On the N-space and tau-space resummation approaches for thrust distribution in e^+e^-

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Motivations

- Extractions of α_s from the thrust distribution performed with analytic treatment of non-perturbative corrections (as opposed to those where NP corrections are performed using Monte Carlo models) yield small values of α_s, in tension with the world average.
- In a recent work Aglietti,Ferrera,Ju,Miao,2025 found important differences between results for the thrust distribution when the resummation is carried out in N-space (or Laplace-space, or conjugate-space) as compared to τ-space ($\tau = 1 - T$, also direct-space), with the N-space result in agreement with the world average.
- In past work Catani,Mangano,Trenradue,P.N.,1996, in the framework of threshold resummation for collider processes, found important differences between direct and conjugate space formulation.

Reminder of the 1996 result

$$\sigma^{(\mathrm{DY})}(\tau)=\int_0^1 \mathrm{d}z\mathrm{d}x\,\mathcal{L}(z)\hat{\sigma}^{(\mathrm{DY})}(x)\delta(z\,x-\tau),\quad \tau=\frac{Q^2}{S},x=\frac{Q^2}{s}.$$

Performing a Mellin transform, convolutions become products:

$$\sigma_N^{(\mathrm{DY})}=\mathcal{L}_N\hat{\sigma}_N^{(\mathrm{DY})},\quad V_N=\int_0^1\frac{\mathrm{d}y}{y}y^N V(y),\quad V=\sigma,\mathcal{L}$$

The Mellin space is more convenient for resummation, since the phase space for soft emissions also factorizes. At the double-logarithmic level (in the $\overline{\mathrm{MS}}$ scheme, and neglecting normalization factors)

$$\hat{\sigma}_N^{(\mathrm{DY})}=\exp(a\log^2 N),\quad a=2C_F\alpha_S/\pi$$

and $C_F\to C_A$ for gluon initiated processes. Remember that

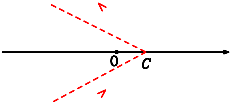
$$\exp(a\log^2 N)=1+a\log^2 N+\frac{1}{2}(a\log^2 N)^2+\dots$$

corresponding to no (soft-collinear) emission, one emission, two emissions, etc. Factorization of soft emissions is facilitated by the use of Mellin space, that is such that the phase space factorizes, which leads nicely to exponentiation.

Inverting from N-space

$$\sigma^{(\mathrm{DY})}(\tau) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \mathcal{L}_N \exp(a\log^2 N)\,\tau^{-N}\mathrm{d}N$$

Deforming the contour into a Hänkel contour H we get a convergent integral independent of C, provided the parton densities are reasonably behaved.



The direct space formula must use directly the short distance DY cross section

$$\hat{\sigma}^{(\mathrm{DY})}(x) = \frac{1}{2\pi i}\int_H \exp(a\log^2 N)\,x^{-N}\mathrm{d}N = \frac{1}{2\pi i}\frac{\partial}{\partial\ell}\int_H \exp(a\log^2 N)\,e^{\ell N}\frac{\mathrm{d}N}{N}$$

where $\ell = \log 1/x$ ($\approx 1 - x$ for $x \to 1$.)

On the Hänkel contour the integral is dominated by $\ell N \approx 1$. So we can define $z = \ell N$ and rewrite the integral as

$$\begin{aligned}\hat{\sigma}^{(\mathrm{DY})}(x) &= \frac{1}{2\pi i}\frac{\partial}{\partial \ell}\int_H \exp\bigl[\bigl[a(\log z - \log \ell)^2 + z\bigr]\,\frac{\mathrm{d}z}{z} \\ &= \frac{\partial}{\partial \ell}\left\{\exp(a\log^2\ell)\frac{1}{2\pi i}\int_H \exp(z - 2a\log\ell + a\log^2 z)\frac{\mathrm{d}z}{z}\right\}\end{aligned}$$

Notice that, up to now, inserting the above formula in the original direct-space formula should lead exactly to the same result of the N-space one. The problem arises if we expand the integral in powers of a:

$$\frac{1}{2\pi i}\int_H \exp(z - 2a\log\ell + a\log^2 z)\frac{\mathrm{d}z}{z} = 1 + \sum_{m>0,\, n\leq m} C_{m,n}a^m\ell^n$$

where the 1 corresponds to the double log approximation in direct space, and the added terms correspond to subleading corrections. The double log term alone can be written as

$$\hat{\sigma}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\exp[a\log^2(1-x)]$$

