheme (Altarelli:1978), (a former Λ_n-scheme (Bace:1978), (Bardeen et al.:1978), (Buras:1980)), the massless W^2-evolution, where $W^2 = Q^2\frac{1-x}{x} + M_p^2$, with
$M_p = 0$ and the scheme-invariant (SI) perturbation theory (SIPT), i..e Grunberg effective charge method (Grunberg:1981,1982).

2.4 Threshold resummation

Threshold (or Sudakov) resummation approach (Catani,Trentadue:1989) resums the large x (and, thus, large n) terms in the coefficient function $C(n, a_s(Q^2))$. The resummation leads to the following representation for $C(n, a_s(Q^2))$ (hereafter we call it Catani-Trentadue (CT) threshold resummation)

$$C(n, a_s(Q^2)) = \tilde{C}(n, a_s(Q^2)) \exp[G(n, Q^2)]$$

where $\tilde{C}(n, a_s(Q^2))$ does not have large n terms and $G(n, Q^2)$ has in the DIS case the following form

$$G(n, Q^2) = \int_0^1 dz \frac{z^n - 1}{1 - z} \left[\int_{Q^2}^{Q^2(1-z)} \frac{dq^2}{q^2} A^{\mathrm{th}}(a_s(q^2)) + B^{\mathrm{th}}(a_s(Q^2(1 - z))) \right],$$

where A^{th} and B^{th} can be represented as series

$$A^{\mathrm{th}}(a_s) = \sum_{i=1} A_i^{\mathrm{th}} a_s^i, \quad B^{\mathrm{th}}(a_s) = \sum_{i=1} B_i^{\mathrm{th}} a_s^i$$

$$A_1^{\mathrm{th}} = 4C_F, \quad A_2^{\mathrm{th}} = 8C_F\left[\left(\frac{67}{18} - \zeta_2\right) C_A - \frac{5f}{9}\right],$$
$$B_1^{\mathrm{th}} = -3C_F, \quad B_2^{\mathrm{th}} = C_F^2\, B_{2F}^{\mathrm{th}} + C_F C_A\, B_{2A}^{\mathrm{th}} + C_F f\, B_{2f}^{\mathrm{th}}\,,$$

with

$$B_{2F}^{\mathrm{th}} = -\frac{3}{2} + 12\zeta_2 - 24\zeta_3, \quad B_{2A}^{\mathrm{th}} = -\frac{3155}{54} + \frac{40}{3}\zeta_2 + 40\zeta_3,$$
$$B_{2f}^{\mathrm{th}} = \frac{247}{27} - \frac{8}{3}\zeta_2$$

Evaluating the q^2-integral in the r.h.s., we have with the accuracy $O(a_s^2(Q^2))$

$$G(n, Q^2) = \int_0^1 dz \frac{z^n - 1}{1 - z}\left[(B_1^{\mathrm{th}} + A_1^{\mathrm{th}}\, L_z) a_s(Q^2)\right.$$
$$\left. + (B_2^{\mathrm{th}} + (A_2^{\mathrm{th}} - \beta_0 B_1^{\mathrm{th}})\, L_z - \frac{\beta_0}{2} A_1^{\mathrm{th}}\, L_z^2) a_s^2(Q^2)\right], \quad L_z = \ln(1 - z)\,.$$

Then by using

$$\int_0^1 dx\, x^n \left(\frac{\ln^{k-1}(1-x)}{1-x}\right)_+ \equiv \int_0^1 dx\, (x^n - 1)\, \frac{\ln^{k-1}(1-x)}{(1-x)} = \frac{(-1)^k}{k}\, D_k(n)\,,$$

where

$$D_1(n) = S_1(n), \quad D_2(n) = S_1^2(n) + S_2(n),$$
$$D_3(n) = S_1^3(n) + 3S_2(n)S_1(n) + 2S_3(n),$$
$$D_4(n) = S_1^4(n) + 6S_2(n)S_1^2(n) + 8S_3(n)S_1(n) + 6S_4(n)\,.$$

with the harmonic sums

$$S_i(n) = \sum_{k=1}^{n} \frac{1}{k^i},\ S_1(n) = \ln n + \gamma_{\mathrm{E}} + O(1/n),\ S_i(n) = \zeta_i + O(1/n),\ (i = 2, \ldots)$$

we have

$$G(n, Q^2) = C_1^{\mathrm{th}}(n) a_s(Q^2) + C_2^{\mathrm{th}}(n) a_s^2(Q^2),$$

where

$$C_1^{\mathrm{th}}(n) = \frac{1}{2} A_1^{\mathrm{th}}\, D_2(n) - B_1^{\mathrm{th}}\, D_1(n)\,,$$
$$C_2^{\mathrm{th}}(n) = \frac{1}{2} (A_2^{\mathrm{th}} - \beta_0 B_1^{\mathrm{th}})\, D_2(n) - B_2^{\mathrm{th}}\, D_1(n) - \frac{\beta_0}{6}\, D_3(n)\,.$$

The results for $\tilde{C}(n, a_s(Q^2))$ can be represented in the form

$$\tilde{C}(n, a_s(Q^2)) = 1 + a_s(Q^2)\tilde{B}_1(n) + a_s^2(Q^2)\tilde{B}_2(n) + \mathcal{O}(a_s^3(Q^2))\,,$$

where

$$\tilde{B}_1(n) = B_1(n) - C_1^{\mathrm{th}}(n),$$
$$\tilde{B}_2(n) = B_2(n) - C_2^{\mathrm{th}}(n) + \frac{1}{2}\left(C_1^{\mathrm{th}}(n)\right)^2 - B_1(n)\,C_1^{\mathrm{th}}(n),$$

It is seen that the terms $\tilde{B}_i(n)$ $(i = 1, 2)$ have no rising terms at $n \to \infty$. Indeed, for example,

$$\tilde{B}_1(n) = -2C_F[2S_2(n) + \frac{1}{n(n+1)}\,S_1(n) + \frac{9}{2} - \frac{3}{2n} - \frac{2}{n+1} - \frac{1}{n^2}]\,.$$

Thus, in this case we resum all the large x contributions without modifying the scale of the strong coupling constant.

2.5 Gottfried sum rule

The CT threshold resummation violates the Gottfried sum rule (Gottfried:1967) i.e. $B_1(n=1)=0$, in this case it is recovered only after re-expansing the term $\exp[G(n,Q^2)]$, since both terms $\tilde{B}_1(n=1)$ and $G(n=1,Q^2)$ have nonzero values.

To restore Gottfried's sum rule, we slightly modify the CT threshold summation as follows:

$$C(n,a_s(Q^2)) = \overline{C}(n,a_s(Q^2))\exp[G(n-1,Q^2)]\,,$$

since $G(n=0,Q^2)=0$. We will call this form as modified CT (mCT) threshold resummation.

The results for $\overline{C}(n,a_s(Q^2))$ can be represented in the form

$$\overline{C}(n,a_s(Q^2)) = 1 + a_s(Q^2)\overline{B}_1(n) + a_s^2(Q^2)\overline{B}_2(n) + \mathcal{O}(a_s^3(Q^2))\,,$$

where

$$\overline{B}_1(n) = B_1(n) - C_1^{\rm th}(n-1),$$
$$\overline{B}_2(n) = B_2(n) - C_2^{\rm th}(n-1) + \frac{1}{2}\left(C_1^{\rm th}(n-1)\right)^2 - B_1(n)\,C_1^{\rm th}(n-1)\,.$$

As in the case of $\tilde{B}_i(n)$ $(i=1,2)$, the terms $\overline{B}_i(n)$ $(i=1,2)$ have no rising terms at $n\to\infty$. Indeed, for example,

$$\overline{B}_1(n) = -2C_F[2S_2(n) - (\frac{1}{n} + \frac{1}{n+1})\,S_1(n) + \frac{9}{2} - \frac{3}{n} - \frac{2}{n+1} - \frac{1}{n^2}]\,.$$

Moreover, $\overline{B}_1(n=1)=0$.

