# Color Superconductivity Revisited

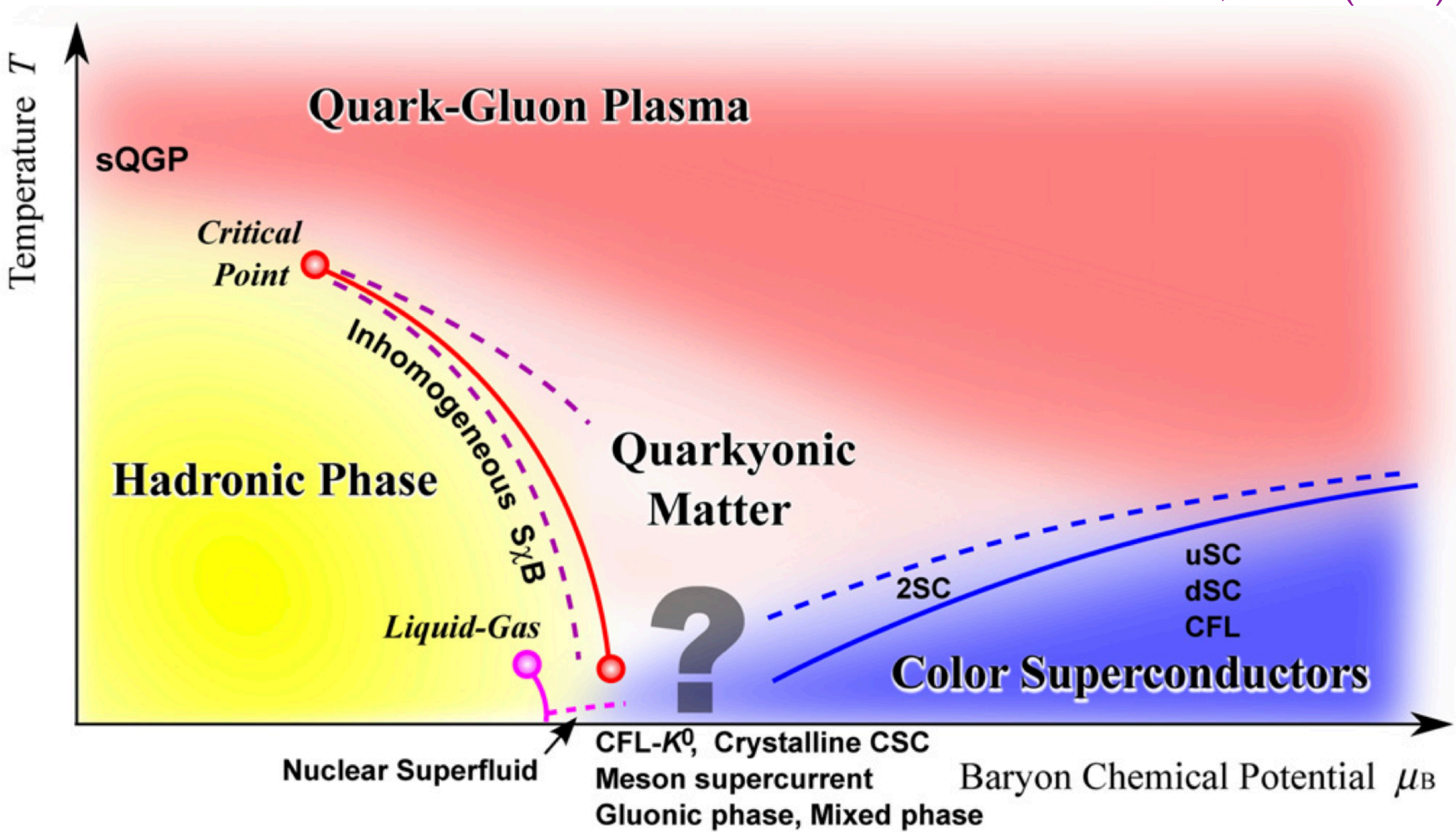
## *~Importance of QCD anomaly and gluons~*

Naoki Yamamoto (Keio University)

ExHIC-d workshop on QCD and hadronic dense matter
November 15, 2025

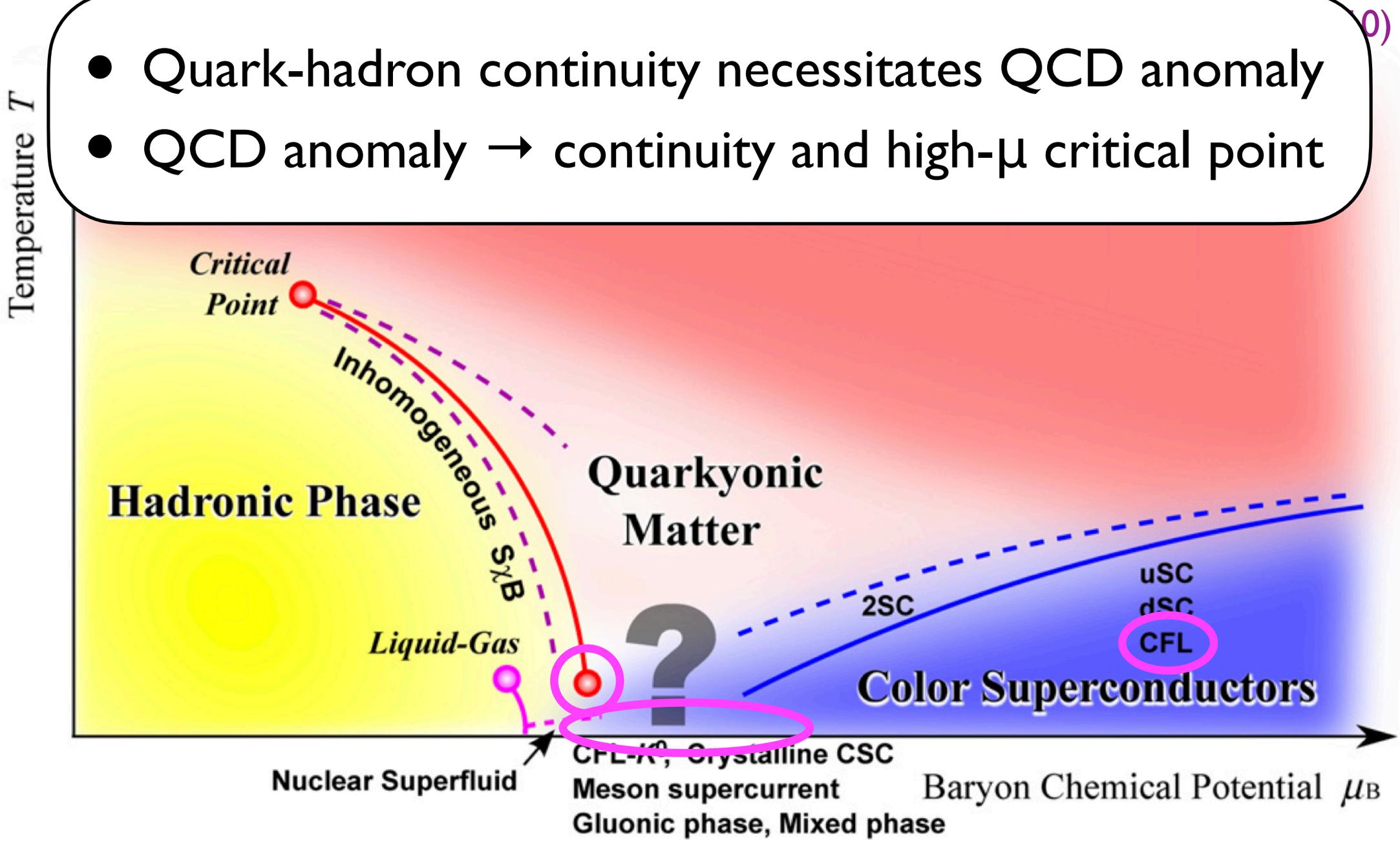
# QCD phase diagram

Fukushima-Hatsuda, PPNP (2010)



