

G.Torrieri



**UNICAMP**

How to make sense of a fluid with  
20 particles?

2504.17152 (PRD) with G.M.Sampaio, G.R.Soares

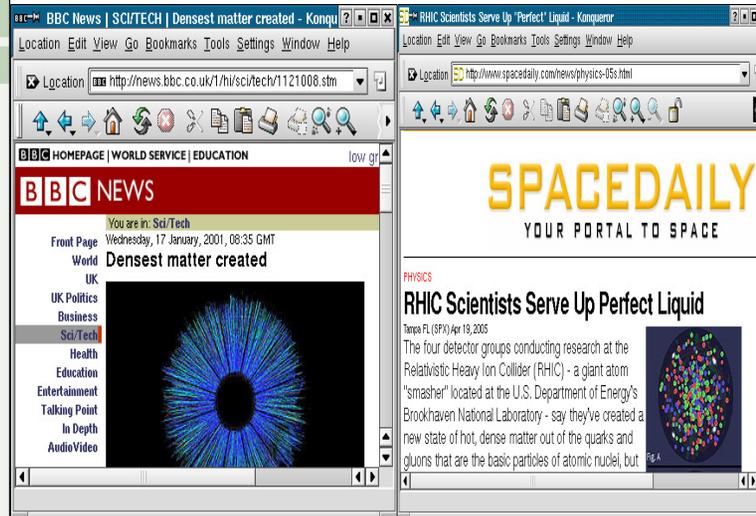
Cover of PRL!!!!

PHYSICAL  
REVIEW  
LETTERS



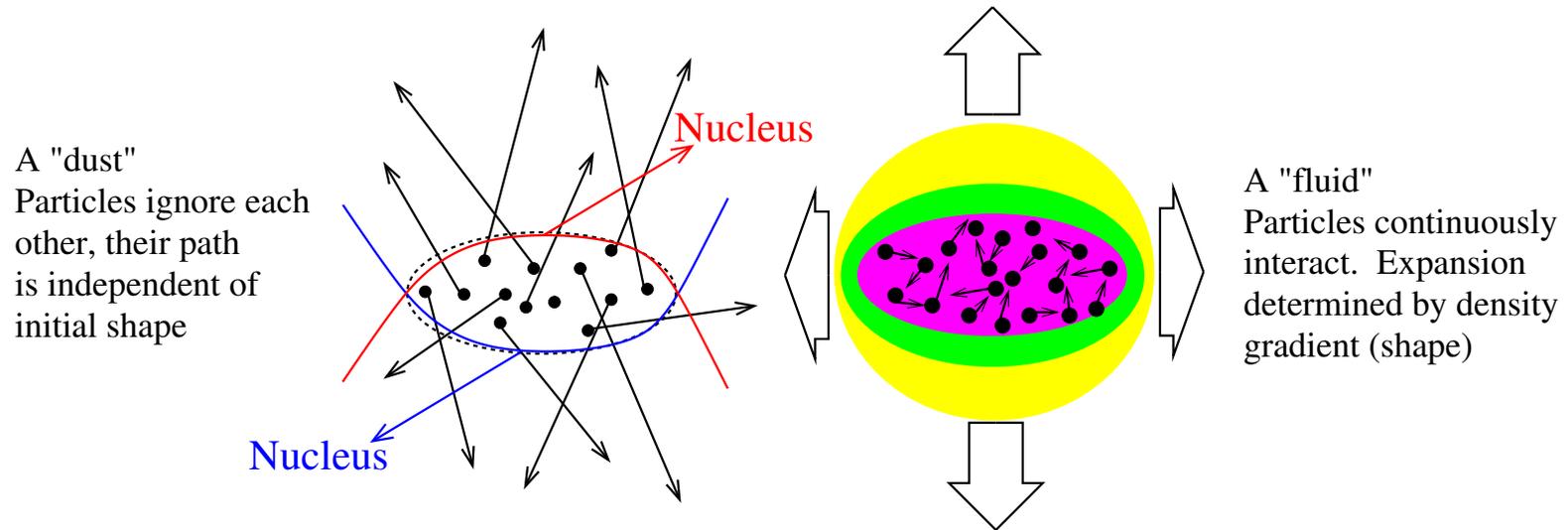
BBC!

SPACE  
DAILY!



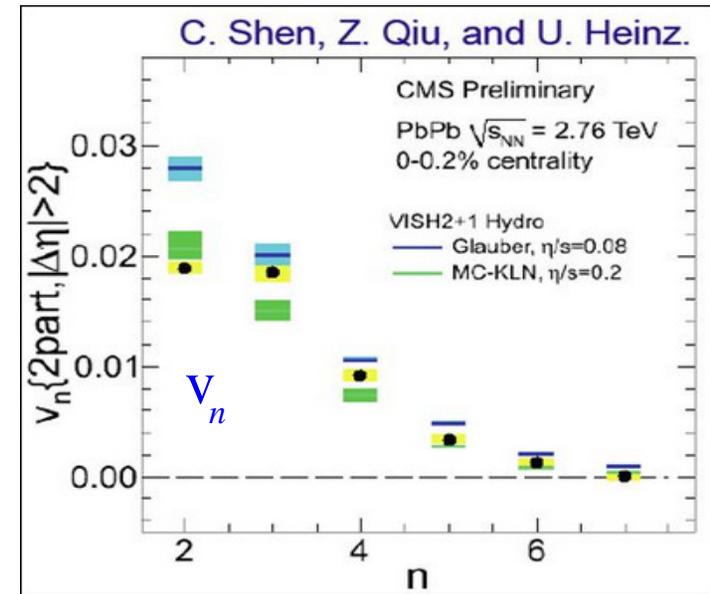
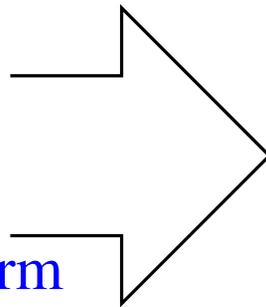
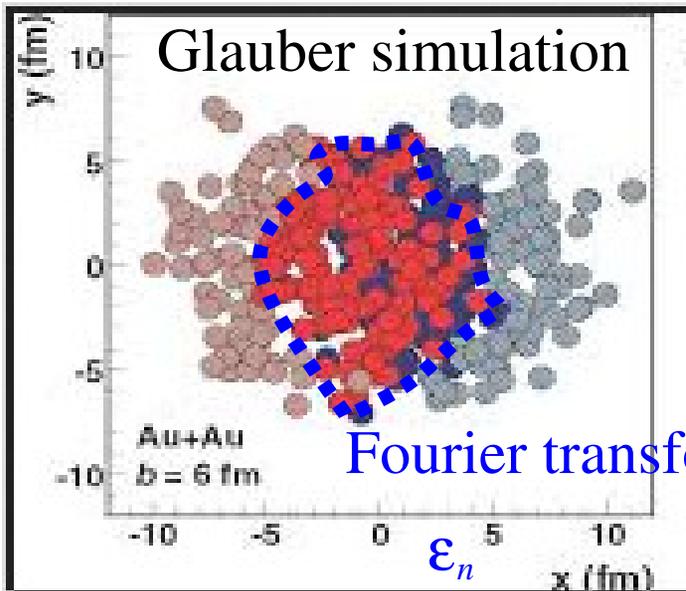
Heavy ion physicists found the perfect liquid! our field redefined by this  
Revived interest in relativistic fluid dynamics, developed in 1960-1980s by  
general relativists and astrophysicists!

## Why do we believe this? And why truly collective?



Observable: 
$$\frac{dN}{p_T dp_T dy d\phi} = \frac{dN}{p_T dp_T dy} [1 + 2v_n(p_T, y) \cos(n(\phi - \phi_0(n, p_T, y)))]$$

Not just local angular momentum or microscopic correlations:  
**ZDC/Spectators, correlations survive large rapidity separations** "True  
 Collectivity" Same  $v_n$  appears in  $\forall$  **n-particle correlations** ,  $\left\langle \frac{dN}{d\phi_1} \frac{dN}{d\phi_2} \dots \right\rangle$



Given reasonable estimate of initial conditions Fits ideal hydro ,  
 fitted upper limit on viscosity low Spurred a lot of theoretical and  
 numerical/phenomenological development of relativistic hydrodynamics.  
 Restarted interest in viscous relativistic hydrodynamics

So what is Hydrodynamics? an "EFT" of averages  $\langle \dots \rangle$  and thermalization

$$\langle T_{\mu\nu} \rangle = (e + P(e))u_\mu u_\nu + P(e)g_{\mu\nu} + \Pi_{\mu\nu} \quad , \quad \langle J^\mu \rangle = \rho u^\mu + q^\mu$$

At rest w.r.t.  $u^\mu$   $\langle T_{\mu\nu} \rangle = \text{Diag}(e(p, \mu), p, p, p)$  ,  $\langle J_\mu \rangle = (\rho(p, \mu), \vec{0})$

Makes system solvable just from conservation laws and EoS:

$$\partial_\mu \langle T^{\mu\nu} \rangle = \partial_\mu \langle J^\mu \rangle = 0, p = p(e, \mu), \rho = \rho(e, \mu)$$

Relaxational equation for  $\Pi_{\mu\nu}$  (entropy  $Ts = \Pi_{\mu\nu} \partial^\mu \beta^\nu$  ,and  $\partial_\mu (su^\mu) \geq 0$  )

$$\mathcal{D}(\tau_\pi, T) \Pi_{\mu\nu} + \Pi_{\mu\nu} + \mathcal{O}(\mathcal{D}^{n \geq 2}, \Pi^{n \geq 2}, T^{n \geq 2}, u^{n \geq 2}) = \eta \times \mathcal{O}(\partial u, \partial T) + \mathcal{O}(\partial^{n > 2})$$

(Navier Stokes  $\tau_\pi \rightarrow 0$  **acausal**  $\Pi_{\mu\nu}$  as a DoF "regularization")

A series whose "small parameter"  $K \sim \frac{l_{\text{micro}}}{l_{\text{macro}}} \sim \frac{\eta}{sT} \nabla u \sim \tau_\Pi \nabla u$  and the transport coefficients calculable from asymptotic correlators (Kubo)