2.6 DIS scheme

Let us consider the case of the so-called DIS-scheme (Altarelli:1978), (it was alternatively called the Λ_n-scheme (Bace:1978), (Bardeen et al.:1978), (Buras:1980))), where NLO corrections to the Wilson coefficients are completely compensated by changing the factorization scale.

NLO

$$a_s(Q^2) \to a_s(k_{\rm DIS}(n)Q^2) \equiv a_n^{\rm DIS}(Q^2),$$
$$k_{\rm DIS}(n) = exp\left(\frac{-2B_1(n)}{\gamma_0(n)}\right) = exp\left(\frac{-r_1^{\rm DIS}(n)}{\beta_0}\right),$$

where

$$r_1^{\rm DIS}(n) = \frac{2B_1(n)\beta_0}{\gamma_0(n)} = \frac{B_1(n)}{d(n)}$$

and $B_1(n) \to B_1^{\rm DIS}(n) = 0$, i.e. $\hat{C}(n, a_{1,n}^{\rm DIS}(Q^2)) = 1 + \mathcal{O}((a_{1,n}^{\rm DIS})^2)$.

The NLO coupling constant $a^{\rm DIS}_{1,n}(Q^2)$ obeys the following equation

$$\frac{1}{a^{\rm DIS}_{1,n}(Q^2)} - \frac{1}{a_1(M_Z^2)} + b_1 \ln\left[\frac{a^{\rm DIS}_{1,n}(Q^2)}{a_1(M_Z^2)} \frac{(1+b_1 a_1(M_Z^2))}{(1+b_1 a^{\rm DIS}_{1,n}(Q^2))}\right]$$
$$= \beta_0 \ln\left(\frac{k_{\rm DIS}(n) Q^2}{M_Z^2}\right) = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right) - r_1^{\rm DIS}(n)\,,$$

which can be obtained from th standart one by substituting $Q^2 \to k_{\rm DIS}(n) Q^2$.

Hereafter the condition that coupling constants in all schemes coincide at $Q^2 = M_Z^2$ is satisfied.

NNLO

$$B_1(n) \to B_1^{\rm DIS}(n) = 0,$$
$$B_2(n) \to B_2^{\rm DIS}(n) = B_2(n) - \left(\frac{1}{2} + \frac{\beta_0}{\gamma_0(n)}\right) B_1^2(n) - \frac{\gamma_1(n)}{\gamma_0(n)} B_1(n)\,.$$

The larger terms $\sim \ln^4(n)$ and $\sim \ln^3(n)$ are cancelled in $B_1^{\rm DIS}(n)$.

We have

$$\hat{C}(n, a_{2,n}^{\mathrm{DIS}}(Q^2)) = 1 + B_2^{\mathrm{DIS}}(n)\left(a_{2,n}^{\mathrm{DIS}}\right)^2 + \mathcal{O}((a_{2,n}^{\mathrm{DIS}})^3)$$

and the NNLO coupling constant $a_{2,n}^{\mathrm{DIS}}(Q^2)$ obeys the equation

$$\frac{1}{a_{2,n}^{\mathrm{DIS}}(Q^2)} - \frac{1}{a_2(M_Z^2)} + b_1 \ln\left[\frac{a_{2,n}^{\mathrm{DIS}}(Q^2)}{a_2(M_Z^2)}\sqrt{\frac{1 + b_1 a_2(M_Z^2) + b_2 a_2^2(M_Z^2)}{1 + b_1 a_{2,n}^{\mathrm{DIS}}(Q^2) + b_2 (a_{2,n}^{\mathrm{DIS}})^2(Q^2)}}\right]$$
$$+ \left(b_2 - \frac{b_1^2}{2}\right) \times \left(I(a_{2,n}^{\mathrm{DIS}}(Q^2)) - I(a_s(M_Z^2))\right) = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right) - r_1^{\mathrm{DIS}}(n)\,,$$

which can be obtained from the standart one by substituting $Q^2 \to k_{\mathrm{DIS}}(n) Q^2$.

Thus, in DIS scheme, the NLO coefficient $B_1(n)$ is exactly compensated by changing the strong coupling argument. Moreover, for large values of n, the NNLO coefficient $B_2^{\mathrm{DIS}}(n)$ contains only terms $\sim \ln^2 n$ whilst the leading terms of the form $\sim \ln^4 n$ and $\sim \ln^3 n$ are cancelled out.

2.7 W^2 scheme

Here we use the fact that in NLO, the basic contibution in the x-space looks like (Bardeen et al.:1978), (Kubar-Andre,Paige:1979)))

$$\tilde{B}_1^W(x) = -\frac{1}{2}\tilde{\gamma}_0(x) \ln \overline{k}_W(x)\,, \quad \overline{k}_W(x) = \frac{1-x}{x}\,, \tag{1}$$

where the quantities marked by tilde do not contain the contribution coming from sum rules. Our standard Wilson coefficient $B_1(n)$ and $\gamma_0(n)$ are obtained from $\tilde{B}_1(n)$ and $\tilde{\gamma}_0(n)$ by adding the sum rule conditions as follows

$$B_1(n) = \tilde{B}_1(n) - \tilde{B}_1(n=1), \quad \gamma_0(n) = \tilde{\gamma}_0(n) - \tilde{\gamma}_0(n=1)\,,$$

with

$$\tilde{B}_1(n) = \int_0^1 dx x^{n-1}\, \tilde{B}_1(x), \quad \tilde{\gamma}_0(n) = \int_0^1 dx x^{n-1}\, \tilde{\gamma}_0(x)\,.$$

The scale $\overline{k}_W(x)$ is related to the massless limit of the effective mass W of a photon–proton cluster

$$W^2 = Q^2 \frac{1-x}{x} + M_P^2 \,,$$

where M_p is a proton mass. In the massless limit ($M_P = 0$) $W^2 = \overline{k}_W(x)\, Q^2$; thus, in this subsection, we use the scale $\overline{k}_W(x)$ in the calculations:

$$Q^2 \to \overline{k}_W(x) Q^2 \,.$$

Since we work in the Mellin moment space, we have $k_W(n)$, which corresponds to $\overline{k}_W(x)$ in the x-space, i.e.

$$Q^2 \to k_W(n) Q^2 \,.$$

NLO In this order,

$$a_s(Q^2) \to a_s(k_W(n)Q^2) \equiv a^W_{1,n}(Q^2),$$

$$k_W(n) = exp\left(\frac{-2B^W_1(n)}{\gamma_0(n)}\right) = exp\left(\frac{-r^W_1(n)}{\beta_0}\right),$$

where

$$r^W_1(n) = \frac{2B^W_1(n)\beta_0}{\gamma_0(n)} = \frac{B^W_1(n)}{d(n)}$$

and

$$\hat{C}(n, a_s(Q^2)) = 1 + a^W_{1,n}(Q^2)\hat{B}^W_1(n) + \mathcal{O}((a^W_{1,n})^2)\,.$$

Here

$$\hat{B}^W_1(n) = B_1(n) - B^W_1(n),$$

with

$$B^W_1(n) = \tilde{B}^W_1(n) - \tilde{B}^W_1(n=1), \quad \tilde{B}^W_1(n) = \int_0^1 dx x^{n-1}\,\tilde{B}^W_1(x),$$

and $\tilde{B}^W_1(x)$ is defined above.

Then, we have ($C_F=(N^2-1)/(2N)$ for SU(N) group)

$$B_1^W(n)=2C_F\left[S_1^2(n)+S_2(n)-\frac{1}{n(n+1)}S_1(n)-\frac{7}{4}\right],\quad S_a(n)=\sum_{m=1}^{n}\frac{1}{m^a},$$

$$\hat{B}_1^W(n)=2C_F\left[-2S_2(n)+\frac{3}{2}S_1(n)-\frac{11}{4}+\frac{3}{2n}+\frac{2}{n+1}+\frac{1}{n^2}-\frac{1}{(n+1)^2}\right].$$

At large x values (and at large n values, respectively), when (ζ_i are Euler ζ-functions)

$$S_1(n)=\ln n+\gamma_{\mathrm{E}}+O(1/n),\quad S_i(n)=\zeta_i+O(1/n),\quad (i=2,...),$$

we have

$$\hat{B}_1^W(n)=2C_F\left[-2\zeta_2+\frac{3}{2}\ln n-\frac{11}{4}+O(n^{-1})\right]\sim 3C_F\ln n+O(n^0)$$

and the results look very similar to those obtained in the previous subsection.