Putting it back in the original direct space formula we get

$$\begin{aligned}\sigma^{(\mathrm{DY})}(\tau) &= \int_0^1 \mathrm{d}x\mathrm{d}z\, \mathcal{L}(z)\hat{\sigma}^{(\mathrm{DY})}(x)\delta(z\, x - \tau) \\ &= \int_\tau^1 \mathrm{d}x\, \exp[a \log^2(1-x)] \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}\left(\frac{\tau}{x}\right).\end{aligned}$$

We find a divergence as $x \to 1$. Expanding in powers of a:

$$\int_0^1 \exp[a \log^2(1-x)]\mathrm{d}x = \sum_{k=0}^{\infty} \frac{a^k}{k!} \int_0^1 \log^{2k}(z)\mathrm{d}z = \sum_{k=0}^{\infty} \frac{a^k (2k)!}{k!} \approx \sum_{k=0}^{\infty} a^k 4^k k!$$

If we introduce the running coupling and look at the size δ of the minimal term of the asymptotic expansion, we find

$$\delta = \left(\frac{\Lambda}{Q}\right)^{\frac{\pi b_0}{4C}}, \qquad C \in \{C_F, C_A\},$$

where the exponent is about 0.72 for Drell-Yan and 0.32 for gluon initiated processes.

Reminder of the 1996 result

In the 1996 paper the same reasoning is extended to the leading log (LL) formula for DY resummation, leading to the same conclusion.

In the case of the 1996 result it was easy to conclude that the divergence and factorial growth of the perturbative coefficients were a spurious artifact of the direct-space approach, generated by the violation of momentum conservation for soft gluon emissions.

It is an example of things that can go wrong when looking at different representations of the resummation formulae.

We thus investigated whether similar problems arise in the case of the resummation for thrust in conjugate-space and in direct-space.

The double log (DL) case.

We assume that the resummed thrust distribution is given in the conjugate space as

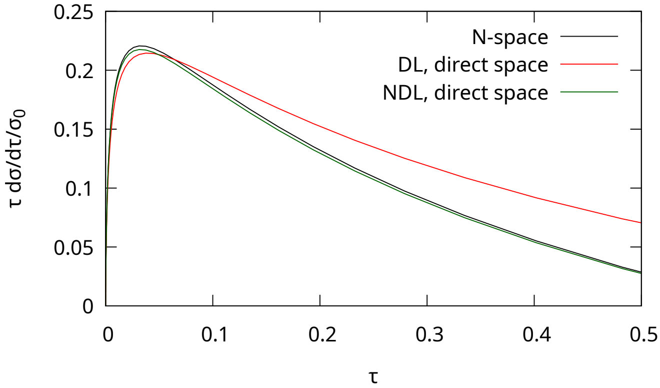
$$R_T(\tau) = \frac{1}{2\pi i}\int_H \frac{\mathrm{d}N}{N} e^{N\tau} e^{-a\log^2 N}, \tag{1}$$

where $R_T = \int_0^\tau \mathrm{d}\sigma$ is the cumulant, $a = C_F\alpha_s/\pi$. We assume that there are no subleading corrections to this formula. As before $\tau N \lessapprox 1$ dominates the integral. We introduce a variable $x = \tau N$ and rewrite, as before

$$R_T(\tau) = e^{-a\log^2\tau}\left[\frac{1}{2\pi i}\int_H \frac{\mathrm{d}x}{x}\exp[-2a\log x\log(1/\tau) - a\log^2 x + x]\right]$$

Again, the factor in front of the square bracket is the DL approximation, and expanding in a we find non-vanishing subleading terms. But now this is the whole answer. The square bracket expression is an analytic function of a, also if we perform a logarithmic expansion, i.e. we expand in a by keeping $\lambda = a\log 1/\tau$ constant. It is easy to check this by showing that its derivative with respecto to a is finite for all (complex) values of a, as long as x is integrated on the Hankel contour.

So we expect a nicely convergent logarithmic expansion:



where we have chosen $a = C_F/\pi \times 0.118$.
(at NNDL we get a line that is indistinguishable from the N-space result).