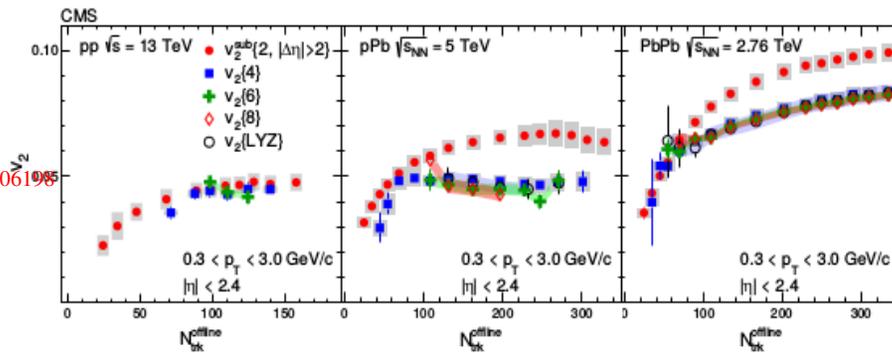
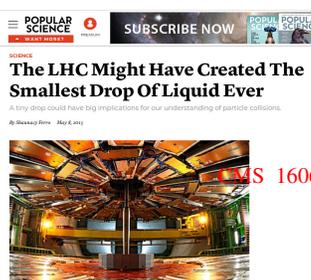
Non-relativistic version still considered beautiful and profound, but with relativity... Issues with causality and diffusion give complications

$u_\mu$  **ambiguous** many definitions: Landau  $u_\mu \propto s_\mu$  Eckart  $u_\mu \propto J_\mu$  BDNK: No relaxational DoF,  $\Pi_{\mu\nu} \propto \partial u$  Price: complicated anisotropic  $u_\mu(T_{\mu\nu})$   
We think flow is "clear", so this is a bit strange . choices supposed to be field redefinitions but give slightly different dynamics  
Geroch, Lindblom,...: when "corrections small" all theories good, when large none good. But no rigorous understanding of this!

$\Pi_{\mu\nu}$  **ambiguous** can even be eliminated as a DOF ( $\sim \partial u$  by carefully choosing  $u_\mu$  (BDNK)) Is it a physical quantity? An observable?

**Entropy is ambiguous** it's definition depends on the definitions above. Yet from statistical mechanics , as long as microstates are local, it should not be ambiguous! of course entropy related to fluctuations

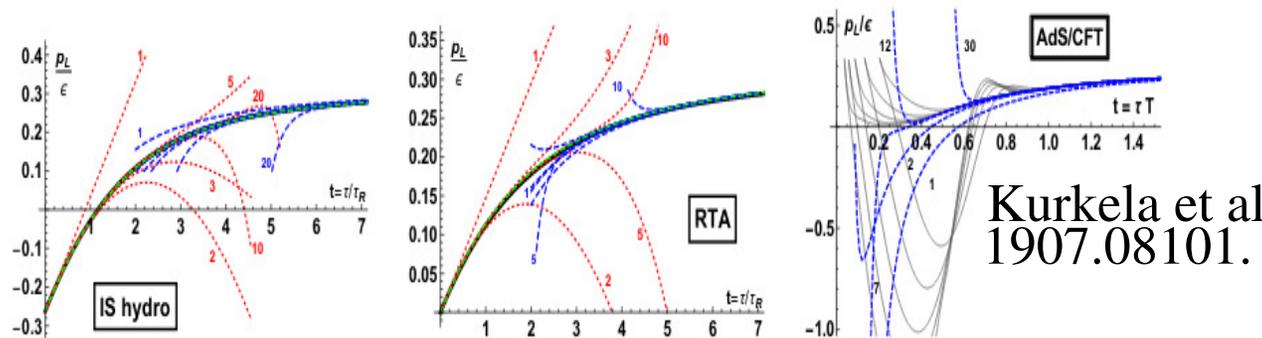
But then LHC switched on and we got a surprise and a conceptual challenge!



1606.06198 (CMS) : When you consider geometry differences and multi-particle cumulants (remove momentum conservation), hydro with  $\mathcal{O}(20)$  particles "just as collective" as for 1000. Fluctuations "irrelevant" even when they dominate! 20205:0 cumulants  $\geq 12, N_{ch} \sim 10$

## Hydrodynamics in small systems: “hydrodynamization” / “fake equilibrium” ?

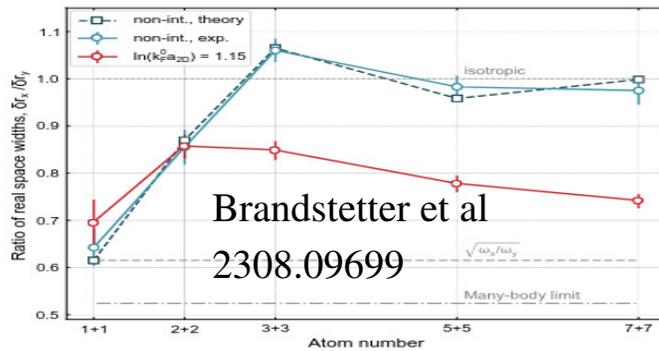
A lot more work in both AdS/CFT and transport theory about “hydrodynamization” / “Hydrodynamic attractors”



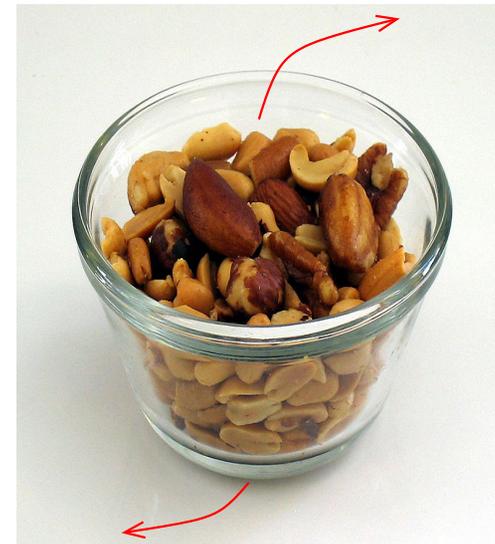
Fluid-like systems far from equilibrium (**large gradients** )! Usually from 1D solution of Boltzmann and AdS/CFT EoMs! “hydrodynamics converges even at large gradients with no thermal equilibrium”

But the issue is **not big gradients** but **small  $N_{dof}$** ! No Molecular chaos/large  $N_c$  , **Ensemble averaging!** ,  $\langle F(\{x_i\}, t) \rangle \neq F(\{\langle x_i \rangle\}, t)$

## Not just in heavy ions

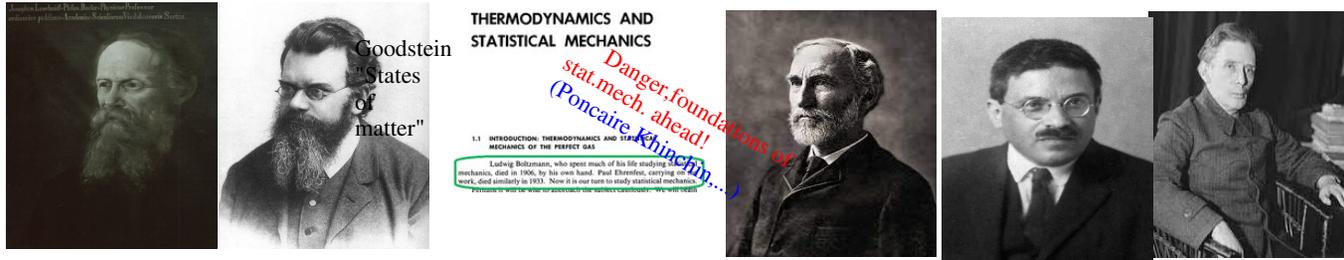


The  
Brazil  
nut effect

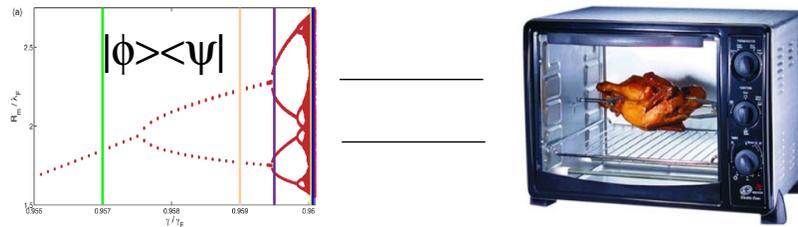


Empirically, strongly coupled systems with enough thermal energy seem to be "fluid" even with a small number of DoFs. EFT does not explain this! The role of fluctuations in hydrodynamics, and of the exact relation of statistical physics and hydrodynamics, are still ambiguous and this is related to experimental puzzles How many DoFs make a fluid?

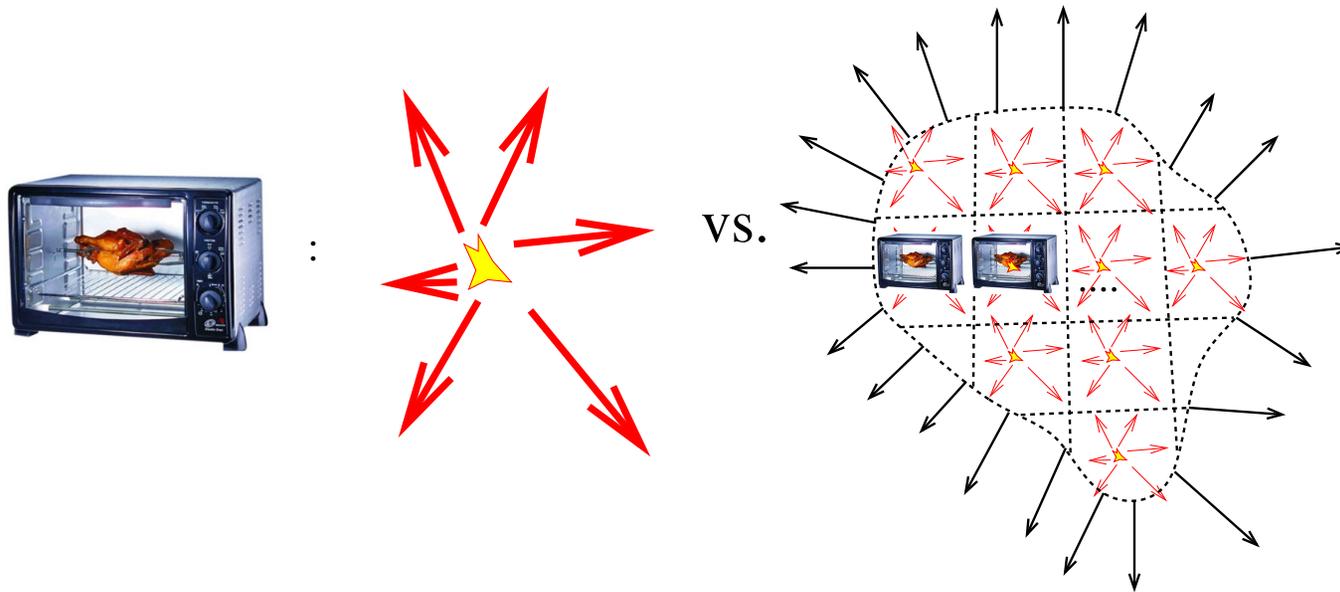
It is worth remembering statistical mechanics is on shaky ground