Note that the NLO coupling constant $a_{1,n}^W(Q^2)$ obeys same equation as $a_{1,n}^{DIS}(Q^2)$ where the replacement $r_1^{\mathrm{DIS}}(n)\to r_1^W(n)$ is done.

NNLO Here,

$$a_s(Q^2) \to a_s(k_W(n)Q^2) \equiv a^W_{2,n}(Q^2),$$

and

$$\hat{C}(n, a_s(Q^2)) = 1 + a^W_{2,n}(Q^2)\hat{B}^W_1(n) + \left(a^W_{2,n}(Q^2)\right)^2 \hat{B}^W_2(n) + \mathcal{O}((a^W_{2,n})^3)\,,$$

with $B^W_1(n)$ and $\hat{B}^W_1(n)$ given above and

$$\hat{B}^W_2(n) = B_2(n) - \left(\frac{1}{2} + \frac{\beta_0}{\gamma_0(n)}\right) B^W_1(n)[2B_1(n) - B^W_1(n)] - \frac{\gamma_1(n)}{\gamma_0(n)} B^W_1(n)\,.$$

Since the DIS-scheme and W^2-evolution are close to each other, it is convenient to express $\hat{B}^W_2(n)$ through $B^{\mathrm{DIS}}_2(n)$:

$$\hat{B}^W_2(n) = B^{\mathrm{DIS}}_2(n) + \left(\frac{1}{2} + \frac{\beta_0}{\gamma_0(n)}\right) \left(\hat{B}^W_1(n)\right)^2 + \frac{\gamma_1(n)}{\gamma_0(n)} \hat{B}^W_1(n)\,,$$

where $B^{\mathrm{DIS}}_2(n)$ is given above.

Note that the leading terms $\sim \ln^4(n)$ are cancelled out in $B^W_2(n)$.

The NNLO coupling $a^W_{2,n}(Q^2)$ obeys same equation as $a^{DIS}_{2,n}(Q^2)$ where the replacement $r^{\mathrm{DIS}}_1(n) \to r^W_1(n)$ is carried out.

Thus, in the W^2-evolution, for large n values the NLO and NNLO coefficients $\hat{B}_1^W(n)$ and $\hat{B}_2^W(n)$ contain only the terms $\sim \ln n$ and $\sim \ln^3 n$, respectively. The most important terms are completely cancelled.

Note that $k_W(n)$ (or equally well $\overline{k}_W(x)$) can be multiplied by an additional factor e^δ:

$$\overline{k}_\delta(x) = e^\delta\, \overline{k}_W(x), \quad k_\delta(n) = e^\delta\, k_W(n)\,.$$

In the case when $\delta = -3/4$, we have

$$\hat{B}_1^{(\delta=-3/4)}(n) = 2C_F\left[-2S_2(n) - \frac{13}{8} + \frac{9}{4n} + \frac{5}{4(n+1)} + \frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$$

and $\hat{B}_1^{(\delta=-3/4)}(n) \sim O(n^0)$ at large n values. The corresponding NNLO coefficient $\hat{B}_2^{(\delta=-3/4)}(n) \sim \ln^2 n$ and it is seen that the case $\delta = -3/4$ is very close to the DIS-scheme.

2.8 Grunberg approach

In this subsection, we consider the Grunberg effective charge method, which is a fairly popular approach. In a sense, it is closely related to the so-called scheme-invariant perturbation theory (SIPT), which is as well widely in use (see, for example, (Vovk: 1990), (Kotikov,Parente,Sanchez Guillen:1993), (Parente, Kotikov, Krivokhizhin: 1994)).

In this approach, all contributions beyond LO are completely canceled by changes in the factorization and renormalization scales, while beyond NLO it is also required to modify the coefficients β_i $(i \geq 2)$ of the QCD β function, i.e. $\beta_i \; (i \geq 2) \to \; \tilde{\beta}_i(n)$.

In order to apply Grunberg approach, it is rather convenient to rewrite the Q^2-dependence of SF moments as follows

$$M(n, Q^2) = \frac{\overline{C}(n, a_s(Q^2))}{\overline{C}(n, a_s(Q_0^2))} \cdot M(n, Q_0^2)\,,$$

where

$$\overline{C}(n, a_s(Q^2)) = a_s^{d(n)}(Q^2)\left[1 + C_1 a_s(Q^2) + C_2 a_s^2(Q^2) + ...\right]$$

contains contributions coming from both coefficient function $C(n, a_s(Q^2))$ and PDF evolution. Indeed,

$$C_1(n) = B_1(n) + Z_1(n), \quad C_2(n) = B_2(n) + Z_1(n)B_1(n) + Z_2(n)\,.$$

The normalization $M_n(Q_0^2)$ is linked with $\mathbf{f}_{NS}(n, Q_0^2)$ as

$$M_n(Q_0^2) = R(f) \cdot \overline{C}(n, a_s(Q_0^2)) \cdot \mathbf{f}(n, Q_0^2)\,. \qquad (2)$$

The effective charge method:

An observable

$$m(Q^2) = a_s(Q^2)[1 + \tilde{c}_1 a_s(Q^2) + \tilde{c}_2 a_s^2(Q^2) + \tilde{c}_3 a_s^3(Q^2) + ...]$$

can be considered as a new coupling constant, containing the new scale $Q^2 \to e^{-\tilde{c}_1/\beta_0} Q^2$ and new (nonuniversal!) coefficients β_i $(i \geq 2)$ of β-function: $\beta_i \to \tilde{\beta}_i$ and $\tilde{\beta}_i$ depend on $\tilde{c}_i$

.

In our case,

$$M(n, Q^2) \sim \left(a_s(Q^2)\right)^{d(n)} [1 + ...]$$

and we can use $m_n(Q^2) = \left(M_n(Q^2)\right)^{1/d(n)}$, where $d(n) = \gamma_0(n)/(2\beta_0)$.

NLO In this order, the strong coupling constant is modified as follows:

$$a_1(Q^2) \to a_1(k_{\rm SI}(n)Q^2) \equiv a_{1,n}(Q^2),$$
$$k_{\rm SI}(n) = exp\left(\frac{-2C_1(n)}{\gamma_0(n)}\right) = exp\left(\frac{-r_1(n)}{\beta_0}\right),$$

where

$$r_1(n) = \frac{2C_1(n)\beta_0}{\gamma_0(n)} = \frac{C_1(n)}{d(n)}.$$

With the above choice of the scale, we have

$$C_1(n) = 0\,, \quad \text{i.e.} \;\; \overline{C}(n) = a_{1,n}^{d(n)}(Q^2)\left[1 + O(a_{1,n}^2)\right].$$

The NLO coupling $a_{1,n}(Q^2)$ obeys same equation as $a_{1,n}^{DIS}(Q^2)$ with the replacement $r_1^{\rm DIS}(n) \to r_1(n)$ is done.