LL result

Our starting formula is

$$R_T(\tau) = \frac{1}{2\pi i} \int_H \frac{\mathrm{d}N}{N} \exp\left[\frac{1}{\alpha_s \beta_0} g_1(\lambda)\right]$$

where $\lambda = a_s \beta_0 \log N$, $a_s = \alpha_s/\pi$, $\beta_0 = b_0 \pi$. We have

$$g_1(\lambda) = -\frac{C_F}{\beta_0}\left[(1-2\lambda)\log(1-2\lambda) - 2(1-\lambda)\log(1-\lambda)\right].$$

Notice that

$$g_1(\lambda) = -\frac{C_F}{\beta_0}\lambda^2 + \mathcal{O}(\lambda^3)$$

so that the leading term in a_s is the DL term.
The integral is dominated by $N\tau$ not much larger than 1. Thus we define $x = N\tau$, and break λ into a τ and an x dependent part:

$$\lambda = \lambda_x + \lambda_\tau, \quad \lambda_z = a_s \beta_0 \log z, z = x \,\mathrm{or}\, \tau$$

Remember: $\lambda_\tau \approx 1$ in the logarithmic counting,
and $\lambda_x \approx a_s$ since x is not much larger than 1

This procedure allows us to write a formal expression that separates large and small terms. We begin with

$$R_T(\tau) = \frac{1}{2\pi i}\int_H \frac{\mathrm{d}x}{x}\exp\left[\frac{1}{\alpha_s\beta_0}g_1(\lambda_\tau) + \frac{\lambda_x}{\alpha_s\beta_0}g_1'(\lambda_\tau) + H(\lambda_\tau,\lambda_x)\right]$$

where the reminder

$$H(\lambda_1,\lambda_x) = \frac{1}{a_s\beta_0}\left[g_1(\lambda_\tau+\lambda_x) - g_1(\lambda_\tau) - \lambda_x g_1'(\lambda_\tau)\right]$$

is defined in such a way that its expansion in λ_x begins with λ_x^2. So

$$R_T(\tau) = \exp\left[\frac{g_1(\lambda_\tau)}{a_s\beta_0}\right]\exp\left(H\left(\lambda_\tau, a_s\beta_0\frac{\mathrm{d}}{\mathrm{d}g_1'(\lambda_r)}\right)\right)\frac{1}{2\pi i}\int_H \frac{\mathrm{d}x}{x}e^{\frac{\lambda_x}{a_s\boldsymbol{\beta_0}}g_{\mathbf{1}}'(\lambda_\tau)}$$

and

$$\frac{1}{2\pi i}\int_H \frac{\mathrm{d}x}{x}e^{\frac{\lambda_x}{a_s\boldsymbol{\beta_0}}g_{\mathbf{1}}'(\lambda_\tau)} = \frac{1}{\Gamma(1-g_1'(\lambda_\tau))}$$

Defining the direct space approximation as

$$R_T^{(d,n)}(\tau) = \exp\left[\frac{g_1(\lambda_\tau)}{a_s\beta_0}\right]\sum_{i=0}^{n} D_i(\lambda_\tau)a_s^i$$

(or variants of this) and computing the D_i algebraically for i up to 60, we find that, to a good approximation, the terms of the series go as

$$a_s^n D_n \approx \frac{(a_s\pi)^n\, Lf(n)}{(1-2\lambda_\tau)^n} \times P_n \approx \pi^n a_s^n(Q\tau)\, Lf(n) \times P_n$$

where P_n is an oscillating function of n, approximately given by $\cos(n\pi/4)$, and

$$Lf(n) = \Pi_{j=2}^{n}\log(j)$$

that we call logfactorial function.

Thus:

- The direct-space expansion is not convergent.
- The growth of the terms is by a factor of log n
- The divergence is much weaker than factorial
- The terms are oscillating: thus a resummation is possible without ambiguity, and is given by the N-space formula.
- The minimal term of the series is easily found to be of order

$$\delta \approx \exp\left[-\frac{(\tau Q/\Lambda)^{2b_0}}{2b_0\log(\tau Q/\Lambda)}\right]$$

  i.e. is suppressed by more than any power of Q.

We take the above as evidence that no “dangerous” factorial growth arises when going from one representation to another.