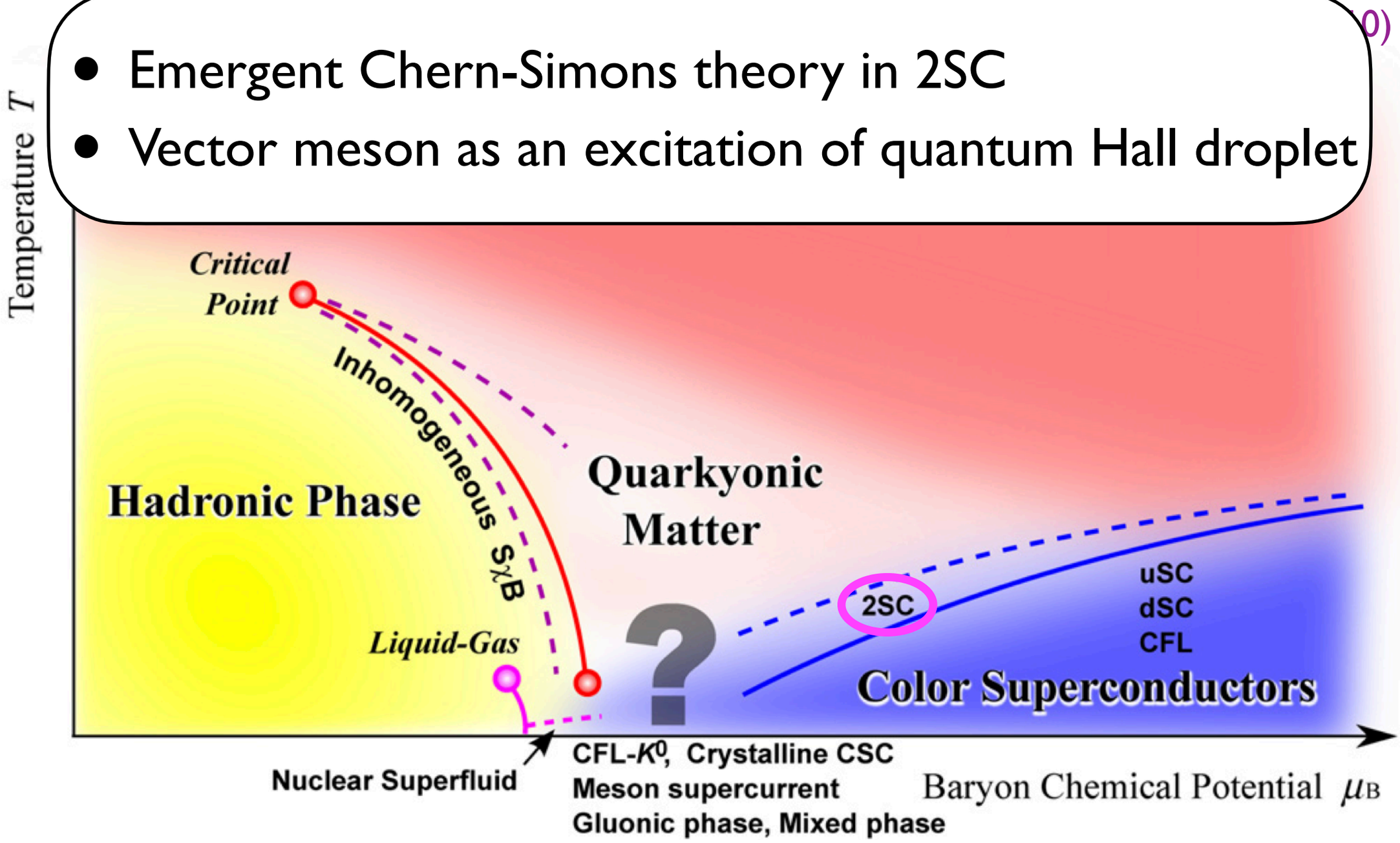
# Contents

- Quark-hadron continuity necessitates QCD anomaly
- QCD anomaly → continuity and high-μ critical point



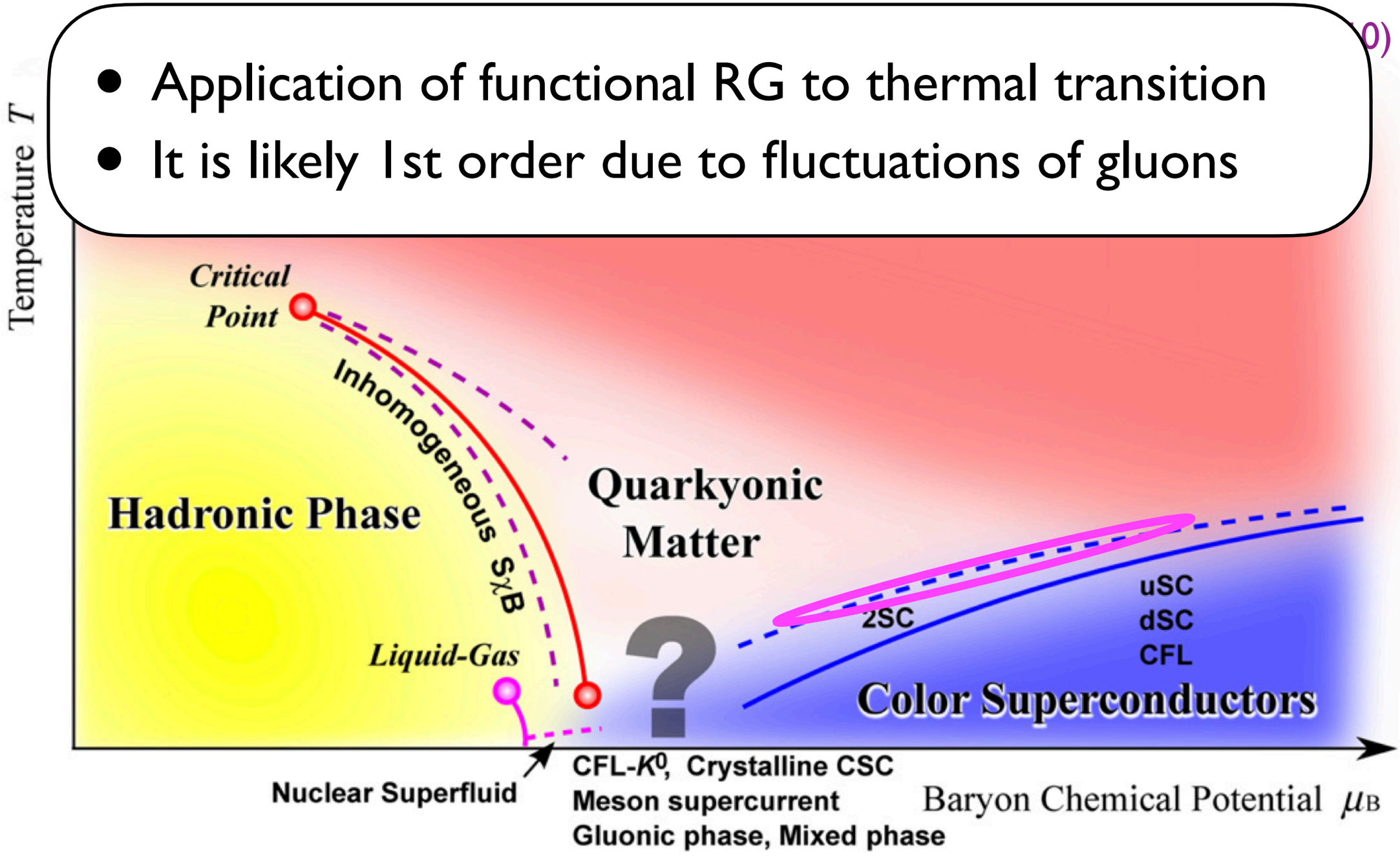
# Contents

- Emergent Chern-Simons theory in 2SC
- Vector meson as an excitation of quantum Hall droplet



# Contents

- Application of functional RG to thermal transition
- It is likely 1st order due to fluctuations of gluons