NNLO Here,

$$a_2(Q^2) \to a_2(k_{\rm SI}(n)Q^2)) \equiv a_{2,n}(Q^2),$$

where the NNLO coupling $a_{2,n}(Q^2)$ obeys the following equation

$$\frac{1}{a_{2,n}(Q^2)} - \frac{1}{a_2(M_Z^2)} + b_1 \ln\left[\frac{a_{2,n}(Q^2)}{a_2(M_Z^2)}\sqrt{\frac{1+b_1 a_2(M_Z^2)+b_2 a_2^2(M_Z^2)}{1+b_1 a_{2,n}(Q^2)+\tilde{b}_2(n) a_{2,n}^2(Q^2)}}\right]$$
$$+\left(\tilde{b}_2(n) - \frac{b_1^2}{2}\right)\cdot(\tilde{I}(a_{2,n}(Q^2)) - \tilde{I}(0)) - \left(b_2 - \frac{b_1^2}{2}\right)\cdot(I(a_s(M_Z^2)) - I(0))$$
$$= \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right) - r_1(n)\,,$$

where

$$\tilde{I}(a_{2,n}(Q^2)) = I(a_2(Q^2) \to a_{2,n}(Q^2), b_2 \to \tilde{b}_2(n))\,.$$

A modified factor is found to be

$$\tilde{b}_2(n) = \frac{\tilde{\beta}_2(n)}{\beta_0}, \quad \tilde{\beta}_2(n) = \beta_2 - r_1(n)\,\beta_1 + (r_2(n) - r_1^2(n))\,\beta_0,$$

with

$$r_2(n) = \frac{C_2(n)}{d(n)} - \frac{d(n)-1}{2}\,r_1^2(n)\,.$$

With the above choice of the scale, we have

$$C_1(n) \to 0, \quad C_2(n) \to 0,$$

i.e. at the NNLO level:

$$\overline{C}(n) = a_{2,n}^{d(n)}(Q^2)\,[1 + O(a_{2,n}^3)]\,.$$

3. A fitting procedure

With the QCD expressions for the Mellin moments $M(n, Q^2)$ analytically calculated according to the formulas, given above the SF $F_2(x, Q^2)$ is reconstructed by using the Jacobi polynomial expansion method:

$$F_2^{\mathrm{LT}}(x, Q^2) = x^a(1-x)^b \sum_{n=0}^{N_{max}} \Theta_n^{a,b}(x) \sum_{j=0}^{n} c_j^{(n)}(a,b) M(j+2, Q^2)\,,$$

where $\Theta_n^{a,b}$ are the Jacobi polynomials:

$$\Theta_n^{a,b}(x) = \sum_{j=0}^{n} c_j^{(n)}(a,b)\, x^j\,,$$

a, b are the parameters fitted. A condition is the requirement of the error minimization while reconstructing the structure functions.

MINUIT program is used to minimize the variable

$$\chi^2_{SF} = |\frac{F_2^{exp} - F_2^{th}}{\Delta F_2^{exp}}|^2\,.$$

The moments $M(n, Q_0^2)$ and $f(n, Q_0^2)$ at some Q_0^2 are theoretical input to the analysis. In the fits of data with the cut $x \geq 0.25$ imposed, only the nonsinglet F_2 structure function and the nonsinglet parton density is worked with and the following patametrization at the normalization point is used

$$M(n, Q_0^2) \;=\; \int_0^1 dx x^{n-2} F_2^{\mathrm{LT}}(x, Q_0^2), \quad f(n, Q_0^2) \;=\; \int_0^1 dx x^{n-2}\, \tilde{f}(x, Q_0^2)$$

with

$$F_2^{\mathrm{LT}}(x, Q_0^2) = \tilde{A}\, x^{\tilde{\lambda}}(1-x)^{\tilde{b}}, \quad \tilde{f}(x, Q_0^2) = A x^{\lambda}(1-x)^{b},$$

where $\tilde{A}$, $\tilde{\lambda}$ and $\tilde{b}$ and also A, λ and b are the parameters that need to be fitted within the Grunberg approach and other approaches, respectively.

4. Results

We use free data normalizations for various experiments. The most stable BCDMS hydrogen data are used as a reference set at the initial beam energy value $E_0 = 200$ GeV. Unlike in our previous analyses, the cut $Q^2 \geq 2$ GeV^2 is used throughout, since for lower Q^2 values equations for coupling constants have no real solutions.

The starting point of the Q^2 evolution is the value of $Q_0^2 =$ 90 GeV^2, which is close to the average Q^2 value (on a logarithmic scale) of the data under study. Based on previous investigations (see (Krivokhizhin et al.: 1987,1990)), the maximum number of moments used in the analyses is N_{max} =8. The cut $0.25 \leq x \leq 0.8$ is also imposed on the data.

We work within the framework of the variable flavor number scheme (VFNS). The threshold crossing point is taken at $Q_f^2 = m_f^2$. In order to emphasize the effect of changing the sign for twist four corrections, the results obtained in the fixed flavor number scheme (FFNS) with $n_f = 4$ are shown as well.

Following our previous analysis carried out in (Kotikov,Krivokhizhin,Shaitkhatden
2022)), we expect a large x resummation only slightly change the strong coupling normalization, at the same time greatly modify the twist four values. Since the latter depend significantly on which data to be analyzed, here we will limit ourselves to dealing with exclusively hydrogen data.

Bellow we examine the effect of resumming large x logarithmic contributions using three different resummation procedures discussed above. But first, we show the results of standard analysis and their dependence on the factorization μ_F and renormalization μ_R scales.

Table 1. Twist four $\tilde{h}_4(x)$ parameter values obtained while fitting hydrogen data (total 314 points, $Q^2 \geq 2$ GeV^2). Calculations are carried out within VFNS (FFNS).

x	NLO $\chi^2 = 246(259)$ $\alpha_s(M_Z^2) = 0.1195$ (0.1192)	NNLO $\chi^2 = 241(254)$ $\alpha_s(M_Z^2) = 0.1177$ (0.1170)
0.275	-0.25±0.02 (-0.26±0.03)	-0.19±0.02 (-0.20±0.02)
0.35	-0.24±0.02 (-0.25±0.02)	-0.19±0.03 (-0.19±0.02)
0.45	-0.19±0.02 (-0.19±0.02)	-0.17±0.03 (-0.16±0.01)
0.55	-0.12±0.03 (-0.10±0.03)	-0.17±0.05 (-0.14±0.03)
0.65	0.05±0.08 (0.12±0.08)	-0.14±0.14 (-0.05±0.06)
0.75	0.34±0.12 (0.48±0.12)	-0.11±0.19 (0.06±0.10)

Table 2. Parameter values of $\tilde{f}(x, Q_0^2)$. Calculations are carried out within VFNS (FFNS).

	A	λ	b
NLO	2.51±0.08 (2.51±0.07)	0.02±0.02 (0.02±0.02)	4.04±0.02 (4.04±0.02
NNLO	2.81±0.01 (2.82±0.01)	0.06±0.01 (0.06±0.01)	4.17±0.01 (4.18±0.01

It is seen that the NS QCD analysis of SLAC, NMC and BCDMS experimental data for SFF_2 gives the following result at the reference point:

$$\alpha_s(M_Z^2) = 0.1177 + \left\{ \begin{array}{l} \pm 0.0003 \text{ (stat)} \pm 0.0018 \text{ (syst)} \pm 0.0007 \text{ (norm)} \\ \pm 0.0020 \text{ (total exp.error)} \end{array} \right. .$$

Thus one can observe that these results look quite similar to those presented earlier.

Moreover they are agree with word-average value:

$$\alpha_s(M_Z^2) = 0.1180 \pm 0.0009$$

4.1 Scale dependence

Let us study the dependence of the results on a different choice of factorization $\mu_F = k_F Q^2$ and renormalization $\mu_R = k_R Q^2$ scales. As it is usual, we select three values $(1/2,\ 1,\ 2)$ for the coefficients k_F and k_R.