Notice also that the presence of the Landau pole does not yield a factorial growth associated with power corrections. This was the reason why this prescription for the Mellin inversion was called “minimal” in the 1996 paper. This was in contrast with ways to write up the threshold resummation (Korchemski,Sterman,1995) that yield linear power corrections to the DY total cross section (shown to be incorrect also by Beneke,Braun,1995).

Full N^{4}LL result

We now want to examine the behaviour of the conjugate and direct space results up to the highest order available, and compare the two results.

We begin with

$$R_T(\tau) = \frac{1}{2\pi i} \int_H \frac{\mathrm{d}N}{N} \exp\left[g(\lambda)\right]$$

where

$$g(\lambda) = \frac{1}{a_s \beta_0} g_1(\lambda) + f_2(\lambda) + a_s f_3(\lambda) + a_s^2 f_4(\lambda) + a_s^3 f_5(\lambda)$$

We get a formula similar to the LL case:

$$R_T(\tau) = \exp\left[\frac{g(\lambda_\tau)}{a_s \beta_0}\right] \exp\left(\frac{1}{a_s \beta_0} H\left(\lambda_\tau, a_s \beta_0 \frac{\mathrm{d}}{\mathrm{d} g_1'(\lambda_r)}\right)\right) \frac{1}{\Gamma[1 - g_1'(\lambda_r)]}$$

where

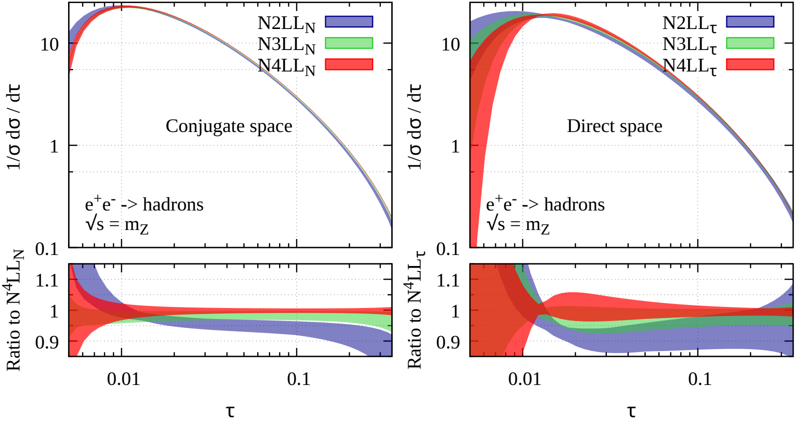
$$H(\lambda_1, \lambda_x) = g(\lambda_\tau + \lambda_x) - g(\lambda_\tau) - \frac{\lambda_x}{a_s \beta_0} g_1'(\lambda_\tau).$$

There are several possible ways to define the direct space resummed expression. We adopt for the N^{k}LL formula

$$R_T^{(k,d)}(\tau) = C^{(k-1)}(a_s)\exp\left[g^{(k+1)}(\lambda_\tau) + \sum_{i=1}^{k-1} d_i(\lambda_\tau)a_s^i\right]$$

where the notation $g^{(i)}$ means that we include up to the f_i term in g, $C^{(i)}$ that we include up to the term of order a_s^i in C. Notice that different arrangements of the terms (i.e. different definitions of the resummation formula) are always possible. They differ by subleading terms.