QM to rescue? Berry/Bohigas/Eigenstate thermalization



$E_{n \gg 1}$  of quantum systems whose classical correspondent is chaotic have density matrices that look like pseudo-random. If off-diagonal elements oscillate fast or observables simple, indistinguishable from MCE! Maximally decohered state (Popescu et al also Kharzeev,...: Same picture

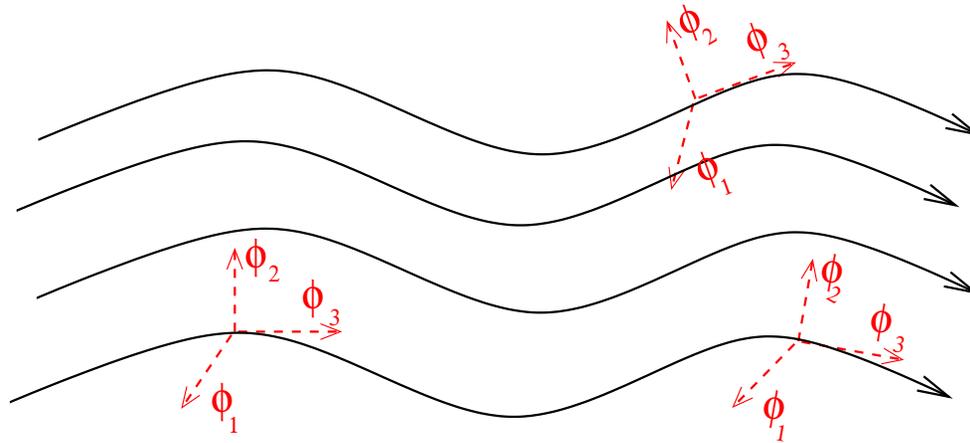


But need to coarse-grain, impose causality, and build hydro-like EFT out of this. could be very different from usual EFT expansion!

Thermodynamic limit relativistically ambiguous: “large volumes“ can fluctuate into non-equilibrium sub-volumes that need “long times” to re-equilibrate . Grad limit/large  $N_c$  hides this! Numerical implementation:Jadna’s poster

Let's look at this ambiguity a bit deeper: Lagrangian and Eulerian hydrodynamics Hydro as fields: (Nicolis et al, 1011.6396 (JHEP))

Continuum mechanics (fluids, solids, jellies,...) is written in terms of 3-coordinates  $\phi_I(x^\mu), I = 1...3$  of the position of a fluid cell originally at  $\phi_I(t = 0, x^i), I = 1...3$ . (Lagrangian hydro . NB: no conserved charges)



The system is a **Fluid** if its Lagrangian obeys some symmetries (Ideal hydrodynamics  $\leftrightarrow$  Isotropy in comoving frame) Excitations (Sound waves, vortices etc) can be thought of as "Goldstone bosons"

**Translation invariance** at Lagrangian level  $\leftrightarrow$  Lagrangian is a function of  $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$  Now we have a “continuous material”!

**Homogeneity/Isotropy** the Lagrangian is a function of  $B = \det B^{IJ}, \text{diag} B^{IJ}$  fluid cell interior has no “preferred” direction  $\Leftarrow SO(3)$

**Invariance under Volume-preserving diffeomorphisms** means the Lagrangian must be a function of  $B$  In all fluids a cell can be infinitesimally deformed

$$\rho = F(B), \quad p = F(B) - 2F'(B)B, \quad u^\mu = \frac{1}{6\sqrt{B}} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K$$

usual hydro energy-momentum tensor follows!  $\sqrt{B}$  is identified with the entropy,  $\sqrt{B} \frac{dF(B)}{dB}$  with microscopic temperature.  $u^\mu$  fixed by  $u^\mu \partial_\mu \phi^{\forall I} = 0$

## Conserved charges (Dubovsky et al, 1107.0731(PRD))

Within Lagrangian field theory a scalar chemical potential is added by adding a  $U(1)$  symmetry to system.

$$\phi_I \rightarrow \phi_I e^{i\alpha} \quad , \quad L(\phi_I, \alpha) = L(\phi_I, \alpha + y) \quad , \quad J^\mu = \frac{dL}{d\partial_\mu \alpha}$$

generally flow of  $b$  and of  $J$  not in same direction. Can impose a well-defined  $u^\mu$  by adding chemical shift symmetry

$$L(\phi_I, \alpha) = L(\phi_I, \alpha + y(\phi_I)) \rightarrow L = L(b, y = u_\mu \partial^\mu \alpha)$$

A comparison with the usual thermodynamics gives us

$$\mu = y \quad , \quad n = dF/dy$$

obviously can generalize to more complicated groups

This looks a bit like GR and this is not a coincidence!

**4D local Lorentz invariance** becomes local SO(3) invariance

**Vierbein**  $g_{\mu\nu} = \eta^{\alpha\beta} e_{\mu}^{\alpha} e_{\nu}^{\beta}$  is  $\frac{\partial x_I^{\text{comoving}}}{\partial x_{\mu}} = \partial_{\mu} \phi_I$  (with Gauge phase for  $\mu$ )

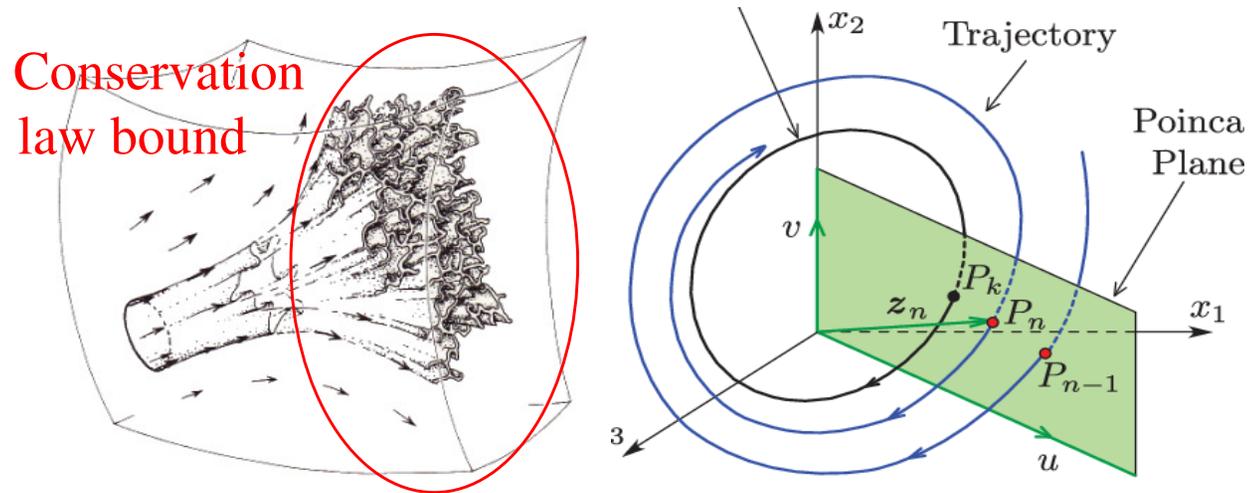
**Entropy**  $\sim \sqrt{b}$ , diffeomorphism invariant

**Killing vector** becomes  $u_{\mu}$

$\mathcal{L} \sim \sqrt{-g} (\Lambda + R + \dots)$  becomes  $\mathcal{L} \sim F(B) \equiv f(\sqrt{-g})$  Just cosmological constant, expanding fluid  $\equiv$  dS space

Very nice... but the ambiguities beyond ideal hydro generally break this .  
Who cares? Should beyond idel hydrodynamics have this general covariance?

## Where does statistical mechanics come from? Ergodicity



Classical evolution via Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x} \quad , \quad \dot{O} = \{O, H\}$$

“Chaos”, conservation laws  $\rightarrow$  phase space more “fractal”, recurring

“After some time”, for any observable ergodic limit applies

$$\int_0^{(large) T} \dot{O}(p, q) dt = \int P(O(p, q)) dq dp$$

where  $P(\dots)$  probability independent of time. This probability can only be given by conservation laws

$$P(O) = \frac{(\sum_i O_i) \delta^4 (\sum_i P_i^\mu - P^\mu) \delta (\sum_i Q_i - Q)}{N}, \quad N = \int P(O) dO = 1$$

this is the microcanonical ensemble. In thermodynamic limit

$$P(O) \rightarrow \delta(O - \langle O \rangle)$$

Hydrodynamics is “thermodynamics in every cell

$$\int_0^{(large) T} \dot{O}(p, q) dt \rightarrow \frac{\Delta\phi}{\Delta t}$$

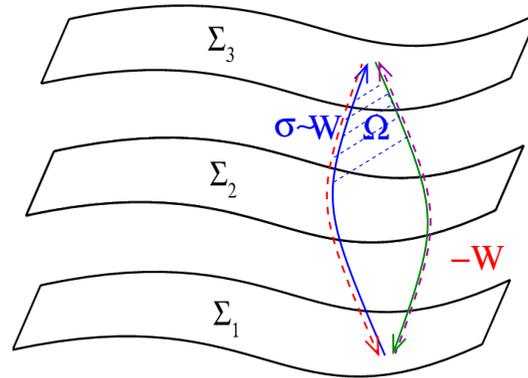
where  $\phi$  is some local observable.

$$\left. \frac{\Delta\phi}{\Delta t} \right|_{t-t'=\Delta} \simeq \frac{1}{d\Omega(Q, E)} \times$$

$$\times \sum \delta_{P^\mu, P^\mu_{macro}(t)}^4 \delta_{Q, Q_{macro}(t)} \delta \left( \sum_j^\infty p_j^\mu - P^\mu \right) \delta \left( \sum_j^\infty Q_j - Q \right)$$

Problem: This is not relativistically covariant!