Results are demonstrated in Table 2. The change in $\alpha_s(M_Z^2)$ value for various k_F and k_R values is denoted by the difference:

$$\Delta\alpha_s(M_Z^2) \;=\; \alpha_s(M_Z^2) - \alpha_s(M_Z^2)|_{k_F=k_R=1}\,.$$

Table 3. NNLO (NLO) $\alpha_s(M_Z^2)$ for a set of k_F and k_R coefficients, (314 points, $Q^2 \geq 2$ GeV^2). Calculations are carried out within VFNS.

k_R	k_F	$\chi^2(F_2)$	$\alpha_s(M_Z^2)$	$\Delta\alpha_s(M_Z^2)$
1	1	241 (246)	0.1177 (0.1195)	0
1/2	1	241 (246)	0.1166 (0.1171)	-0.0011 (-0.0024)
1	1/2	239 (243)	0.1170 (0.1170)	-0.0007 (-0.0025)
1	2	244 (249)	0.1193 (0.1227)	+0.0016 (+0.0032)
2	1	243 (247)	0.1191 (0.1225)	+0.0014 (+0.0030)

As can be seen from this table, the theoretical uncertainties for the maximum and minimum values of the coupling constant corresponding to $k_i = 2$ and $k_i = 1/2$ ($i = F, R$), respectively, are equal to $+0.0021$ ($+0.0044$) and -0.0013 (-0.0035) for the case of NNLO (NLO)

Thus, the present analysis gives the following theoretical error for the result presented above

$$\alpha_s(M_Z^2) = 0.1177 \pm 0.0020 \text{ (total exp.error)} + \left\{ \begin{array}{c} +0.0021 \\ -0.0013 \end{array} \right. \text{(theor)}\,.$$

4.2 CT (mCT) Resummation

Table 4. Same as in Table 1 but carried out within CT threshold resummation.

x	NLO $\chi^2 = 246(228)$ $\alpha_s(M_Z^2) = 0.1256$ (0.1226)	NNLO $\chi^2 = 241(238)$ $\alpha_s(M_Z^2) = 0.1177$ (0.1167)
0.275	0.12±0.02 (0.12±0.06)	-0.22±0.04 (-0.16±0.02)
0.35	0.22±0.06 (0.23±0.06)	-0.14±0.04 (-0.13±0.02)
0.45	0.34±0.06 (0.36±0.06)	-0.13±0.04 (-0.11±0.02)
0.55	0.37±0.06 (0.41±0.06)	-0.22±0.10 (-0.12±0.03)
0.65	0.40±0.06 (0.47±0.06)	-0.20±0.16 (-0.12±0.08)
0.75	0.35±0.12 (0.43±0.12)	-0.40±0.25 (-0.34±0.15)

Table 5. Same as in Table 4 but carried out within mCT threshold resummation.

x	NLO $\chi^2 = 238(248)$ $\alpha_s(M_Z^2) = 0.1244$ (0.1256)	NNLO $\chi^2 = 242(255)$ $\alpha_s(M_Z^2) = 0.1161$ (0.1166)
0.275	-0.23±0.03 (-0.25±0.03)	-0.16±0.02 (-0.19±0.02)
0.35	-0.17±0.03 (-0.19±0.03)	-0.14±0.02 (-0.17±0.02)
0.45	-0.06±0.03 (-0.08±0.02)	-0.12±0.04 (-0.13±0.04)
0.55	0.00±0.05 (0.00±0.04)	-0.12±0.08 (-0.12±0.08)
0.65	0.12±0.10 (0.13±0.07)	-0.13±0.15 (-0.10±0.15)
0.75	0.14±0.20 (0.17±0.13)	-0.37±0.26 (-0.31±0.27)

Table 6. Same as in Table 2 but carried out within CT and mCT threshold resummations.

	A	λ	b
NLO (CT)	2.93±0.10 (2.93±0.09)	0.07±0.02 (0.07±0.02)	4.15±0.03 (4.15±0.03)
NNLO (CT)	2.95±0.08 (3.11±0.01)	0.09±0.02 (0.12±0.01)	4.22±0.03 (4.28±0.01)
NLO (mCT)	2.91±0.10 (2.91±0.09)	0.09±0.02 (0.09±0.02)	4.15±0.03 (4.15±0.03)
NNLO (mCT)	3.08±0.01 (3.02±0.09)	0.11±0.01 (0.10±0.01)	4.26±0.01 (4.25±0.03)

Using the CT and mCT threshold resummations (other applications of the CT resummation can be found in (Corcella,Magnea:2005) (Bonvini et al.:2015)), we repeat our previous analyses in VFNS and FFNS. The results are presented in Tables 4, 5 and 6 and illustrated in Figs. 1 and 2.

In Table 6, we see PDF suppression at large x values when CT and mCT resummations are used. Indeed, the values of the parameter b extracted here are greater than those found earlier and presented in Table 2. This is completely consistent with the results obtained earlier for the CT resummation in (Corcella,Magnea:2005)(Bonvini et al.:2015).

Moreover, Tables 1–5 show that at NNLO the normalized value of the strong coupling constant and the shape of the twist-four corrections, with the exception of a large x range, remain virtually unchanged when switching to the CT and mCT schemes. In the region of large x values, at NNLO the form of twist four corrections varies noticeably. Moreover, it seems that they rise to negative values as $1/(1-x)$ but this observation needs additional investigations.

The situation is more complicated at NLO. In fact, there is a discrepancy between the normalized values of the strong coupling constant in the standard case and in the CT and mCT schemes. Moreover, in the CT threshold resummation the twist four corrections are quite large. Note that these results for the strong coupling constant and the twist four corrections are quite similar in the both schemes: VFNS and FFNS.

Now we consider three other schemes of resummation.

4.3 Resummation

Now we repeat NS QCD analysis performed in this Section above (whose results are shown in Table 1), this time by using the schemes that contain effective resummation of large x logarithms, which, in turn, contribute to the Wilson coefficient functions. These schemes are the DIS-scheme, W^2-evolution and the Grunberg effective charge method, which are presented in detail in Section 4 above.

Table 7. Same as in Table 1 but carried out within VFNS only

x	NLO DIS scheme (W^2-evolution)[SIPT] $\chi^2 = 238(251)[245]$ $\alpha_s(M_Z^2) = 0.1177(0.1179)[0.1178]$	NNLO DIS scheme (W^2-evolution)[SIPT] $\chi^2 = 242(249)[249]$ $\alpha_s(M_Z^2) = 0.1160(0.1163)[0.1178]$
0.275	-0.18±0.01 (-0.17±0.02) [-0.22±0.03]	-0.14±0.01 (-0.13±0.03) [-0.17±0.02]
0.35	-0.11±0.01 (-0.13±0.01) [-0.15±0.02]	-0.13±0.02 (-0.08±0.02) [-0.14±0.02]
0.45	-0.04±0.04 (-0.09±0.01) [-0.07±0.03]	-0.11±0.09 (0.02±0.02) [-0.10±0.02]
0.55	-0.10±0.01 (-0.09±0.04) [-0.12±0.03]	-0.12±0.03 (0.08±0.03) [-0.09±0.02]
0.65	-0.17±0.04 (-0.09±0.05) [-0.14±0.05]	-0.22±0.05 (0.08±0.05) [-0.12±0.04]
0.75	-0.57±0.08 (-0.46±0.18) [-0.51±0.09]	-0.59±0.08 (-0.18±0.10) [-0.42±0.09]

Table 8. Same as in Table 1 but carried out within FFNS only

x	NLO DIS scheme (W^2-evolution)[SIPT] $\chi^2 = 244(243)[245]$ $\alpha_s(M_Z^2) = 0.1180(0.1179)[0.1179]$	NNLO DIS scheme (W^2-evolution)[SIPT] $\chi^2 = 244(245)[245]$ $\alpha_s(M_Z^2) = 0.1160(0.1163)[0.1171]$
0.275	-0.21±0.03 (-0.19±0.03) [-0.22±0.03]	-0.22±0.03 (-0.13±0.03) [-0.22±0.01]
0.35	-0.13±0.02 (-0.11±0.02) [-0.15±0.02]	-0.14±0.02 (-0.08±0.02) [-0.12±0.01]
0.45	-0.05±0.03 (-0.04±0.03) [-0.07±0.03]	-0.03±0.03 (0.02±0.02) [0.02±0.02]
0.55	-0.10±0.03 (-0.10±0.03) [-0.12±0.03]	-0.03±0.03 (0.08±0.03) [0.01±0.04]
0.65	-0.11±0.05 (-0.15±0.05) [-0.14±0.05]	0.00±0.05 (0.08±0.05) [0.05±0.09]
0.75	-0.45±0.09 (-0.52±0.09) [-0.51±0.09]	-0.32±0.09 (-0.18±0.10) [-0.26±0.15]