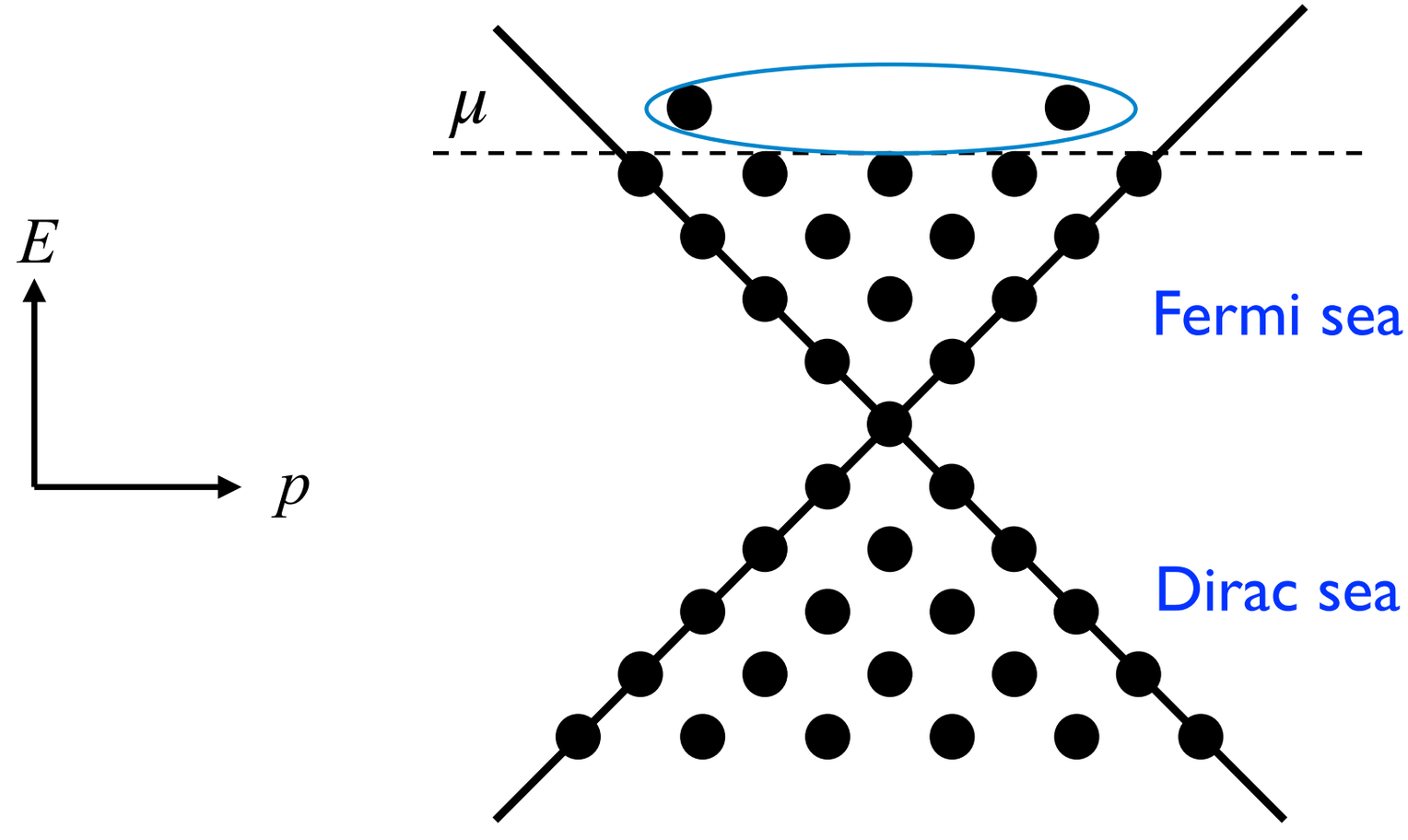

Non-Abelian nature of QCD is essential in all cases

# Quark matter

high density  high-density limit

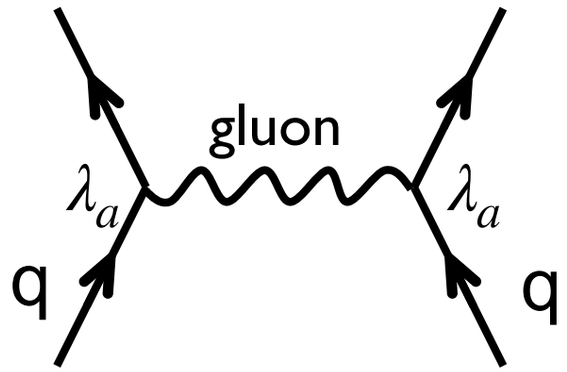
- Nuclear matter → Quark matter → Almost free quark gas

Collins-Perry, PRL (1975)



# Color superconductivity

- $\mu_{\rm B} \gg \Lambda_{\rm QCD}$: one-gluon exchange int. (asymptotic freedom)
- Attractive interaction in the color asymmetric channel




$$(\lambda_a)_{ij}(\lambda_a)_{kl} = \frac{2}{3}(\lambda_{\rm S})_{ik}(\lambda_{\rm S})_{lj} - \frac{4}{3}(\lambda_{\rm A})_{ik}(\lambda_{\rm A})_{lj}$$

cf. quarks are bound to form a nucleon in the vacuum

- Attractive interaction leads to quark-quark pairing (BCS mechanism)

cf. metallic superconductivity: Coulomb repulsion between electrons

Color superconductivity comes from non-Abelian nature of QCD

# Diquark condensate

- QCD has various quantum numbers:

$$\left\langle q^a_{\mathrm{R}i}(\boldsymbol{p}) q^b_{\mathrm{R}j}(-\boldsymbol{p}) \right\rangle = (d_{\mathrm{R}})^c_k \epsilon^{abc} \epsilon_{ijk}$$

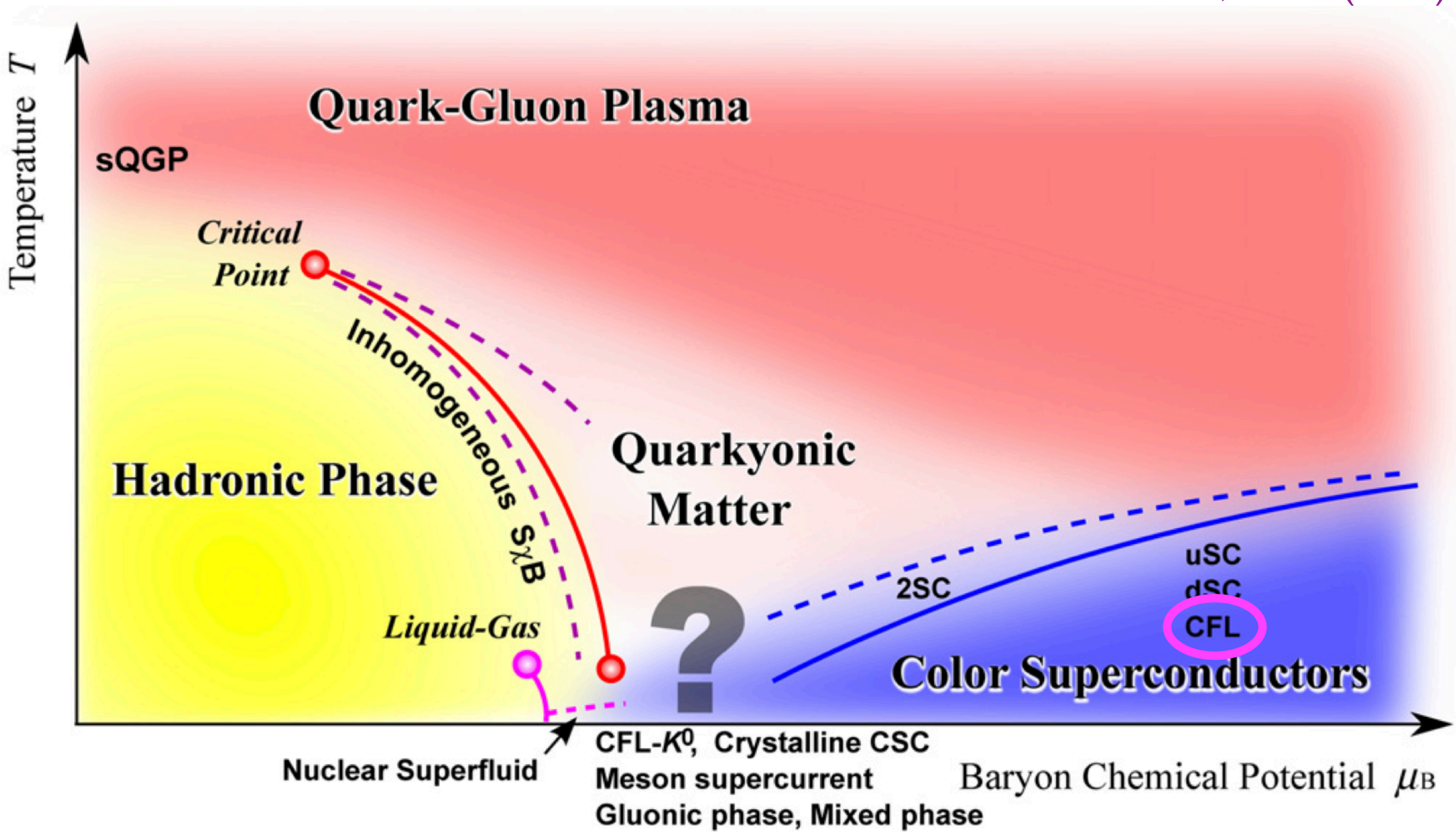
color a, b = R, G, B
flavor i, j = u, d, …

- Opposite momenta (BCS instability)
- s-wave pairing (opposite spins) → same helicity/chirality
- Color antisymmetric (attractive interaction)

  → Flavor antisymmetric (Pauli principle)

# QCD phase diagram

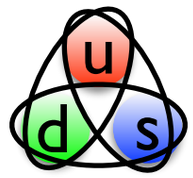
Fukushima-Hatsuda, PPNP (2010)



# Color-flavor locking

- 3-flavor limit: $\langle (q_{\rm R})^a_i (q_{\rm R})^b_j \rangle = (d_{\rm R})^c_k \epsilon^{abc} \epsilon_{ijk}$

color flavor




$(d_{\rm R})^c_k \propto \delta^c_k$ color-flavor licking (CFL): ground state at $\mu_{\rm B} \gg \Lambda_{\rm QCD}$

Alford-Rajagopal-Wilczek, NPB (1999)

- Symmetry breaking: $SU(3)_C \times SU(3)_L \times SU(3)_R \times U(1)_B \to SU(3)_{C+L+R}$
- Color-singlet order parameter: $(\bar{d}^a_{\rm L})(d^a_{\rm R}) \sim \bar{q}_{\rm L} \bar{q}_{\rm L} q_{\rm R} q_{\rm R}$ etc.
- Same symmetry as the hyper-nuclear matter at low density

→ Quark-hadron continuity Schäfer-Wilczek, PRL (1999)

# Importance of QCD anomaly

- Without QCD anomaly, discrete symmetries are different:

  $\mathrm{U}(1)_\mathrm{A} \to \mathbb{Z}_2$ by $\langle \bar{q}_\mathrm{L} q_\mathrm{R} \rangle$ in nuclear matter;
  $\mathrm{U}(1)_\mathrm{A} \to \mathbb{Z}_4$ by $\langle \bar{q}_\mathrm{L} \bar{q}_\mathrm{L} \rangle \langle q_\mathrm{R} q_\mathrm{R} \rangle$ in CFL

Different symmetries → phase transition

- QCD anomaly explicitly breaks $\mathrm{U}(1)_\mathrm{A} \to \mathbb{Z}_{2N_\mathrm{f}} \ (= \mathbb{Z}_6)$:

  $\mathbb{Z}_6 \to \mathbb{Z}_2$ by $\langle \bar{q}_\mathrm{L} q_\mathrm{R} \rangle$ in nuclear matter;
  $\mathbb{Z}_6 \to \mathbb{Z}_{\mathrm{gcd}(6,4)} = \mathbb{Z}_2$ by $\langle \bar{q}_\mathrm{L} \bar{q}_\mathrm{L} \rangle \langle q_\mathrm{R} q_\mathrm{R} \rangle$ in CFL

QCD anomaly (instantons) in CFL is essential for the continuity

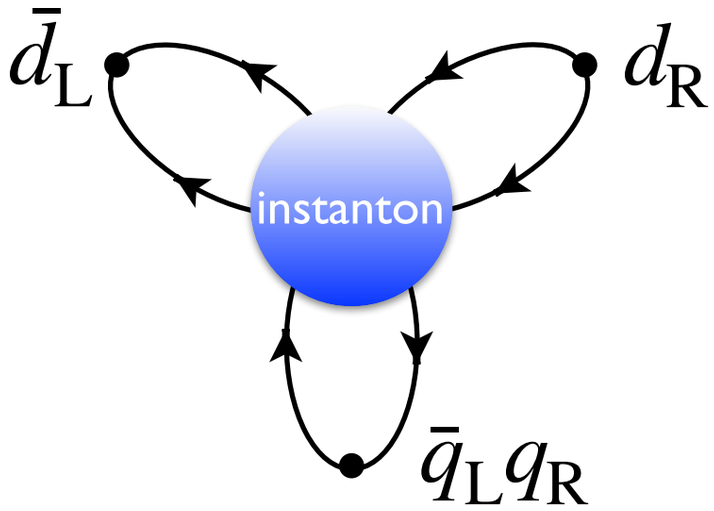
Hatsuda, Baym, Tachibana, Yamamoto, PRL (2006)

# Anomaly-induced continuity

Hatsuda, Baym, Tachibana, Yamamoto, PRL (2006)

- Instanton-induced interaction in CFL

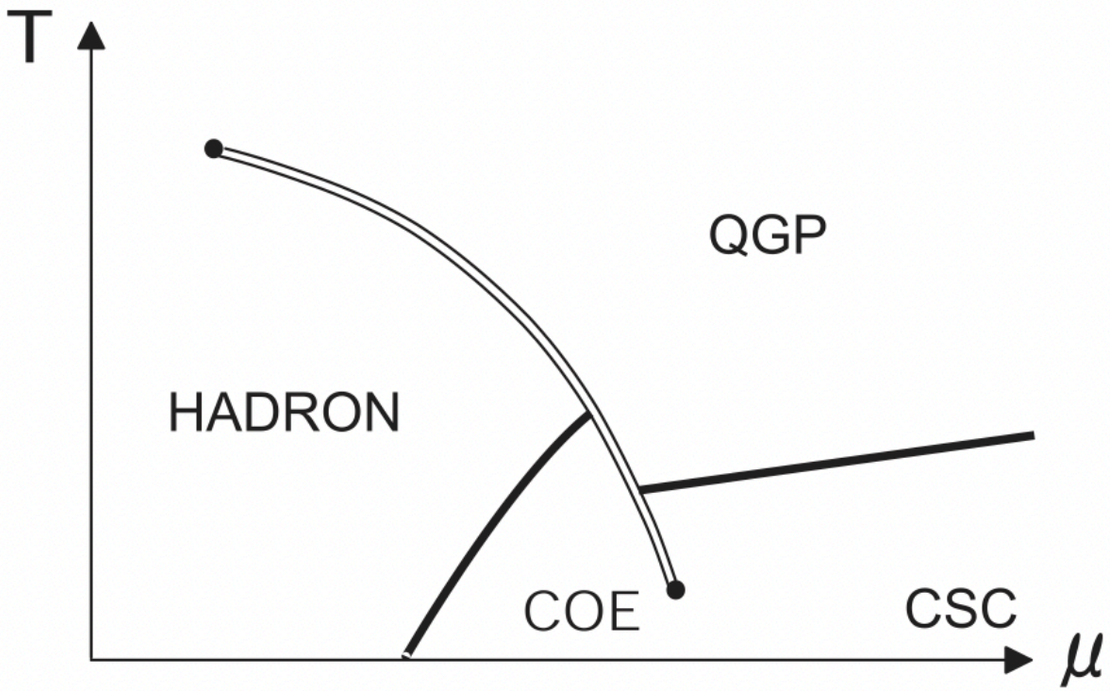
$$\det{}_{\rm f}(\bar{q}^i_{\rm L} q^j_{\rm R}) \sim \underbrace{(\bar{q}_{\rm L}\bar{q}_{\rm L})}_{\bar{d}_{\rm L}}\ \underbrace{(q_{\rm R}q_{\rm R})}_{d_{\rm R}}\ \underbrace{(\bar{q}_{\rm L}q_{\rm R})}_{\text{chiral cond.}}$$




- It acts as a "source" for chiral condensate (like mass) even w/o mass

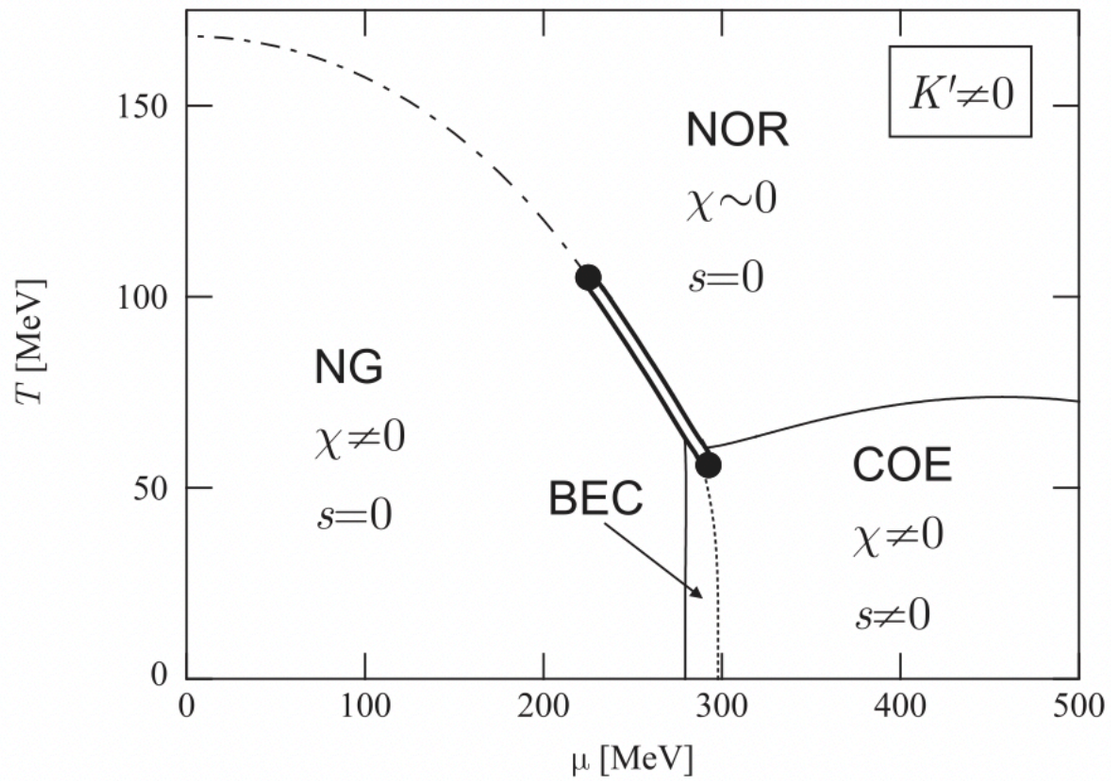
  → leads to crossover (like crossover in finite-T QCD)

# Continuity and critical points




Ginzburg-Landau analysis

Hatsuda, Baym, Tachibana, Yamamoto, PRL (2006)