Solution: Foliation!



$$t \rightarrow \Sigma_0 \quad , \quad x_\mu \rightarrow \Sigma_\mu \quad , \quad \Delta \rightarrow \text{“smooth”} \quad \frac{\partial \Sigma_\mu}{\partial \Sigma_\nu}$$

**Smooth:**  $R_{curvature}$  of metric change smaller than “cell size” (New  $l_{mfp}$  )

$$\frac{\Delta \phi}{\Delta \Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad \Sigma_\mu \rightarrow \Sigma'_\mu \quad , \quad \frac{\Delta \phi}{\Delta \Sigma'_0} = \frac{\Delta \phi}{\Delta \Sigma_0}$$

What kind of effective lagrangian would enforce

$$\frac{\Delta\phi}{\Delta\Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad \frac{\Delta\phi}{\Delta\Sigma'_0} = \frac{\Delta\phi}{\Delta\Sigma_0}$$

with

$$P(\dots) \sim \delta\left(\sum_i P_i^\mu - P\right) \delta\left(\sum_i Q_i - Q\right)$$

Now Remember Noether's theorem!

$$p_\mu = \int d^3\Sigma^\nu T_{\mu\nu} \quad , \quad T_{\mu\nu} = \frac{\partial L}{\partial\partial^\mu\phi} \Delta_\nu\phi - g_{\mu\nu}L \quad , \quad \Delta_\nu\phi(x_\mu) = \phi(x_\mu + dx_\nu)$$

$$Q = \int d^3\Sigma^\nu j_\nu \quad , \quad j_\nu = \frac{\partial L}{\partial\partial^\mu\phi} \Delta_\psi\phi \quad , \quad \Delta_\psi\phi = |\phi(x)| e^{i(\psi(x) + \delta\psi(x))}$$

momentum generates spatial translations, conserved charges generate complex rotations!

## Space-like foliations decompose

$$d\Sigma_\mu = \epsilon_{\mu\nu\alpha\beta} \frac{\partial\Sigma^\nu}{\partial\Phi_1} \frac{\partial\Sigma^\alpha}{\partial\Phi_2} \frac{\partial\Sigma^\beta}{\partial\Phi_3} d\Phi_1 d\Phi_2 d\Phi_3$$

where the determinant (needed for integrating out  $\delta$  – *functions* is only in the volume part

$$\frac{\partial\Sigma'_\mu}{\partial\Sigma_\nu} = \Lambda_\mu^\nu \det \frac{d\Phi'_I}{d\Phi_J} \quad , \quad \det \Lambda_\mu^\nu = 1$$

Physically,  $\Lambda_\mu^\nu$  moves between the frame  $d\Sigma_{rest}^\mu = d\Phi_1 d\Phi_2 d\Phi_3 (1, \vec{0})$

so lets try

$$\underbrace{L(\phi)}_{\text{microscopic DoFs}} \simeq L_{eff}(\Phi_{1,2,3})$$

with

$$\frac{\Delta\phi}{\Delta\Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad P(\dots) = \delta(\dots)\delta(\dots)$$

the general covariance requirement of  $\frac{\Delta\phi}{\Delta\Sigma_0} = \frac{\Delta\phi}{\Delta\Sigma'_0}$  means the invariance of the RHS

$$\begin{aligned} & \frac{d\Omega(dP'_\mu, dQ', \Sigma'_0)}{d\Omega(dP_\mu, dQ, \Sigma_0)} = \\ & = \frac{d\Sigma'_0 \int da_\mu d\psi \delta^4 (d\Sigma^\nu a_\alpha \partial^\alpha (\delta^\mu_\nu L) - dP^\mu(\Sigma_0)) \delta (d\Sigma^\mu \psi \partial_\mu L - dQ(\Sigma_0))}{d\Sigma_0 \int da'_\mu d\psi' \delta^4 (d\Sigma'_\nu a'_\alpha \partial^\alpha (\delta^\mu_\nu L) - dP'_\mu(\Sigma'_0)) \delta (d\Sigma'_\mu \psi' \partial^\mu L - dQ'(\Sigma'_0))} \end{aligned}$$

It is then easy to see, via

$$\delta((f(x_i))) = \sum_i \frac{\delta(x_i - a_i)}{\underbrace{f'(x_i = a_i)}_{f(a_i)=0}} \quad , \quad \phi'_I = \frac{\partial_\alpha \Sigma'_I}{\partial^\alpha \Sigma^J} \Phi_J \quad , \quad \delta^4(\Sigma_\mu) = \det \left| \frac{\partial \Sigma^\mu}{\partial \Sigma^\nu} \right| \delta^4$$

that for general covariance to hold

$$L(\Phi_I, \psi) = L(\Phi'_I, \psi') \quad , \quad \det \frac{\partial \phi_I}{\partial \phi_J} = 1 \quad , \quad \psi' = \psi + f(\phi_I)$$

the symmetries of perfect fluid dynamics are equivalent to requiring the ergodic hypothesis to hold for generally covariant causal spacetime foliations!!!! Quantum:  $\Delta t_{micro-sampling} \rightarrow \rho_{ij} e^{i\Delta t E_{ij}}$  and proof similar!

The crucial question: Does this extend to non-ideal hydrodynamics?

**Generating functionals** , not constitutive relations

Every cell corresponds to a **partition function** , not a **conserved current**

**Near-maximum entropy** related to this, **and diffeo-invariant!**

Covariant w.r.t. Metric  $g_{\mu\nu} \leftrightarrow \partial\Sigma_\mu/\partial\Sigma^\nu$

**Close to local equilibrium** is **not** on gradient expansion but the approximate applicability of fluctuation-dissipation (**not the same!** )

**Refoliations in  $\Sigma_\mu \rightarrow$  Changes in  $g_{\mu\nu} \leftrightarrow$  reshuffling in interpretation**

**NB: Global equilibrium** , defined as  $\text{Max} [\langle \ln \hat{\rho} \rangle]_{\beta_{\mu,\mu}, \dots}$  ill defined if  
 $\nabla\delta_\mu \simeq 1/R, 1/T$  since hydrodynamic turbulence, statistical fluctuations **talk** (“unstable” equilibrium is not in equilibrium!). **local equilibrium** defined above well-defined!, solid basis of an EFT.

In summary, what we need is a hydrodynamics...

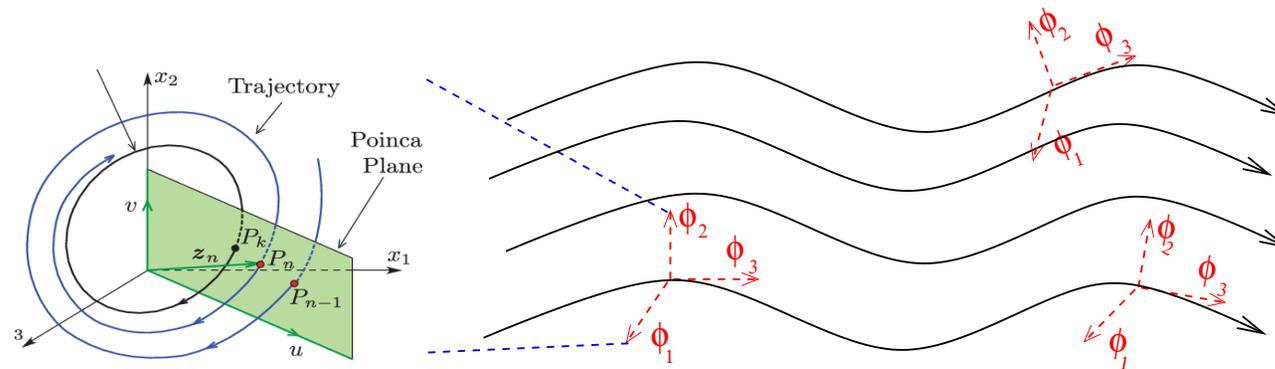
**Manifestly** in terms of probability distributions of observable quantities  $T_{\mu\nu}, J_\mu, \Omega_{\alpha\mu\nu}$ , **Cells** defined by full generating functionals,

**A diffeomorphism-invariant GC ensemble** at the level of fluctuations equivalent  $e, u_\mu, \beta_\mu, \Pi^{\mu\nu}, \dots$  choices leaving  $\langle T_{\mu\nu} \rangle$  invariant! Equivalent to choosing foliations  $\Sigma_\mu$

**Entropy content** a scalar w.r.t.  $\Sigma_\mu$  changes. Possibly order-by order, Different Boltzmannian entropy  $\forall$  counted as Gibbsian entropy

Ambiguity from fluctuations makes system look like a fluid, **If many equivalent choices** of  $e, u_\mu, J^\mu, \Pi^{\mu\nu}, \dots$  likely in one its "small"! Ideal hydro behavior.

Beyond ideal hydro and small systems Ergodicity/Poncaire cycles meet relativity , and most fluctuations equivalent to reparametrizations “ghosts”



Gibbs entropy+relativity : non-equilibrium  $\rightarrow$  “phase loss” of Poncaire cycles. one can see a slightly out of equilibrium cell either as a “mismatched  $u_\mu$ ” (fluctuation) or as lack of equilibrium (dissipation). **Microscopic momentum fluctuations and momentum dissipation locally indistinguishable!**

Fluctuations reducible to  $d\Sigma_\mu \rightarrow d\Sigma'_\mu$ , shift in  $u_\mu, \Pi_{\mu\nu}$  unphysical/“ghost”!

A generally covariant local equilibrium theory:Ingredients

**Local physics at the partition function level** Use Zubarev operator

$$Z = \int \mathcal{D}\phi \exp \left[ -\hat{T}_{\mu\nu} \beta^\mu d\Sigma^\nu \right]$$

dynamics independent of  $d\Sigma_\mu$  general covariance  $g_{\mu\nu} = \partial\Sigma_\mu / \partial\Sigma^\nu$

**Partition functions approximately Gaussian** “ad hoc“, but it works and  
Gaussians are universal **unless critical point!**

**The Gravitational ward identity** For Gaussians, and only for them it  
both enforces general covariance and determines dynamics

**Fluctuation-dissipation** for 2nd law **must be generally covariant!**

The gravitational ward identity (Deser, Boulware, JMP **8** (1967), 1468)

$$\nabla \mathcal{W} = 0 \quad , \quad \mathcal{W} = G^{\mu\nu, \alpha\beta} (\Sigma_\mu, \Sigma'_\nu) - \frac{1}{\sqrt{g}} \delta (\Sigma' - \Sigma) \times$$

$$\times \left( g^{\beta\mu} \langle \hat{T}^{\alpha\nu} (x') \rangle_\Sigma + g^{\beta\nu} \langle \hat{T}^{\alpha\mu} (x') \rangle_\Sigma - g^{\beta\alpha} \langle \hat{T}^{\mu\nu} (x') \rangle_\Sigma \right)$$

$G^{\mu\nu, \alpha\beta}$  is the propagator for energy-momentum tensor (fluctuation!)