Table 9. Parameter values of $\tilde{f}(x, Q_0^2)$ carried out in DIS scheme (W^2-evolution) and parameter values of $F_2^{\rm LT}(x, Q_0^2)$ carried out in SIPT within VFNS only

	A	λ	b
NLO	2.97±0.09 (2.97±0.09)	0.09±0.02 (0.09±0.02)	4.24±0.02 (4.26±0.03)
NNLO	3.20±0.01 (3.29±0.13)	0.13±0.01 (0.14±0.02)	4.29±0.01 (4.32±0.03)
	$\tilde{A}$	$\tilde{\lambda}$	$\tilde{b}$
NLO	2.91±0.09	0.08±0.02	4.23±0.03
NNLO	2.66±0.22	0.02±0.04	4.14±0.06

Table 10. Same as in Table 8 but carried out within FFNS only

	A	λ	b
NLO	3.00±0.09 (2.97±0.09)	0.09±0.02 (0.09±0.02)	4.26±0.03 (4.26±0.03)
NNLO	3.02±0.10 (3.29±0.13)	0.10±0.02 (0.14±0.02)	4.25±0.03 (4.32±0.03)
	$\tilde{A}$	$\tilde{\lambda}$	$\tilde{b}$
NLO	2.91±0.09	0.09±0.02	4.26±0.03
NNLO	3.08±0.01	0.11±0.02	4.25±0.03

From Tables 9 and 10 we see PDF suppression at large values of x, which is completely consistent with the results shown in Table 6, as well as with the results obtained earlier in (Corcella,Magnea:2005)(Bonvini et al.:2015). Indeed, the extracted values of the parameter b are higher than in the $\overline{MS}$ case (see Table 2) and coincide (within the uncertainties) with those extracted by the CT and mCT resummations (see Table 6). Such results exist for all the considered cases, with the exception of the Grunberg approach in VFNS, where the value of the parameter b coincides (within the uncertainty limits) with those indicated in Table 2.

Tables 7 and 8 show that when switching to these three schemes, which effectively take into account resummation in the region of large values of x, the normalized value of the strong coupling constant remains almost unchanged, whereas the shapes of the twist four corrections change noticeably and are quite similar in both PT orders.

We believe that the difference in the shape of the twist four

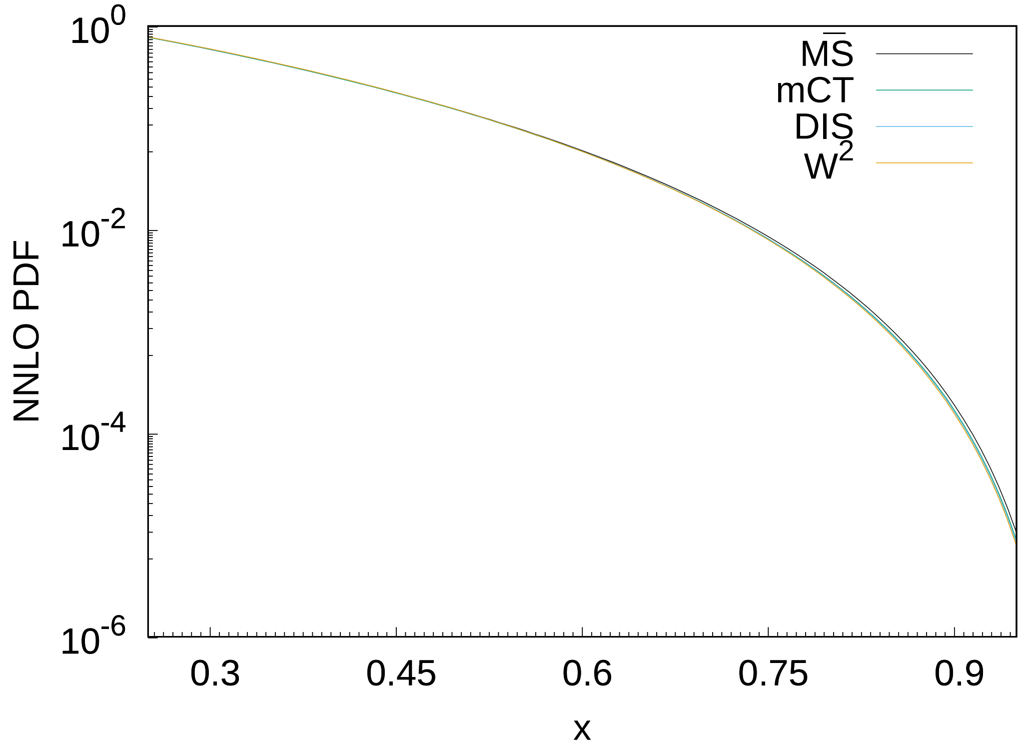

Figure 1: PDF $\tilde{f}(x, Q_0^2)$ curves from Eq. (40) obtained in various approaches.

corrections partially arises in the DIS and W^2 schemes, as well as in the Grunberg approach compared to the standard case, since resumming large contributions at large values of x leads to low values of the arguments of the strong coupling constant, at which the perturbation theory is at the limit of its applicability.

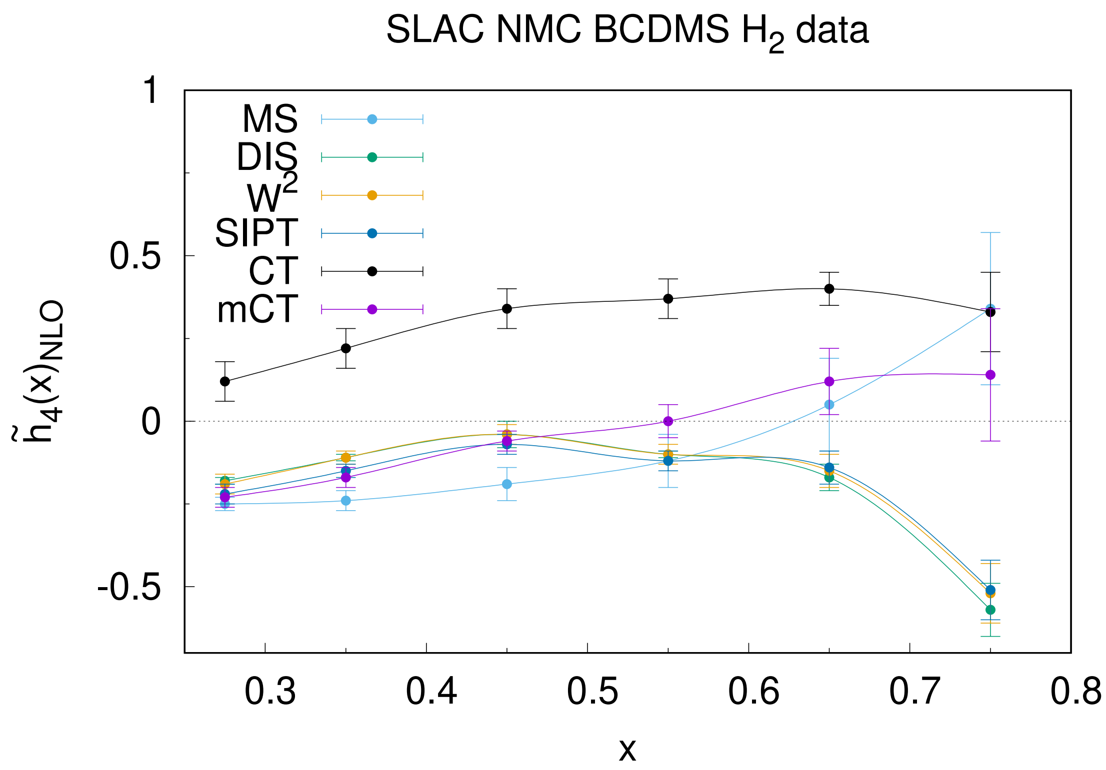

Figure 2: The values of the parameters $\tilde{h}_4(x)$ obtained in the analysis of experimental data within VFNS in NLO and with a conventional definition of the strong coupling constant.