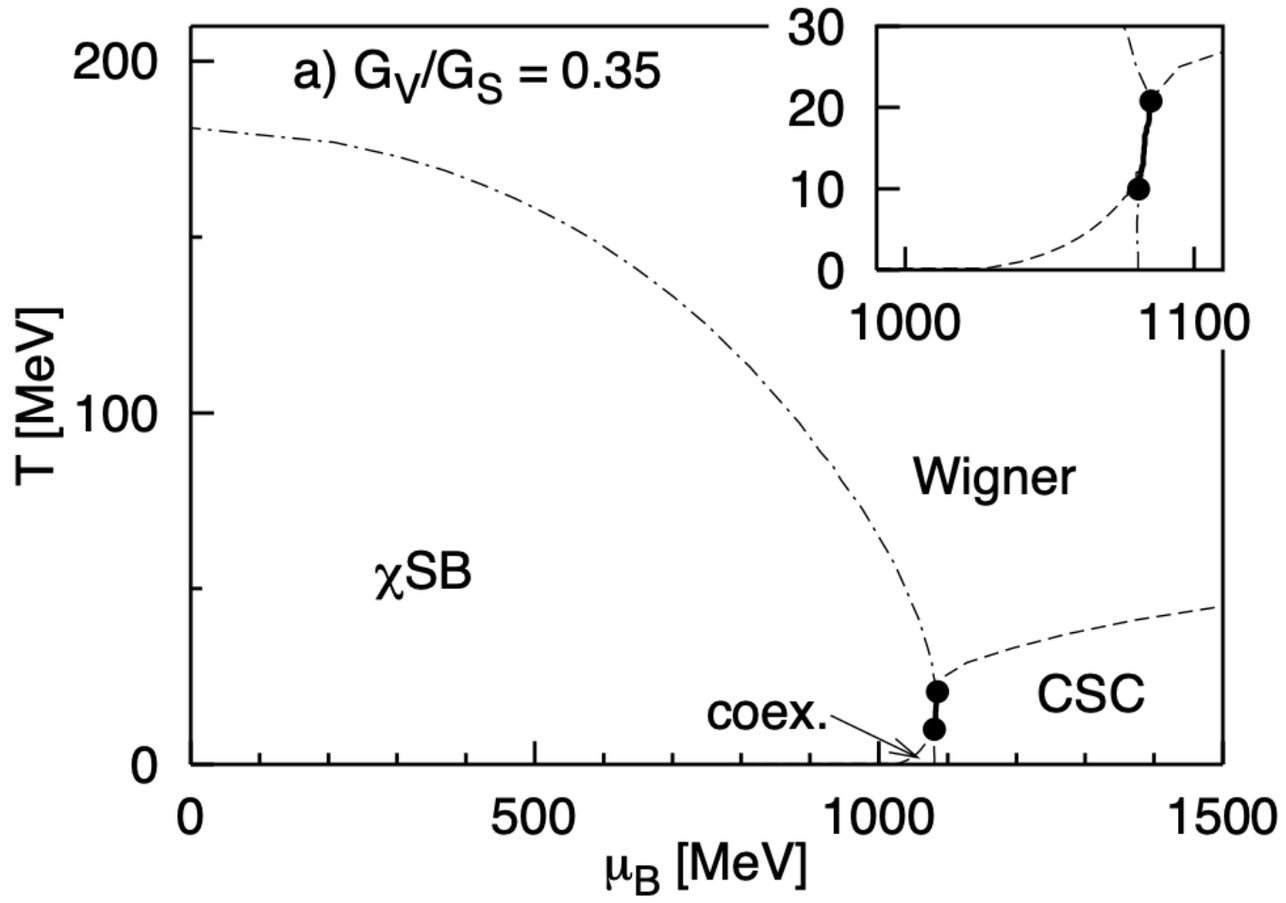

NJL model analysis

Abuki, Baym, Hatsuda, Yamamoto, PRD (2010)

Align with constraints on neutron-star EOS w/o 1st order transition

see, e.g., Baym, Hatsuda, Kojo, Powell, Song, Takatsuka, RPP (2018)

# Caveats

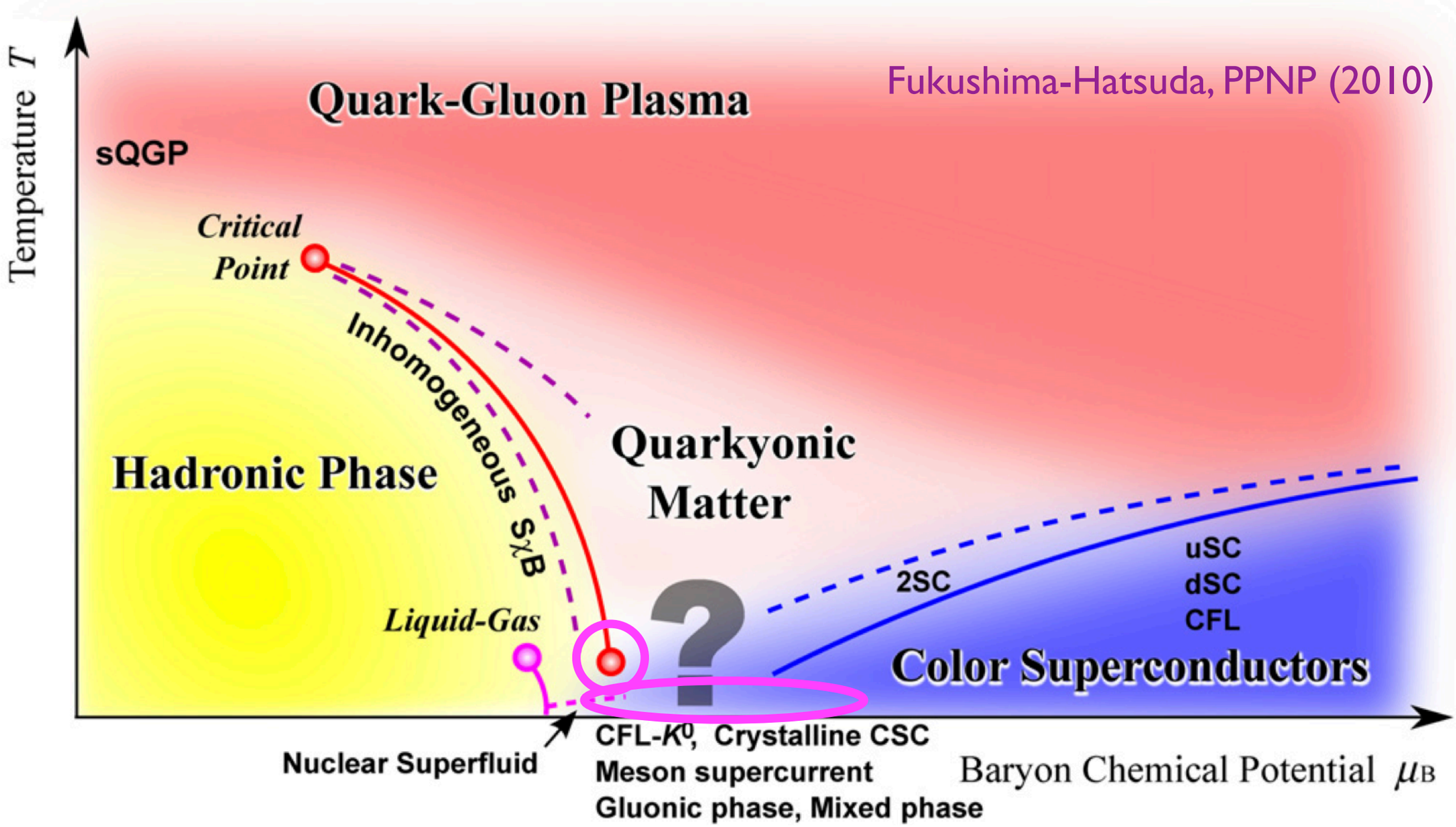



2-flavor NJL model w/ vector interaction

Kitazawa, Koide, Kunihiro, Nemoto, PTEP (2002)

Nuclear matter breaks $U(1)_B$ while 2SC does not break any global symm.; there must be a transition in 2-flavor QCD beyond NJL model