Kubo formulae neglect contact terms by **imposing thermostatic background**

Fancy name but consequence of energy-momentum Noether current

$$\partial_\mu T^{\mu\nu} + \Gamma_{\nu\alpha\beta} T^{\alpha\beta} = 0 \quad , \quad \langle T_{\mu\nu}^n \rangle = \frac{\delta^n}{\sqrt{-g} \delta g^{\mu\nu(n)}} \ln \mathcal{Z}$$

Note: deterministic hydro gradient expansion and linearized fluctuations **inherently break this!** Ward identity specifies dynamics only for Gaussian

Cumulant expansion: Partition function is Gaussian!  $\ln \mathcal{Z} \simeq \ln \mathcal{Z}|_0 -$

$$-\frac{\partial^2 \ln \mathcal{Z}}{\partial \beta_\mu \partial \beta_\nu} \Big|_0 \ln \prod_{\Sigma(x)} \exp \left[ -\frac{1}{2} \langle \Delta T_{\mu\nu}(\Sigma(x')) \rangle C^{\mu\nu\alpha\beta}(\Sigma(x), \Sigma(x')) \langle \Delta T_{\alpha\beta}(\Sigma(x)) \rangle \right]$$

Gaussian  $\ln \mathcal{Z}$  is Solvable, function of diagonalized  $\langle T_{\mu\nu} \rangle$  (e,p) and  $C'^{\mu\nu\alpha\beta}$  in frame that diagonalizes  $\langle T_{\mu\nu} \rangle$

$$\ln \mathcal{Z}(\Sigma_0) \simeq \left( \sqrt{\prod_i \mu_i^2} \sqrt{C'_{\alpha\beta}} \right)^{-1}$$

where

$$C'_{\alpha\beta\iota\xi} = \Lambda_{\mu\nu} \Lambda_{\alpha\beta} \Lambda_{\delta\iota} \Lambda_{\zeta\xi} C^{\mu\alpha\delta\zeta} \quad , \quad \Lambda^{\mu\nu} \Lambda^{\alpha\beta} \langle T_{\mu\beta} \rangle = \text{Diag}(\mu_1, \mu_2, \mu_3, \mu_4)$$

NB: there is an approximation, measure is never Gaussian

$$T^{\mu\nu} - \langle T^{\mu\nu} \rangle = \Lambda_{\alpha}^{\mu} \text{diag}(\lambda_{1,2,3,4})^{\alpha\beta} \Lambda_{\beta}^{\nu} \quad , \quad \Lambda_{\nu}^{\mu} = \exp \left[ \int \frac{i}{2} d\omega_{\alpha\beta} (M^{\alpha\beta})^{\mu}_{\nu} \right]$$

With inspiration from Gaussian ansatzes for strongly coupled systems (**Kogan-Kovner-Milhano**) and central limit treatment to renormalization (**G.Jona-Lasinio**) we hope/conjecture Gaussian propagators are a fixed point.

$$T_0 \rightarrow T_0 + \Delta T_0 \quad , \quad s \rightarrow s + \Delta s \quad , \quad F(s) \rightarrow F(s) + \Delta F(s)$$

$$\left( \sqrt{\prod_i \lambda_i \mu_i} \sqrt{C'_{\alpha\beta}} \right)^{-1} = \left( \sqrt{\prod_i (\lambda_i + \Delta\lambda_i)(\mu_i + \Delta\mu_i)} \sqrt{C'_{\alpha\beta} + \Delta C'_{\alpha\beta}} \right)^{-1}$$

This will be verified in further work

If  $C_{\alpha\beta\gamma\nu}$  evolved via Ward identity from initial conditions **manifestly diffeoinvariant**

$$Q^{\alpha\beta} = \int d\Sigma_\mu \left[ \nabla_\nu C^{\mu\nu\alpha\beta} + \frac{1}{\sqrt{-g}} \sum_i \frac{1}{\lambda_i^2 \sqrt{C'_{\zeta\delta}}} \nabla^\mu C^{\alpha\beta}_{\mu\nu} \right]$$

Physically, “the price” for general covariance is that the propagator is not a function of the Lagrangian, but **evolved deterministically from initial conditions in an ensemble of field configurations** **This propagator can then be used to propagate the ensemble, and vice-versa!**

Lesson to make other stochastic theories generally covariant (Gravity)?

**Temperature** obtainable from covariantization of Maxwell relations

$$\langle E^2 \rangle - \langle E \rangle^2 \equiv C_V T \Rightarrow \left( C'_{\alpha\beta} \right)^{3/2} = \frac{\partial \beta_\mu \partial \beta_\nu}{\partial \Sigma_\mu \partial \Sigma_\nu}$$

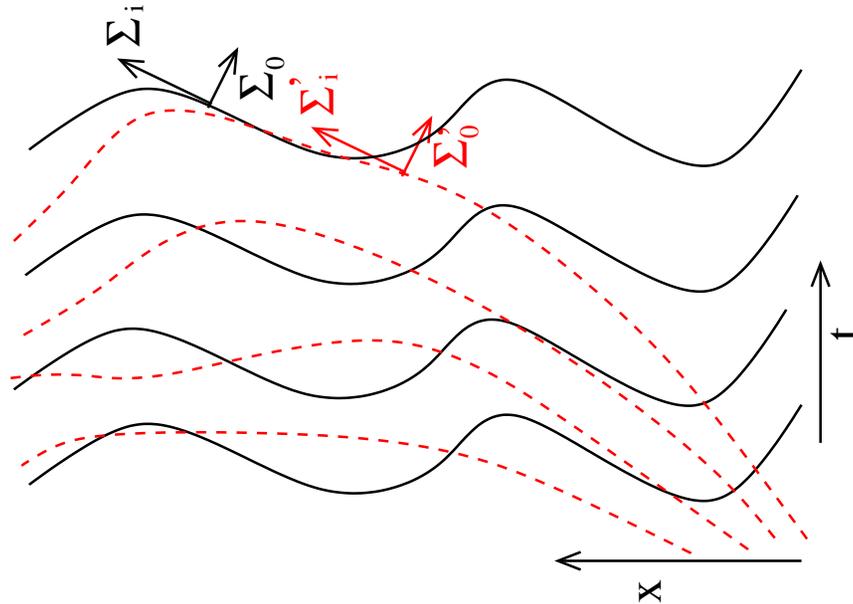
$\tilde{C}'_{\alpha\beta\mu\nu}(k)$  contains EoS and transport coefficients information

$$\eta \sim k^{-1} \lim_{k \rightarrow 0} \text{Im} \tilde{C}'_{xyxy} \quad , \quad c_s^2 \sim k^{-1} \lim_{k \rightarrow 0} \text{Im} \tilde{C}'_{xxxx}$$

$$\tau_\pi \sim k^{-2} \lim_{k \rightarrow 0} \text{Im} \frac{d}{dk} \tilde{C}'_{xyxy} \quad C_V \sim \max_k \text{Re} \tilde{C}'_{xxxx} \quad , \quad \dots$$

But we should underline that the full  $\tilde{C}'_{\alpha\beta\mu\nu}(k)$  is a field that evolves from the initial conditions of an ensemble of configurations of the  $T_{\alpha\beta}$  field, not a functional of the lagrangian alone

## General covariance and linear response



Different foliations are locally causal, but disagree on causality at distant regions. Linear response, Schwinger Keldysh,... fluctuation (equal time correlator) and dissipation (forward correlator) relation complicated but you “know the background“ . general covariance emerges in limit you do not!

fluctuation-dissipation relation From linear response the Gaussian propagator earlier,

$$T_{\mu\nu}(\Sigma) = \int e^{\epsilon\Sigma_0} G^{\mu\nu,\alpha\beta}(\Sigma'_0 - \Sigma_0) \delta g_{\alpha\beta}(\Sigma'_0) d\Sigma_0$$

we can find the time-ordered one

$$\langle \hat{T}^{\alpha\beta} \rangle_{\Sigma+d\Sigma} = i \int \frac{1}{\sqrt{g}} \mathcal{G}^{\mu\nu,\alpha\beta}(\Sigma'_0 - \Sigma_0) \langle T \rangle_{\mu\nu}(\tau, x) d\Sigma_0$$

$$\tilde{\mathcal{G}}^{\mu\nu\alpha\beta} = \frac{1}{2i} \left( \frac{\tilde{G}_T^{\alpha\beta\mu\nu}(\Sigma_0, k)}{\tilde{G}_T^{\alpha\beta\mu\nu}(-i\epsilon\Sigma_0, k)} - 1 \right)$$

These are Standard fluctuation-dissipation techniques (Forster, Kadanoff, ... ) but are they generally covariant

But Can it be Generally covariant? Seemingly no! in  $d\omega$  and because

$$dV dt \rightarrow \sqrt{|\det [-g_{\mu\nu}]|} d^4x \quad , \quad \prod_i \lambda_i \mu_i C'_{\mu\nu}{}^{\mu\nu} \sim Volume$$

restored if volume preserving diffeomorphisms invariance

$$\Sigma_\mu = (t(x), \vec{x}) = (t(\phi_I), \phi_{I=1,2,3}) \quad , \quad \phi_I \rightarrow \phi'_I(\phi_J) \quad , \quad \det_{IJ} \partial\phi_I / \partial\phi'_J = 1$$

hydro EFTs based on Schwinger-Keldysh break this symmetry beyond ideal limit (Haehl et al,1502.00636,also Grozdanov,GT+Montenegro) , which shows role of general covariance . Physically Local equilibrium  $\rightarrow$  dissipative evolution locally indistinguishable from an isentropic fluctuation. so symmetries of ideal hydro should remain! flow in ideal limit is defined as a Killing vector,  $u^\mu \partial_\mu \phi_I = 0$  Physically, once fluctuations are taken into account in a generally covariant way,  $u_\mu$  deviation from this is locally indistinguishable from a fluctuation, encoded in the structure of  $C_{\mu\nu\alpha\beta}$  .