Indeed, used resummation schemes change slightly the twist four terms in the area of relatively small x values while these corrections in the region of large x values change their sign (see also Fig. 1).

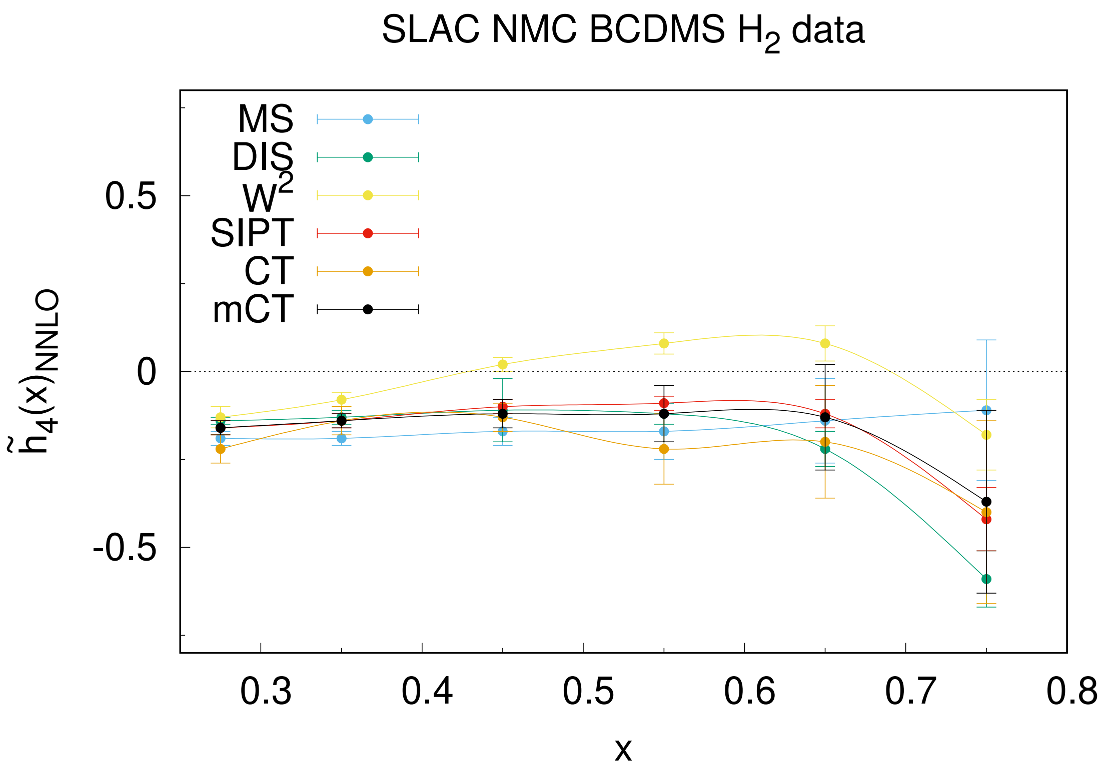

Figure 3: Same as in Fig. 1 but carried out in NNLO.

These changes in the values of twist four corrections are nearly independent (??) of both the chosen resummation scheme and the order of perturbation theory. Moreover, it seems that they rise as $1/(1-x)$ at large x but this observation needs additional investigations.

Such a behavior is in sharp contrast with the analyses performed in $\overline{\mathbf{MS}}$ scheme, where twist four corrections are mostly positive at large x and rise as $1/(1-x)$ (see also Table 1 and Fig. 1).

Negative values of twist four corrections for large x, obtained in schemes with resummation of large x logarithms, can lead to the following phenomenon: at least part of the (negative) power terms can be absorbed by the difference between the usual strong coupling and the analytic one, if we use the analytic coupling constant in our analyses. Of course, such a phenomenon was absent in the case of the $\overline{\mathbf{MS}}$ scheme, where the use of analytic QCD coupling simply increases the magnitude of the twist four corrections.

Note that in previous works (see (Vovk: 1990), (Kotikov,Parente, Sanchez Guillen:1993), (Parente, Kotikov, Krivokhizhin: 1994)), where resummation at large x values was performed within the framework of the Grunberg approach, only decrease in the twist four contributions was observed, since the corresponding contributions were not studied in detail.

4.4 "Frozen" coupling constan

In the soft case (see, for example, (Badelek et al.:1996), the argument of the strong coupling constant is modified. It contains a shift $Q^2 \to Q^2 + M^2$, where M is an additional scale that strongly changes the infrared properties of $\alpha_s(Q^2)$. For massless produced quarks, the value of M is usually taken to be the mass of the ρ meson m_ρ, i.e. $M = m_\rho$. In the case of massive quarks with mass M_i, the value $M_i = 2m_i$ is usually used.

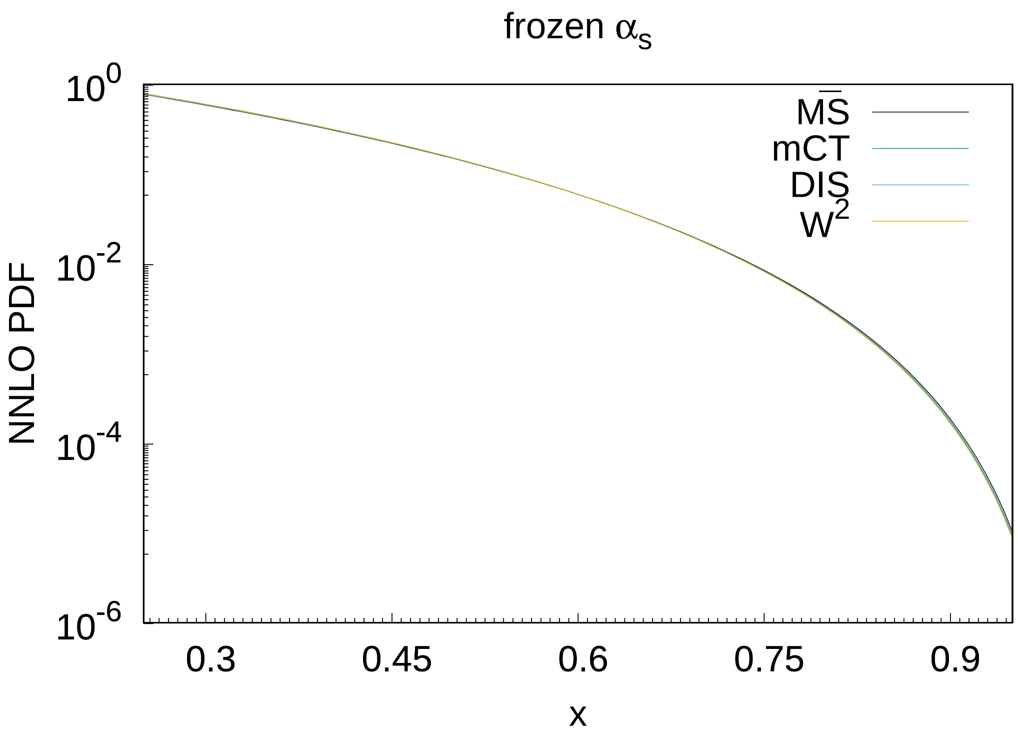

Figure 4: PDF $\tilde{f}(x, Q_0^2)$ curves from Eq. (40) obtained with a "frozen" coupling constant in various appoaches.