# QCD phase diagram



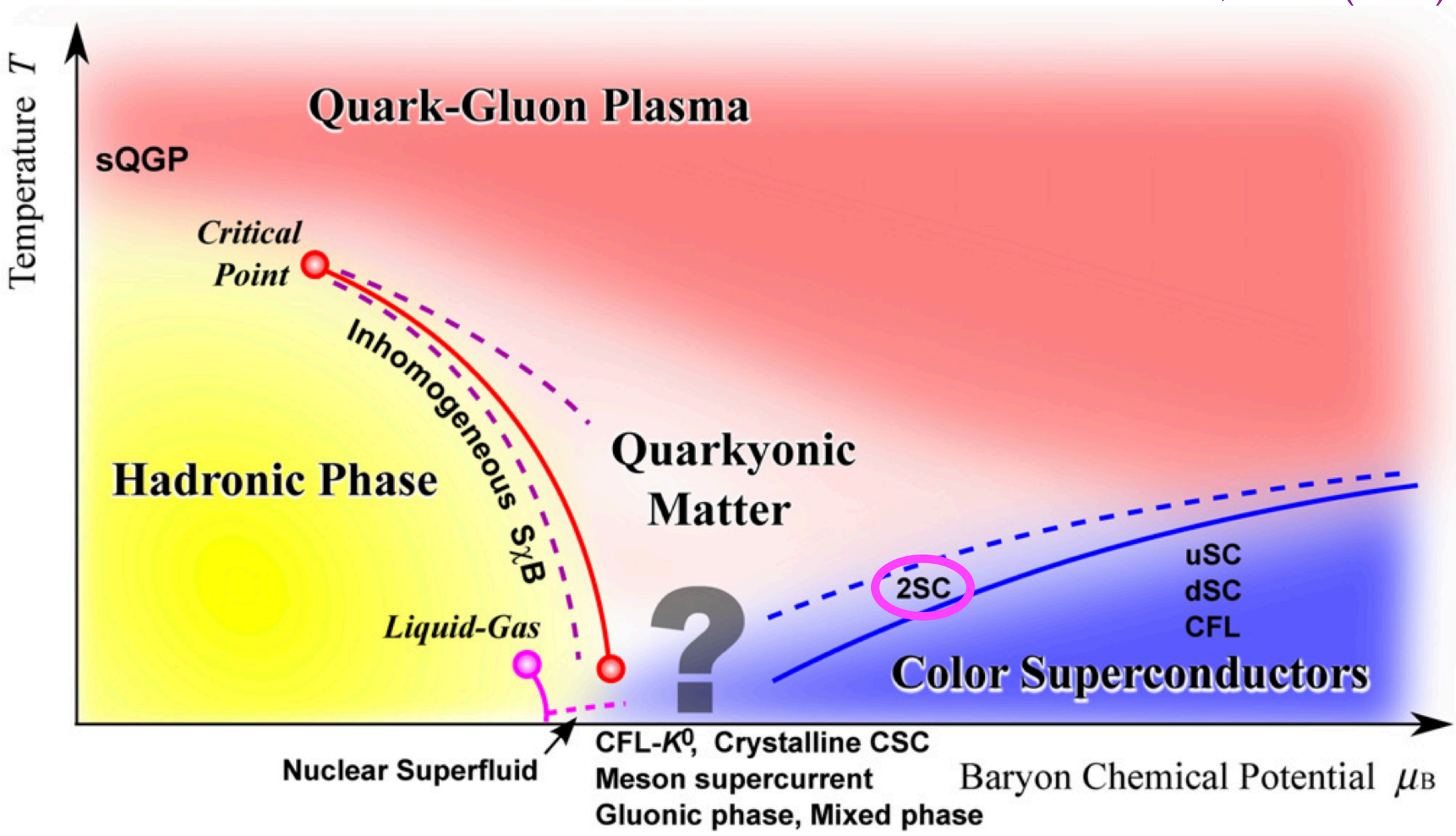

Fukushima-Hatsuda, PPNP (2010)

- Existence of high-T CP & continuity → existence of high-μ CP
- New dynamic universality due to $U(1)_B$ breaking Sogabe, Yamamoto, PRD (2017)

# Quantum Hall droplet in 2-flavor color super

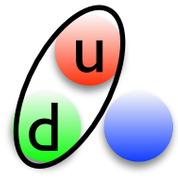
# QCD phase diagram

Fukushima-Hatsuda, PPNP (2010)



# 2-flavor color superconductivity

- 2-flavor limit: $\langle (q_{\rm R})^a_i (q_{\rm R})^b_j \rangle = d_{\rm R} \epsilon^{ab3} \epsilon_{ij}$ (color, flavor)




see, e.g., Alford-Rajagopal-Wilczek, PLB (1998)

- "Symmetry breaking": $SU(3)_C \to SU(2)_C$ by gauge fixing
  - $SU(2)_C$ sector: pure Yang-Mills theory at low energy → confined
- $\eta$ ($\bar{q}\bar{q}qq$ state) becomes lighter at higher density

  cf. $\eta$ is a $\bar{q}q$ state in the vacuum
- Low-energy d.o.f.s: unpaired quarks, confined SU(2) gluons, $\eta$ meson

N.B. $\eta$ does not couple to unpaired quarks at leading order

# Effective theory for η

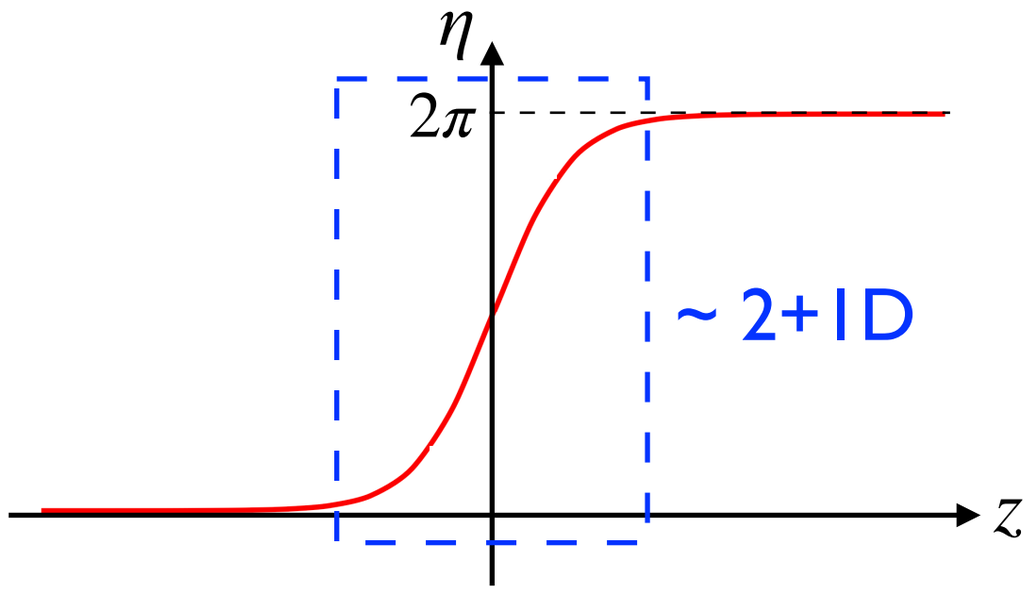
Son-Stephanov-Zhitnitsky, PRL (2001)

- $U(1)_A$ symmetry: $\eta \to \eta + \text{const}.\ (\eta + 2\pi \sim \eta)$

$$\mathcal{L} = f^2[(\partial_t \eta)^2 - v^2(\boldsymbol{\nabla}\eta)^2] + A\cos\eta$$

($f, v, A$ computable by weak-coupling analysis at high density)

- Domain-wall solution from $\eta = 0$ $(z = -\infty)$ to $\eta = 2\pi$ $(z = \infty)$:




Topological charge:

$$Q_{\rm DW} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \partial_z \eta = 1$$

EFT on 2+1D domain wall?
Low-energy d.o.f.s coupled to η?

# Domain Walls of High-Density QCD

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[2]*Department of Physics, University of Illinois, Chicago, Illinois 60607-7059*

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(Received 17 December 2000)

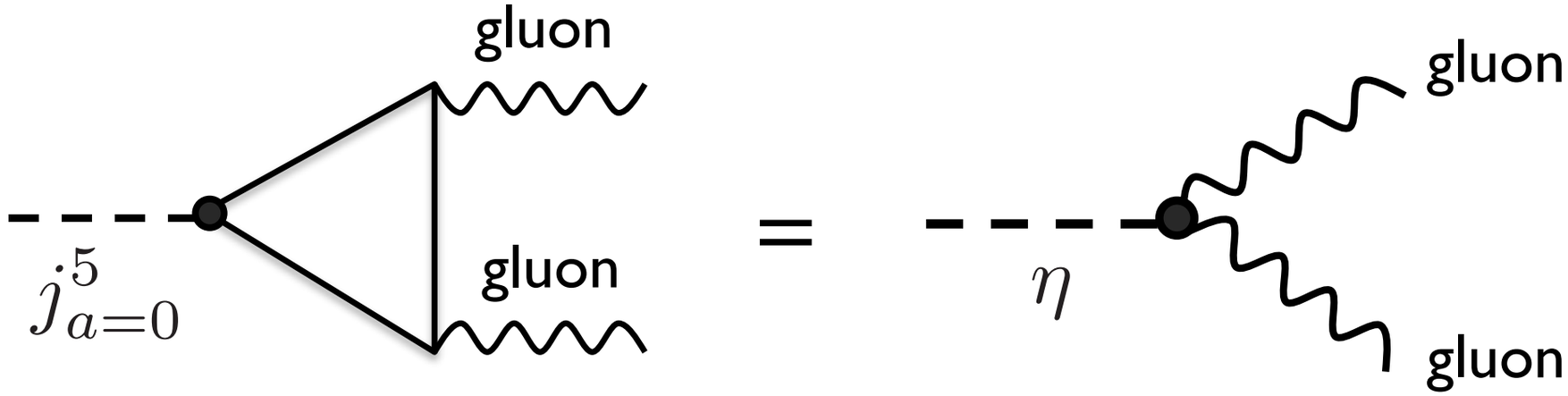
We show that in very dense quark matter there must exist metastable domain walls where the axial U(1) phase of the color-superconducting condensate changes by $2\pi$. The decay rate of the domain walls is exponentially suppressed and we compute it semiclassically. We give an estimate of the critical chemical potential above which our analysis is under theoretical control.