The mere fact that thermodynamic quantities can be described via a  $\ln \mathcal{Z}$  gives rise to the Gibbs-Duhem relation

$$s = T \ln \mathcal{Z} = P + e - \mu n$$

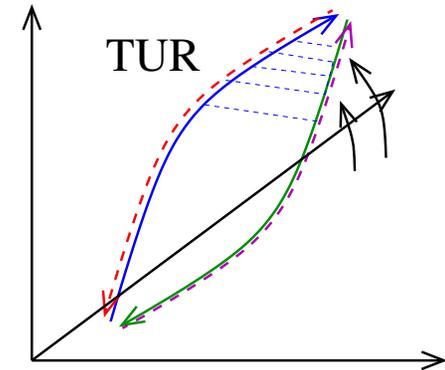
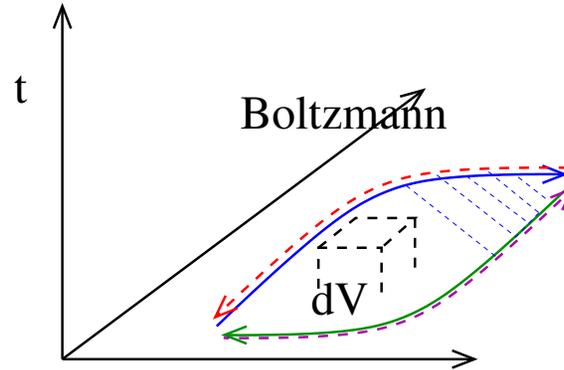
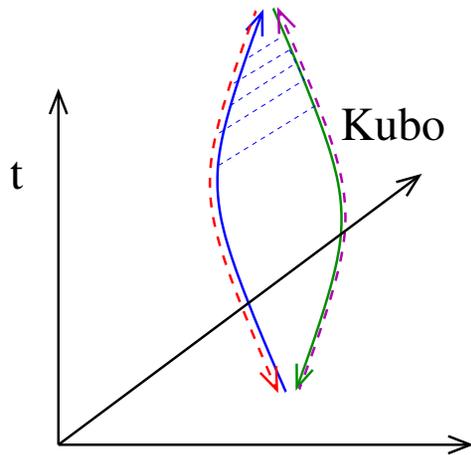
Enforce invariance under  $\Sigma_\mu$  refoliations, a scalar  $\ln \mathcal{Z}$

$$-\Delta \ln \mathcal{Z} = -\beta_\nu J^\nu \Delta \mu + P^i \Delta \beta_i - \Delta \Sigma^0 \beta_0 \int_0^{P^0} c_s^2(e) de, \quad P_{\alpha=0, i=1,2,3} \equiv T_{\alpha\beta} d\Sigma^\beta$$

Crooks theorem becomes

$$\frac{\mathcal{P} \left\{ P_\mu|_\tau \rightarrow P_\mu|_{\tau+\Delta\tau} \right\}}{\mathcal{P} \left\{ P_\mu|_{\tau+\Delta\tau} \rightarrow P_\mu|_\tau \right\}} \sim \exp[\ln \mathcal{Z}|_{\tau+\Delta\tau} - \ln \mathcal{Z}|_\tau], \quad \Delta\tau = \beta_\mu \frac{\Delta^3 \Sigma^\mu}{\Delta^3 \phi_{i=1,2,3}}$$

Zubarev statistical operator + Crooks theorem (Work  $\sim \int T_{\mu\nu} d^3 \Sigma^\mu \beta^\mu d\tau$ )



**Crooks fluctuation theorem** Relates fluctuations, entropy, work

$$P(W)/P(-W) = \exp[\Delta S] \quad , \quad W = T_{\mu\nu} d\Sigma^\mu \beta^\nu$$

reproduces Boltzmann entropy and Kubo for right contours and...

**ideal hydro emerges** for small fluctuations and vanishing viscosity!

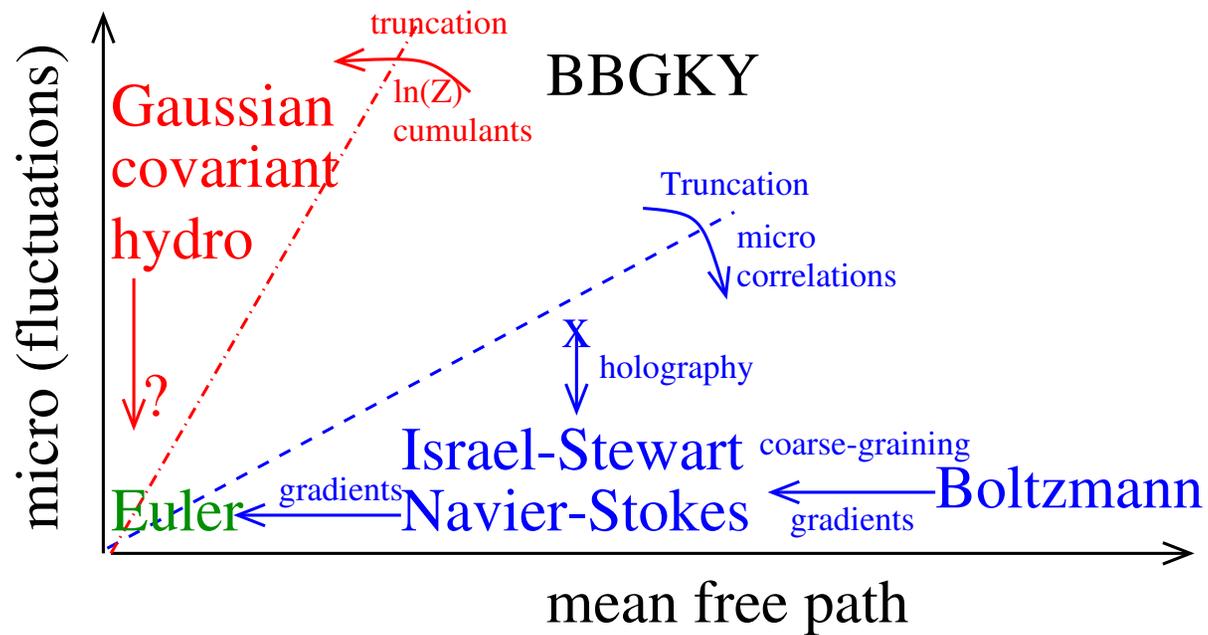
$$\lim_{\Delta S \rightarrow \infty} \frac{P(W)}{P(-W)} = e^{\Delta S} \equiv \delta(d\Sigma_\mu(su^\mu)) = 0 \Rightarrow n^\mu \partial_\mu(su^\mu) = 0$$

**Singular** limit but regularizeable **Fluctuation constraint becomes singular** when **general covariance determines EoM!**

**Characteristic volume** of volume preserving diffeomorphisms  $\rightarrow 0$

**Viscous corrections break volume-preserving diffeos** because fluctuation-dissipation violated. Any structure of a distribution invisible when it collapses to a  $\delta$ -function. **Fluctuation and dissipation appear together** in a way satisfying the Ward identity!

Those scales again:  $l_{micro} \ll l_{mfp} \ll L_{macro}$   
 $\sim s^{-1/3}, n^{-1/3} \quad \sim \eta/(sT)$



Preponderance of “chaotic” regime could mean general covariant hydro is both more likely and “closer in dynamics” to Euler!

When is the system is a good fluid? Usual approach,  $Kn$ , we propose  $\xi$

$$Kn \equiv \min_{\beta} \frac{(T^{\mu\nu} - T_0^{\mu\nu}) \partial_{\mu} \beta_{\mu}}{T_{\alpha}^{\alpha}} \sim \frac{\eta}{(e+p)R} \sim \frac{\tau_{\Pi}}{R}, \xi \equiv \frac{\min_{\beta} [\langle T^{\mu\nu} - T_0^{\mu\nu} \rangle \partial_{\mu} \beta_{\mu}]}{\sqrt{C'_{\alpha\beta}}}$$

**Flow ambiguity** at most by the possibility to "choose" the vector  $\beta$

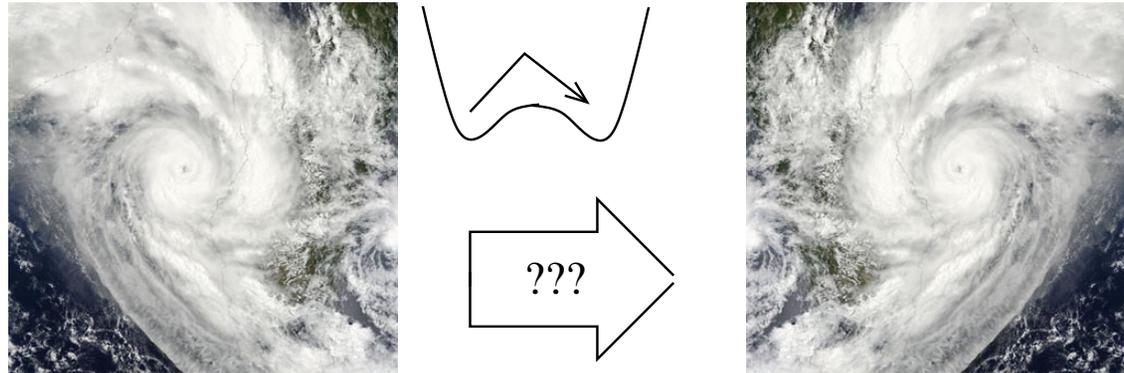
**fluctuations** some fluctuation-dissipation relation  $Kn \sim N_{dof}^{-1}$

**Phenomenology** breaks this

$\xi$  qualitatively depends in a radically different way from on  $N_{dof}$  : As  $N_{dof} \rightarrow \infty$  Gibbs-Duhem  $\rightarrow \xi \sim Kn$  , but away from this limit  $Kn$  is bound to increase with the number of DoFs while  $\xi$  could well decrease.