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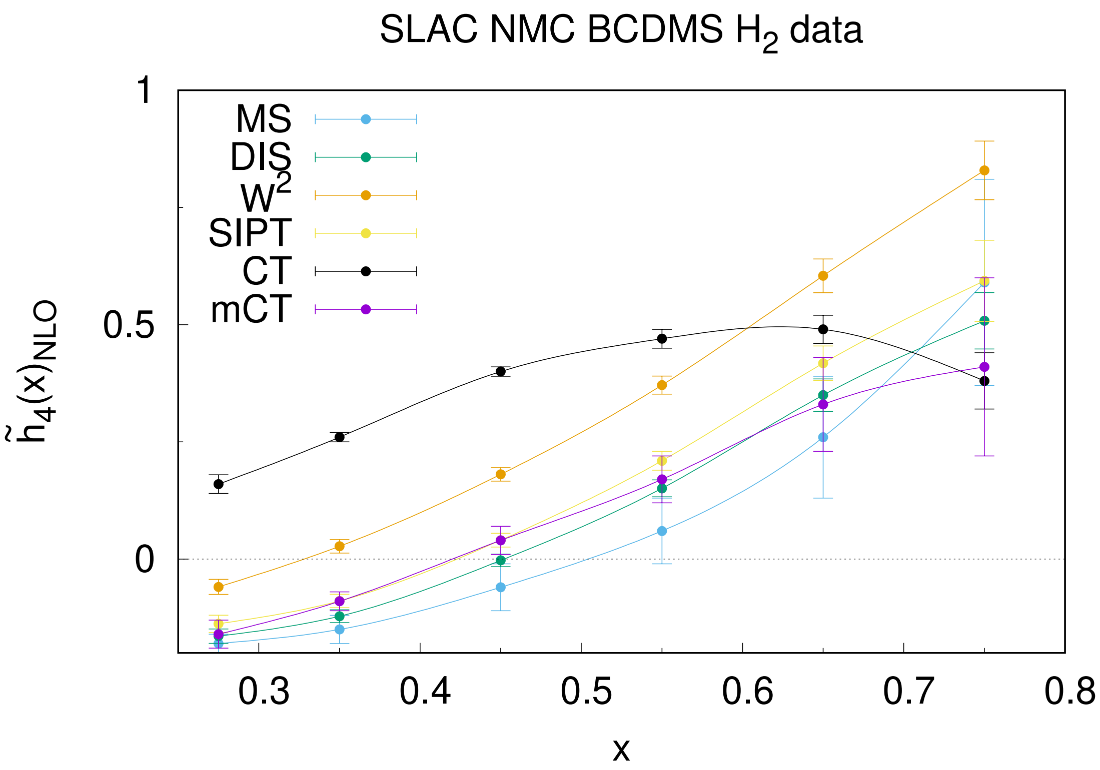

Figure 5: Same as in Fig. 1 but carried out with “frozen” coupling constant.

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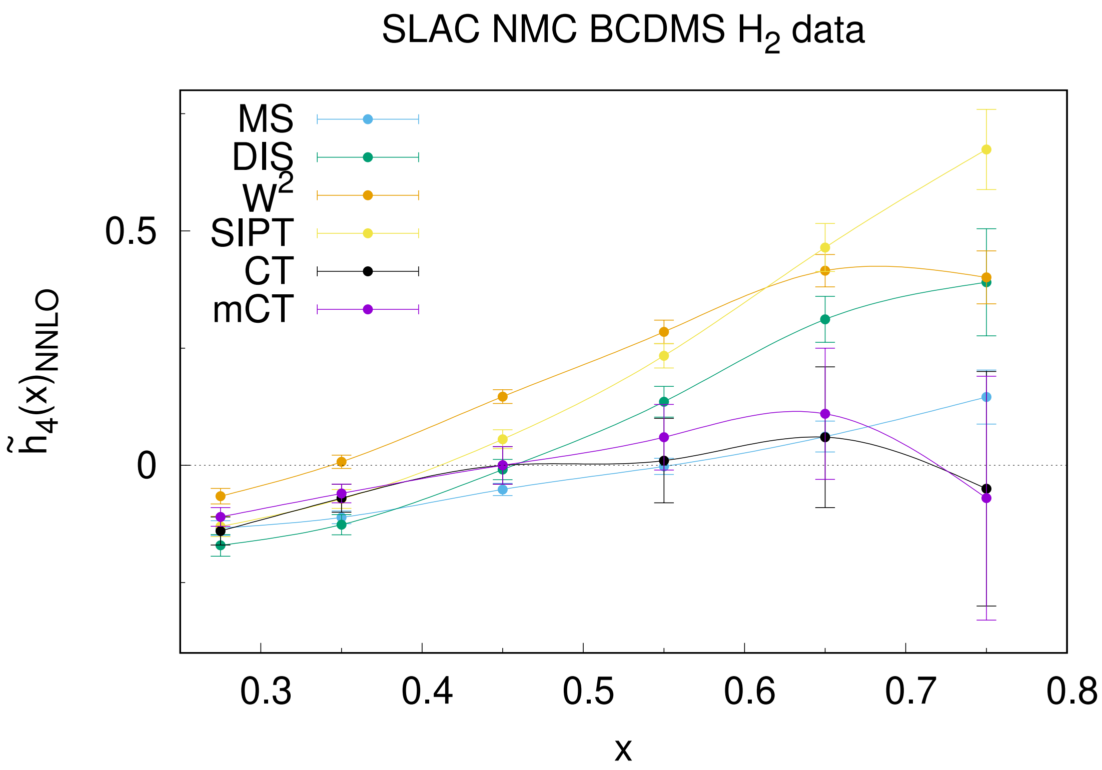

Figure 6: Same as in Fig. 2 but carried out with “frozen” coupling constant.

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7. Summary

We analyzed the experimental data collected by BCDMS, SLAC and NMC collaborations for DIS SF $F_2(x, Q^2)$ by resumming large logarithms at large x values into the corresponding Wilson coefficient function. For this matter we applied the famous CT resummation scheme (Catani,Trentadue:1989),(Catani:2014),Catani:2019) and its modification mCT, which takes into account the Gottfried sume rule. We also used three additional schemes: the DIS scheme, the W^2 evolution and the Grunberg effective charge method.

In the $\overline{MS}$-scheme, we got the strong coupling constant:

$$\alpha_s(M_Z^2) = 0.1177 \pm 0.0020 \text{ (total exp.error)} + \left\{ \begin{matrix} +0.0021 \\ -0.0013 \end{matrix} \right. \text{ (theor)}\,,$$

which is in agreement with word-average value:

$$\alpha_s(M_Z^2) = 0.1180 \pm 0.0009$$

It has been shown that in NNLO, the use of the CT and mCT threshold resummations leads to results very similar to those obtained within the framework of the $\overline{MS}$ scheme (in NLO, the CT and mCT resummations lead to significantly higher values for normalization of the strong coupling constant).

In all the cases studied, the only difference from the standard $\overline{MS}$ approach (without summing up large contributions for large x values) is the PDF suppression for large x values, as evidenced by the growth of the extracted values of the parameter b in the PDF forms. This is completely consistent with the results obtained earlier in (Corcella,Magnea:2005)(Bonvini et al.:2015).

Moreover, in all the cases, except for the CT case in NLO, the shape of the twist four corrections is quite similar, but the CT case in NLO leads to significantly high values of the twist four corrections.

Using three other schemes, namely DIS, W^2 and Grunberg approach, where large contributions at large x effectively resum into the argument of the strong coupling constant, leads to the same suppression of the PDF for large x.

Moreover, using these schemes does not lead to a noticeable change in the values of the strong coupling constant $\alpha_s(M_Z^2)$.

However, the values of the twist four corrections become large and negative, which contradicts the results obtained within the $\overline{MS}$, CT and mCT schemes.

It was noted that the difference in the shape of the twist four corrections is due to the small values of the arguments of the strong coupling constant, which arise from the resummation of large contributions at large values of x in these three schemes.

Indeed, when using one of the most popular infrared modifications of the coupling constant: the "frozen" version, for all the cases considered the results do not really change, with the exception of the shapes of the twist-four corrections for the DIS, W^2 and Grunberg schemes, where the twist-four corrections take a more familiar shape, increasing with increasing values of x.

In the future, we plan to extend the analysis to N^{3}LO accuracy. Note that there are already several N^{3}LO studies (Blumlein, Bottcher:2008), (Khorramian et al.: 2010), (Blumlein, Saragnese, 2021), (McGowan et al.,2023), (Ball et al., 2024), (Harland-Lang et al., 2025).