*Discussion.*—It would be interesting to investigate possible astrophysical consequences of the high-density QCD walls. In particular, one would like to know if such walls can be created inside neutron stars. To describe the motion of the wall, one may need more than just the effective Lagrangian (8): the coupling of $\eta$ to ungapped quarks and $SU(2)_c$ gluons could be important. The moving wall may

η-gluon coupling was ignored without any justification:
low-energy EFT is incomplete

# QCD anomaly

Coupling of η to gluons emerges through the QCD anomaly




UV (quark)

IR (hadron)

$$\partial_\mu j^{\mu 5} = -\frac{1}{4\pi^2}\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

$$S_{\mathrm{anom}} = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\ \eta\ \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

$$\because \eta \sim \bar{q}\bar{q}qq \rightarrow Q_5(\eta) = 4Q_5(q)$$

# Topological field theory on the wall

Nishimura, Yamamoto, Yokokura PRD (2025)

- η couples to confined SU(2) gluons via QCD anomaly:

$$S_{\rm anom} = -\frac{1}{16\pi^2}\int {\rm d}^4x\ \eta\ {\rm tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \qquad \boxed{F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma + \cdots)}$$

$$\to -\frac{1}{8\pi^2}\int_{-\infty}^{\infty}{\rm d}z\partial_z\eta\int {\rm d}^3x\epsilon^{\mu\nu\rho}{\rm tr}(A_\mu\partial_\nu A_\rho + \cdots) =: S_{\rm CS}[A]$$

$2\pi$ for DW

- $SU(2)_{-1}$ Chern-Simons (CS) theory on the wall:

$$S_{\rm CS}[A] = -\frac{1}{4\pi}\int {\rm d}^3x\ \epsilon^{\mu\nu\rho}{\rm tr}\left(A_\mu\partial_\nu A_\rho - \frac{2{\rm i}}{3}A_\mu A_\nu A_\rho\right)$$

SU(2) gauge field

$\rightleftarrows$ $U(1)_2$ CS theory (level-rank duality):

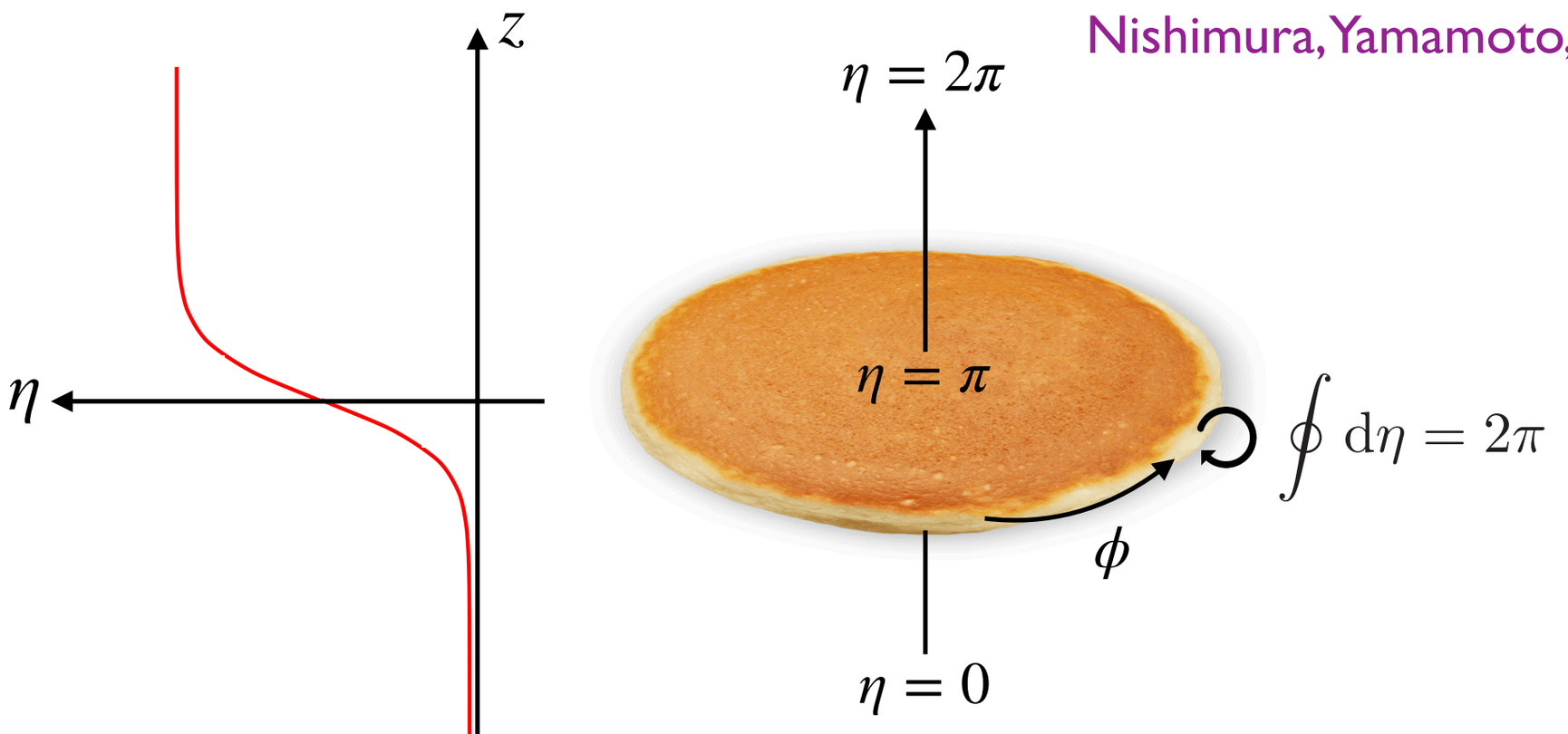
$$S_{\rm CS}[a] = \frac{2}{4\pi}\int {\rm d}^3x\ \epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho$$

U(1) gauge field

# Quantum Hall droplet

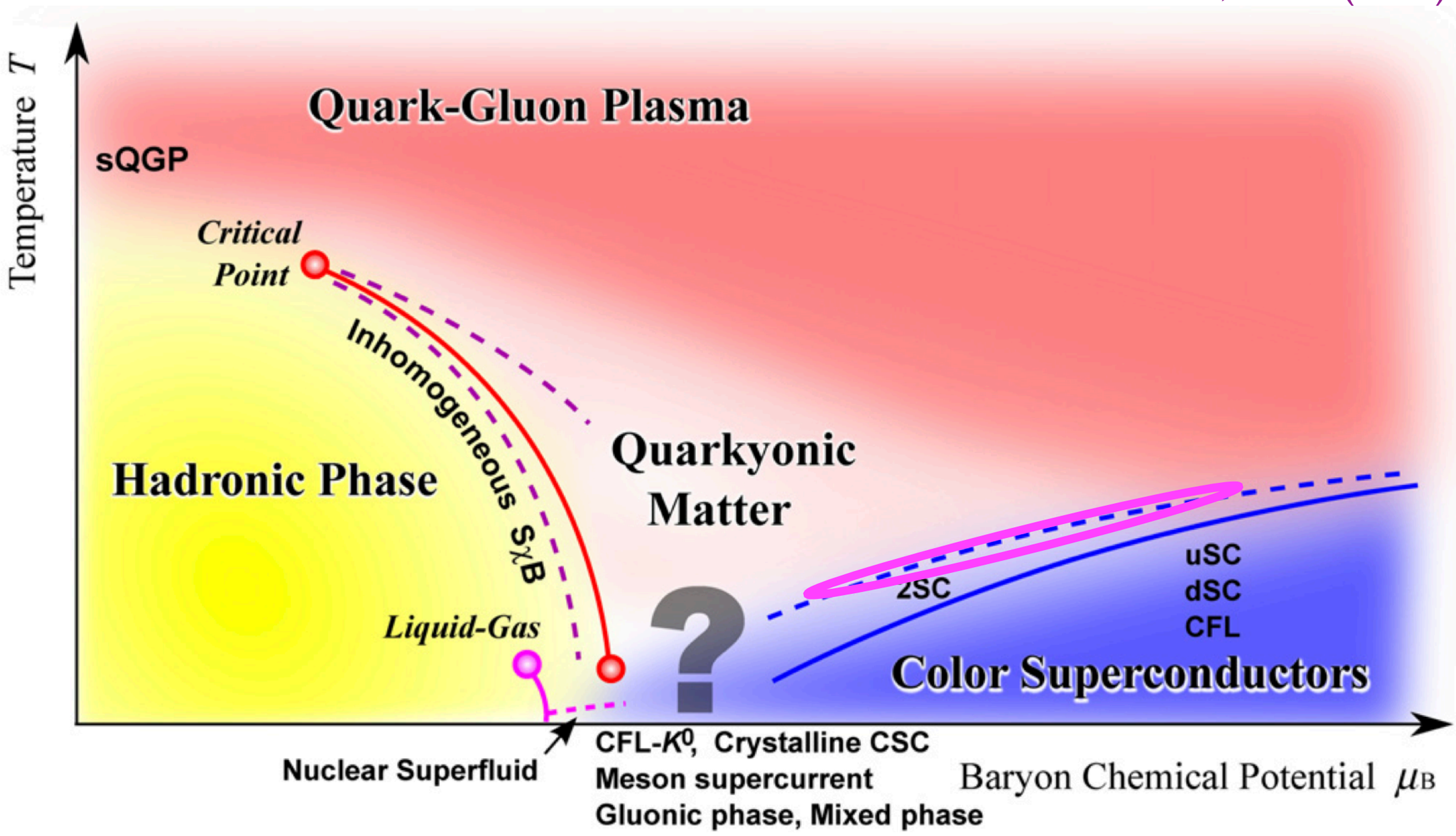
- Edge theory $S_{\rm edge}[\phi]$: 1+1D CFT for massless scalar $\phi$
  - Energy ~ $1/R$, spin: 2/2 = 1
- Total energy of the system: $E(R) = \pi R^2 T_{\rm DW} + 2\pi R T_{\rm string} + \dfrac{C}{2\pi R}$
- Droplet stabilized by the edge mode ~ flavor-singlet vector meson

Nishimura, Yamamoto, Yokokura PRD (2025)



# QCD phase diagram

Fukushima-Hatsuda, PPNP (2010)