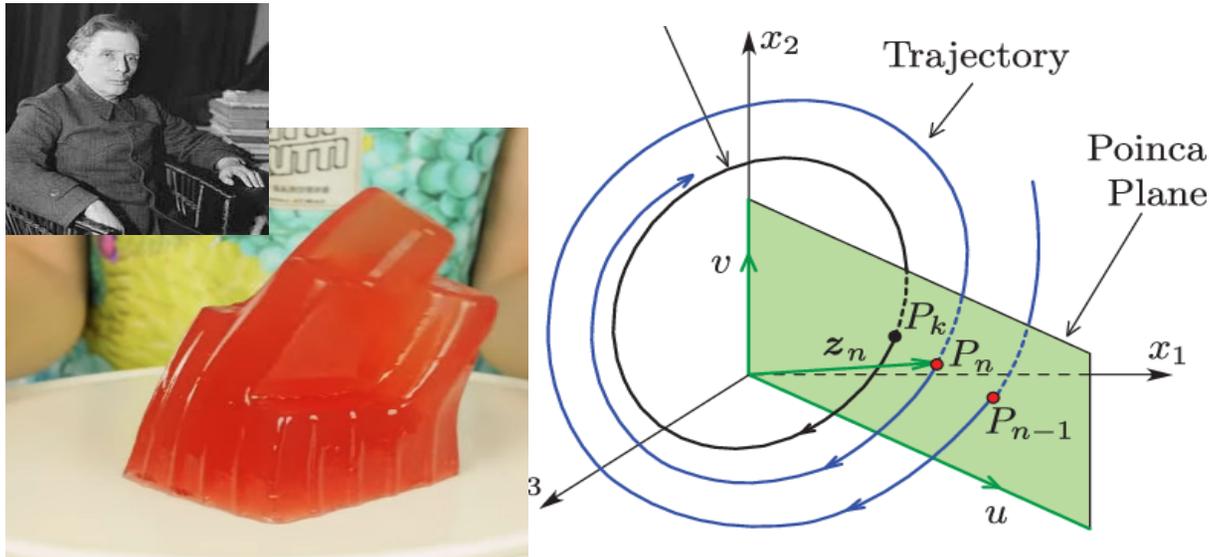
For small strongly coupled systems  $\xi \ll 1$ ?

## Hydrodynamics is universal I: local rather than global equilibrium!



**Global equilibrium** , defined as  $\text{Max} [\langle \ln \hat{\rho} \rangle]_{\beta_{\mu}, \mu, \dots}$  in presence of forces (rotations, fields, acceleration) ill defined if  $\nabla \delta_{\mu} \simeq 1/R, 1/T$  since hydrodynamic turbulence, statistical fluctuations talk (“unstable” equilibrium is not in equilibrium!). **local equilibrium** well-defined!, solid basis of an EFT. This ambiguity, reflecting the tension of **thermodynamic limit** and **relativity** , is due to entropy in Global equilibrium being Boltzmannian (“micro” Dofs) and not Gibbsian (covariantly “coarse-grained” Dofs, fluctuation-generated soundwaves, ...)

## Hydrodynamics is universal II: Solids, jellies etc and local equilibrium



Khinchin showed ergodicity impossible unless phase space indecomposable, something problematic for systems with broken volume preserving diffeos. Anything except fluids has a global equilibrium but no EFT built around local equilibrium stable against local fluctuations. **A complicated way of saying non-fluids are brittle (long correlations determine dynamics)**

## Hydrodynamics is universal III: Hydrodynamic stability and causality

**Stability?** It's all about those scales

$$l_{micro} \ll l_{mfp} \ll L_{macro}$$

When statistical mechanics applies, one expects dynamics to be absolutely chaotic near the **first scale, where Poincare cycles occur** and partially chaotic in the middle of the chain of inequalities **where turbulence occurs**

**Local causality?** criterion must be changed to

$$\int dx e^{ikx} \langle u_\mu(x) u_\nu(x') \rangle \rightarrow \int dx e^{ikx} T [T_{\mu\nu}(x), T_{\alpha\beta}(x')]$$

fluctuations or signals between unobservables don't have to be causal!

## Onto spin hydrodynamics?

STAR  
collaboration  
1701.06657

**NATURE**  
August 2017

Polarization by vorticity  
in heavy ion collisions



Could give new talk about this, but will mention hydro with spin not developed and a lot of conceptual debates. **does hydrodynamics with spin (and fluctuations) depend on the pseudo-gauge?** **entropy current** seems to!

What is a pseudogauge?

**Mathematically** (S. Jeon, 2310.11269 ) Let  $\Phi^{\alpha\beta\gamma}$  be fully antisymmetric

$$T_{\mu\nu} \rightarrow T'_{\mu\nu} + \frac{1}{2} \partial_\lambda (\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda}) \quad , \quad \partial^\mu T_{\mu\nu} = \partial^\mu T'_{\mu\nu} = 0$$

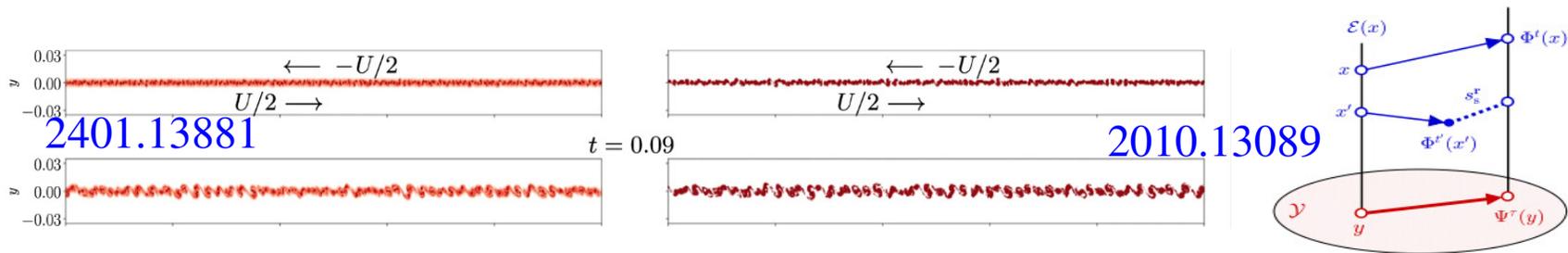
Can move around spin and angular momentum, shows the ambiguity of localized currents

**Physically** (T. Brauner, 1910.12224) a microscopic field redefinition with a non-inertial transformation preserving the action

$$x^\mu \rightarrow x^\mu + \epsilon \zeta^\mu(x) \quad , \quad \psi_a \rightarrow \psi_a + \epsilon \psi'_a \quad , \quad \mathcal{S} \rightarrow \mathcal{S}$$

So in current approach  $\ln \mathcal{Z}$  derivatives (dynamics and observables) exactly pseudo-gauge invariant! Work in progress, need Ward identity with torsion

## What happens in non relativistic limit?



- No unique Lorentz  $\rightarrow$  Galileo expansion, need extra assumptions. I think
  - $c_s \rightarrow \infty \Rightarrow$  incompressible matter
  - This way Local hydro Lorentz invariance becomes the Galileo group
  - Spontaneous stochasticity conjecture [2401.13881](#)
  - Hidden spatiotemporal symmetries in turbulence [2010.13089](#)
- But regarding ergodic hypotheses, what is the Stat.mech of incompressible matter? Probably singular

**Fluctuations in non-ideal hydrodynamics** not well understood

**Intimately related** to entropy current, double counting of DoFs  
Could alter fluctuation-dissipation expectation, "fluctuations help dissipate", in analogy to Gauge theory

**Approximate local equilibrium** not understood in Gibbsian picture  
Our proposal: applicability of fluctuation-dissipation

**Need a covariant** description purely in terms of observable quantities  
Ergodicity works in ideal hydro, Crooks theorem/K-K beyond it?

**Could be relevant for** hydro in small systems

**A non-relativistic limit?** (Brazil nut effect) all depends on time dilation, so a bit at a loss!

SPARE SLIDES

## Fluctuations... what is fluctuating?

One can always decompose  $T_{\mu\nu}$  into “near-equilibrium”  $e, p, n, u_\mu$  and non-equilibrium “rest”  $\Pi_{\mu\nu}, J_\mu$ , its an algebraic operation, provided you have a matching condition eg

$$T_{\mu\nu}u^\mu = eu_\nu \quad , \quad T_{\mu\nu} = (e + p(e))u_\mu u_\nu + p(e)g_{\mu\nu} + \Pi_{\mu\nu}$$

At deterministic level this is good but consider an ensemble  $\hat{T}_{\mu\nu}...$

$T_{\mu\nu} \rightarrow \hat{T}_{\mu\nu}$  so  $e, p, n \rightarrow \hat{e}, \hat{p}, \hat{n}$ , ensemble elements which fluctuate event-by-event.  $u_\mu \rightarrow \beta_\mu$ , Lagrange multiplier for momentum. Mixing these mixes apples and Legendre-transformed apples, lose all connection to stat-mech!

**It would mean** transversality condition  $u_\mu \Pi^{\mu\nu} = 0$  applies event-by-event rather than on average, **as expected from statistical mechanics** .

**Anisotropy, transport and statistical mechanics** Anisotropic hydrodynamics justified within transport via improved relaxation time

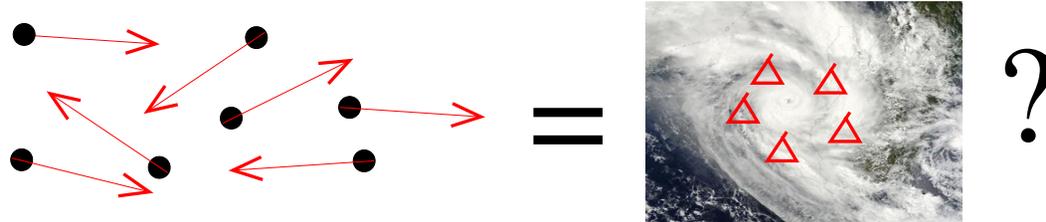
$$f(x, p) = f_{eq} (1 + \phi(x, p)) \rightarrow f_{eq} (1 + \phi(x, p) + a_\mu(x) p^\mu)$$

Problem: Boltzmann is an approximation where  $f(x, p)$  represents **an infinity of particles** . Fundamentally, hydrodynamics comes from Kubo

$$\eta = \lim_{k \rightarrow 0} \frac{1}{k} \text{Im} \int dx \langle \hat{T}_{xy}(x) \hat{T}_{xy}(y) \rangle \exp [ik(x - y)]$$

Usually semiclassical approximation yields Boltzmann equation than relaxation time, which guarantees the Kinchin condition to be fulfilled. Above demonstration reliable only in that limit

The basic problem with  $f(x,p)$



Let's solve the simplest transport equation possible: Free particles

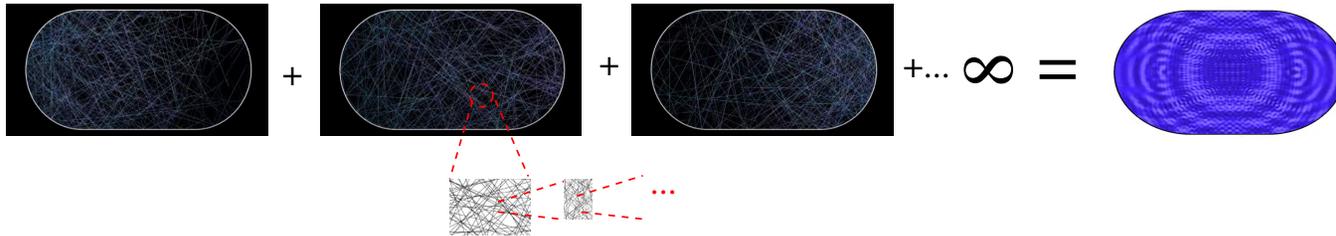
$$\frac{p^\mu}{m} \partial_\mu f(x, p) = 0 \rightarrow f(x, p) = f\left(x_0 + \frac{p}{m}t, p\right)$$

obvious solution is just to propagate

**What is weird** is that "hydro-like" solution possible too (eg vortices)!

$$f(x, p) \sim \exp[-\beta_\mu p^\mu] \quad , \quad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

But obviously unphysical, **no force!** **What's up?**



This paradox is resolved by remembering that  $f(x, p)$  is defined in an ensemble average limit where the number of particles is not just “large” but **uncountable** . **curvature from continuity!**

BUt this suggests Boltzmann equation disconnected from  $N_{dof} \leq \infty$  !