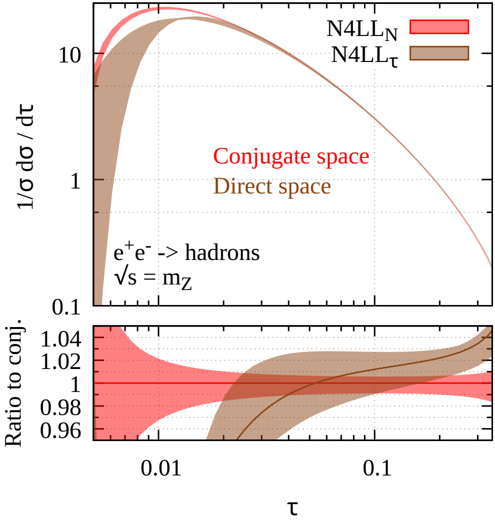
Results with error bands

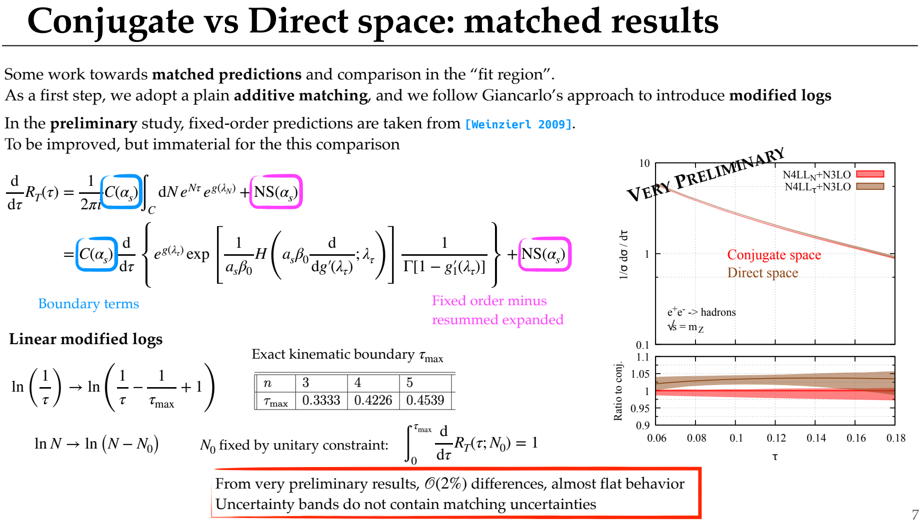


Error bands are 7-point scale variations for the renormalization scale μ_R and the resummation scale μ_L, defined as

$$\log\frac{1}{\nu} \to log\,\frac{\mu_L}{Q\nu} - \log\frac{\mu_L}{Q}, \qquad \nu \in \left\{\tau, \frac{1}{N}\right\}$$

Results with error bands





Conjugate vs Direct space: matched results

Some work towards **matched predictions** and comparison in the “fit region”.
As a first step, we adopt a plain **additive matching**, and we follow Giancarlo’s approach to introduce **modified logs**

In the **preliminary** study, fixed-order predictions are taken from `[Weinzierl 2009]`.
To be improved, but immaterial for the this comparison

$$\frac{\mathrm{d}}{\mathrm{d}\tau}R_T(\tau) = \frac{1}{2\pi i}\boxed{C(\alpha_s)}\int_C \mathrm{d}N\, e^{N\tau}\, e^{g(\lambda_N)} + \boxed{\mathrm{NS}(\alpha_s)}$$

$$= \boxed{C(\alpha_s)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ e^{g(\lambda_\tau)} \exp\left[\frac{1}{a_s\beta_0} H\left(a_s\beta_0 \frac{\mathrm{d}}{\mathrm{d}g'(\lambda_\tau)};\lambda_\tau\right)\right] \frac{1}{\Gamma[1-g_1'(\lambda_\tau)]}\right\} + \boxed{\mathrm{NS}(\alpha_s)}$$

Boundary terms

Fixed order minus resummed expanded

Linear modified logs

$$\ln\left(\frac{1}{\tau}\right) \to \ln\left(\frac{1}{\tau} - \frac{1}{\tau_{\max}} + 1\right)$$

Exact kinematic boundary $\tau_{\max}$

n	3	4	5
$\tau_{\max}$	0.3333	0.4226	0.4539

$$\ln N \to \ln\left(N - N_0\right)$$

N_0 fixed by unitary constraint: $\int_0^{\tau_{\max}} \frac{\mathrm{d}}{\mathrm{d}\tau}R_T(\tau;N_0) = 1$

From very preliminary results, $\mathcal{O}(2\%)$ differences, almost flat behavior
Uncertainty bands do not contain matching uncertainties

From Buonocore’s talk at “α_S determination from thrust workshop”, CERN, Nov.27-28 2025

Conclusions and further remarks

- From the asymptotics of the transformation we do not find clear indications that one of the two schemes is patological
- Looking at the sequence of $N^K LL$ corrections with k up to 4, the conjugate-space formulation seems to perform better.
- Differences of the two approaches are sensible, i.e. the error band don't overlap.
- Comparisons with Aglietti,Ferrera,Ju,Miao,2025 are under way.
- Disclaimer: this work is not yet finish. In particular the $N^k LL$ predictions are still preliminary, and may change by the time we publish our results ...