**In Anisotropic hydro**  $\beta_\mu$  not Killing vector . So no reason to assume ensemble average/thermal fluctuations sampled fairly close to equilibrium! Boltzmann equilibrium and Gibbs-type thermal equilibrium could be very different. **lets work with the latter**

## Vlasov and Boltzmann in a classical world

Villani , <https://www.youtube.com/watch?v=ZRPT1Hzze44>

**Vlasov equation** contains all classical correlations, instability-ridden, “filaments”, cascade in scales.

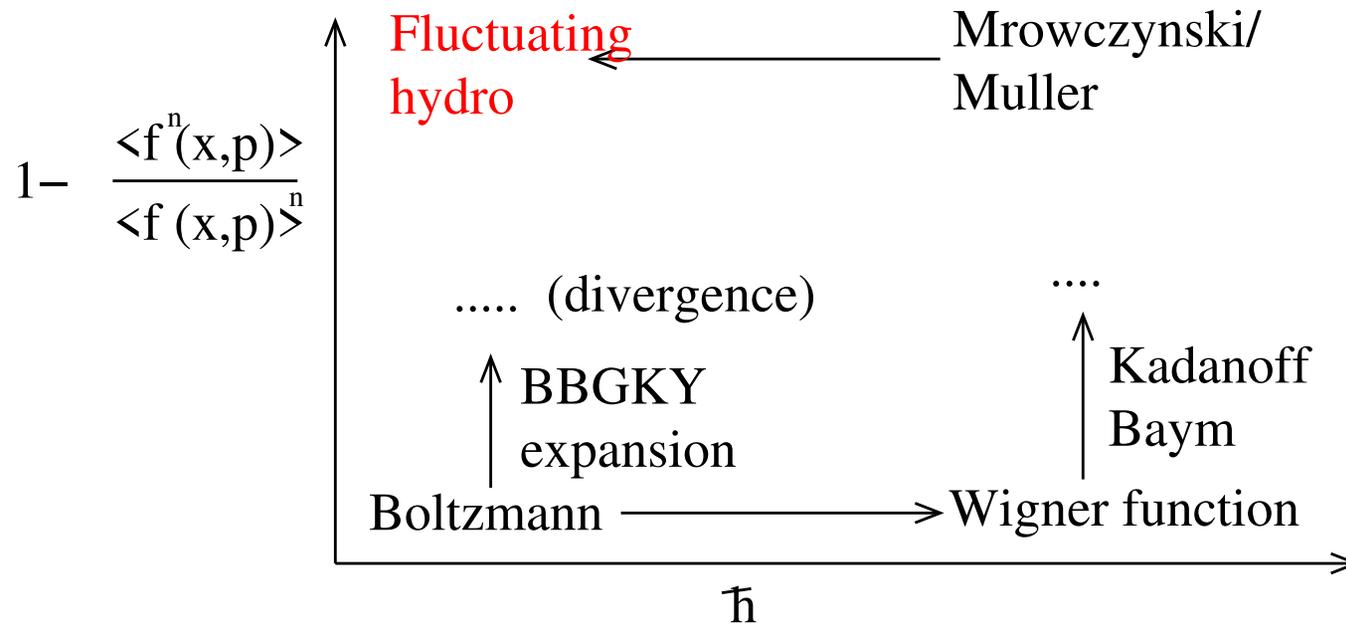
$N_{DOF} \rightarrow \infty$  invalidates KAM theorem stability

**Boltzmann equation** “Semi-Classical UV-completion” of Vlasov equation, first term in BBGK hierarchy, written in terms of Wigner functions.

Infinitely unstable jerks on infinitely small scales Random scattering

But if number of particles  $N \ll \infty$  Correlations important! .

## Boltzmann equation, BBGKY and limits



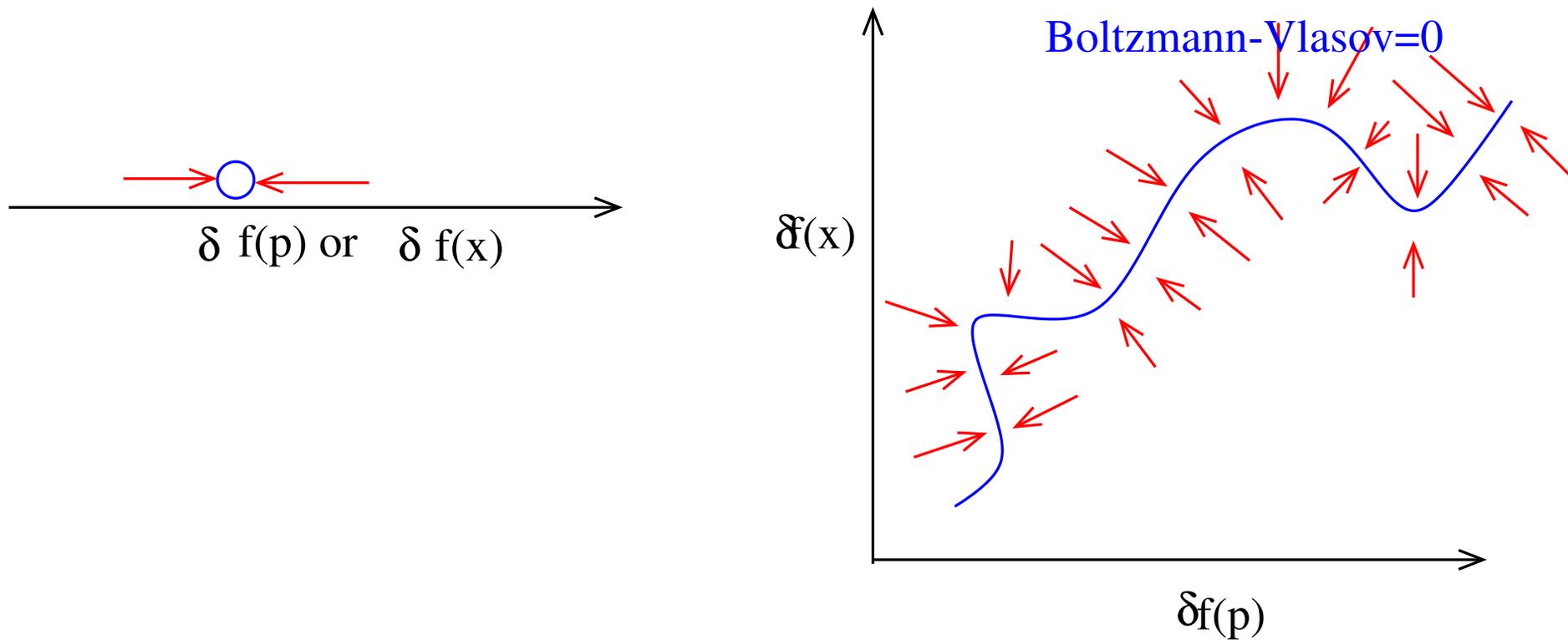
Boltzmann equation emerges as a double limit from **microscopic correlations**,  $\hbar \rightarrow 0$ . Relaxing the latter limit would destroy statistical independence **CHSH relations**, so probably not relevant (phases "chaotic"). But fluctuating hydro "non-perturbative" in correlations

Finite number of particles:  $f(x, p)$  not a function but a functional  
 $(\mathcal{F}(f(x, p)) \xrightarrow{\text{Boltzmann}} \delta(f' - f(x, p)))$ , incorporating continuum of  
 functions and all correlations. Perhaps solvable!

$$\frac{p^\mu}{\Lambda} \frac{\partial}{\partial x^\mu} f(x, p) = \left\langle \underbrace{\hat{C}[\tilde{W}(\tilde{f}_1, \tilde{f}_2)] - g \frac{p^\mu}{\Lambda} \hat{F}^{\mu\nu}[\tilde{f}_1, \tilde{f}_2] \frac{\delta}{\delta \tilde{f}_{1,2}} \tilde{W}(\tilde{f}_1, \tilde{f}_2)}_{\text{How many } A-B=0?} \right\rangle$$

Wigner functional to  $\mathcal{O}(\hbar^0)$ . What is the effect? If only Boltzmann term  
**not much!**

If Both Vlasov and Boltzmann terms, redundancy-ridden!



One can deform  $f(x, p)$  by  $\delta f(x)$  or  $\delta f(p)$  so that  $\hat{C} - \hat{W}$  cancels. In ensemble average deformation makes no sense, but away from it it does!

Discretize  $x, p \rightarrow$  random matrix problem!

$$\dot{f}_{ij} - \left[ \frac{\vec{p}_k}{\Lambda} \cdot \Delta_k \right] f_{ij} = \langle \hat{\Omega} \rangle$$

$$\hat{\Omega} \sim d [f'_{i_1 j_1}] \left[ \mathcal{W}_{i_1 j_1 i j} \left( \mathcal{C}_{j j_1} (f_{ij} f'_{i_1 j_1} - f_{i j_1} f'_{i j}) - \mathcal{V}_{i i_1}^\mu f_{ij} f'_{i_1 j_1} \frac{\Delta f_{ij}}{\Delta p^\mu} \right) \right]$$

- Theorems of random matrix theory can be used to prove limit very different from RTA!
- can be tested numerically with a **lattice Boltzmann** algorithm
- connects to Zubarev Gibbs-Duhem relation
 
$$\ln \mathcal{Z} = \ln \left[ \prod_{i=1}^N \exp \left( \Delta^3 \Sigma_\mu (\beta_\nu T^{\mu\nu} - \mu_i J^\mu) \right) \right]$$