# Thermal phase transition

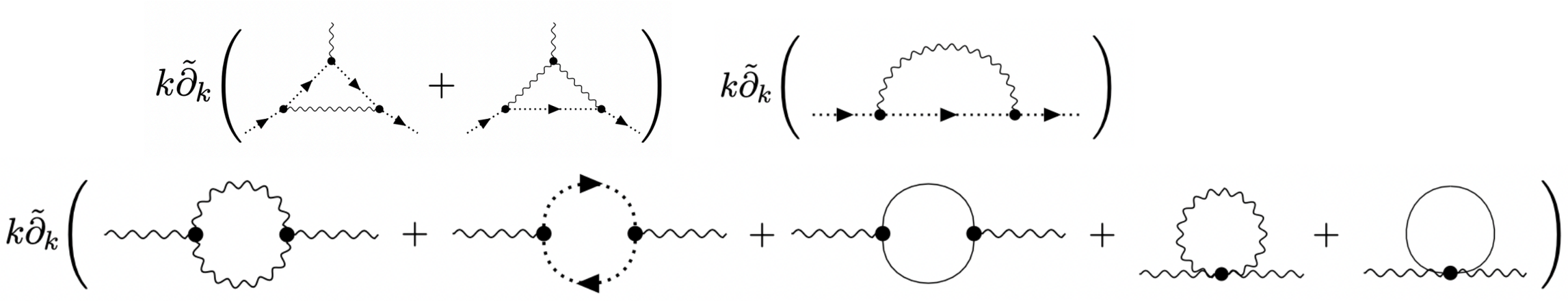
- Phase transitions are typically 2nd order at the mean-field level
- Fluctuations may induced 1st order transition (Coleman-Weinberg)
- Metallic superconductivity exhibits 2nd order even w/ fluctuations
- How about color superconductivity?
  - Thermal gluon fluctuations → 1st order transition at weak coupling

Giannakis, Hou, Ren, Rischke, PRL (2004)

Gluon fluctuations not incorporated in NJL model are important

# Functional RG analysis

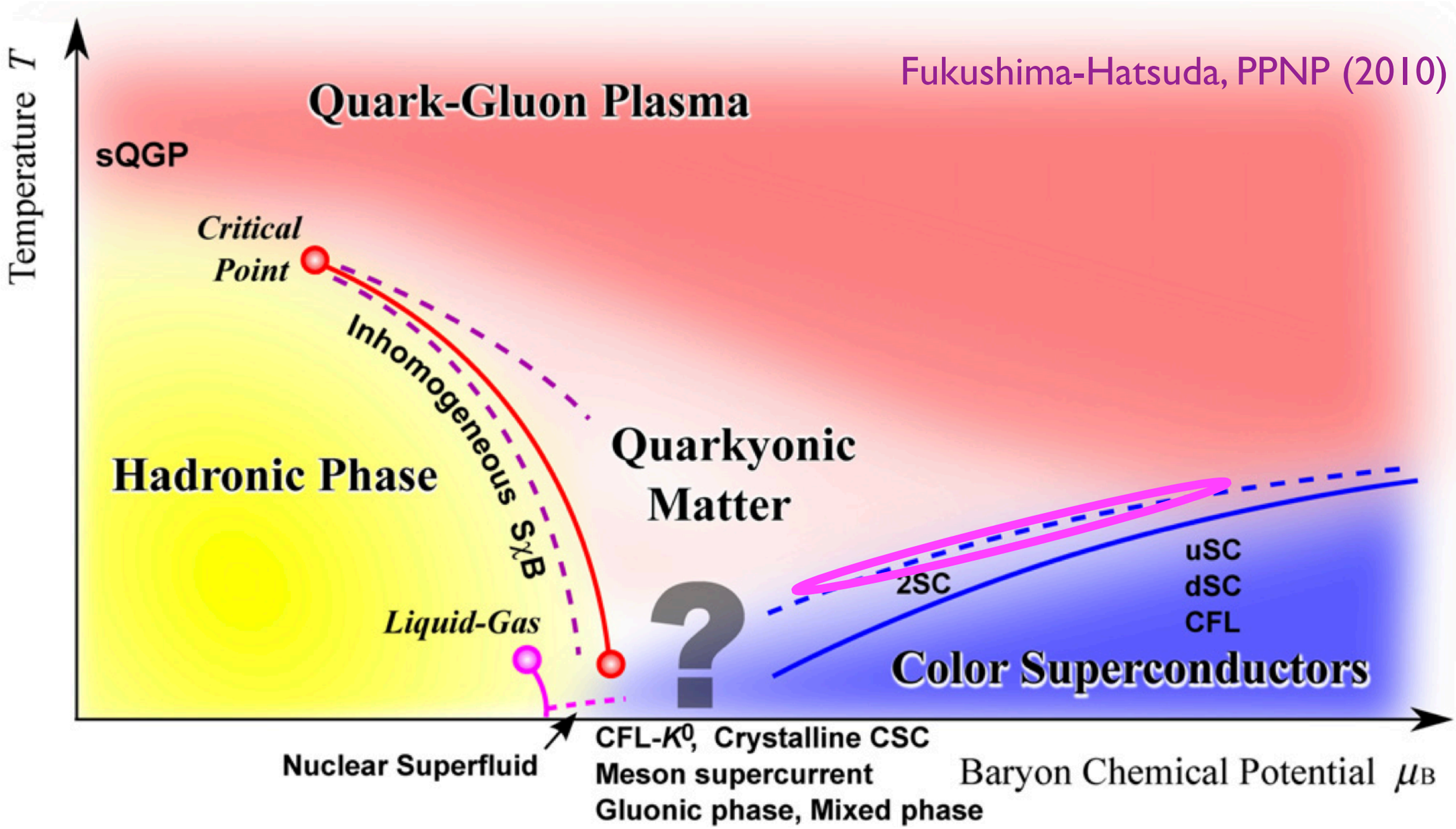
- Ginzburg-Landau theory: $\mathcal{L}_0 = -\frac{1}{2}\,\mathrm{Tr}\,(F_{ij}F_{ij}) + \mathrm{Tr}\left[(D_i\phi)^\dagger(D_i\phi)\right] + V(\phi)$
- Flow equation: $k\partial_k\Gamma_k = \frac{1}{2}\int k\tilde{\partial}_k\,\mathrm{Tr}\,\log\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)$ Wetterich, PLB (1993)
- Regulator: $R_k(q) = (k^2 - q^2)\Theta(k^2 - q^2)$ Litim, PRD (2001)



$\beta(g) < 0$ for all $g$ if $N_c = 3$ and $N_f \leq 55$ Fejös, Yamamoto, JHEP (2019)

No IR fixed point → 1st order (related to asymptotic freedom)

# QCD phase diagram




Fukushima-Hatsuda, PPNP (2010)

Presence of 2nd order transition is relevant for observables, e.g., critical dilepton enhancement Nishimura-Kitazawa-Kunihiro, PTEP (2022)

# Conclusion & Outlook

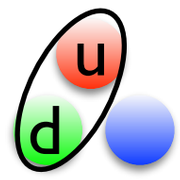
- Quark-hadron continuity is possible only w/ QCD anomaly
  - High-T critical point & continuity → High-μ critical point
    - New dynamic universality class → possible observables?
- Chern-Simons theory naturally emerges in 2SC phase
  - Quantum Hall droplet as a vector meson
- Thermal color super transition likely 1st order by gluon effects
  - Possibility of 2nd order has to be carefully examined

# Backup slides

# 2-flavor color superconductivity

- 2-flavor limit: $\langle (q_{\rm R})^a_i (q_{\rm R})^b_j \rangle = d_{\rm R} \epsilon^{ab3} \epsilon_{ij}$

color flavor




see, e.g., Alford-Rajagopal-Wilczek, PLB (1998)

- "Symmetry breaking": $SU(3)_C \times U(1)_B \to SU(2)_C \times U(1)_{\tilde{B}}$

$$\tilde{B} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Baryon number is carried only by unpaired (blue) quarks

# Confined SU(2) sector

Rischke-Son-Stephanov, PRL (2001)

- Action: $S = \dfrac{1}{g^2}\displaystyle\int \mathrm{d}^4x \left(\frac{\epsilon}{2}\boldsymbol{E}^a\cdot\boldsymbol{E}^a - \frac{1}{2\lambda}\boldsymbol{B}^a\cdot\boldsymbol{B}^a\right)$
- Weak-coupling analysis: $\epsilon = 1 + \dfrac{g^2\mu^2}{18\pi^2\Delta^2} \gg 1\,, \quad \lambda = 1$
- Manifestly Lorentz-covariant form: $S = -\dfrac{1}{2g'^2}\displaystyle\int \mathrm{d}^4x' \mathrm{Tr}(F'_{\mu\nu}F'^{\mu\nu})$

$$(x^0)' := \epsilon^{-1/2}x^0,\ (A_0^a)' := \epsilon^{1/2}A_0^a, \text{ and } g' := \epsilon^{-1/4}g$$

- Effective coupling constant: $\alpha'_{\mathrm{s}} := \dfrac{g'^2}{4\pi} \simeq \dfrac{3}{2\sqrt{2}}\dfrac{g\Delta}{\mu}$
- Emergent confinement scale: $\Lambda'_{\mathrm{QCD}} \sim \Delta \exp\left(-\dfrac{3\pi}{11\alpha'_{\mathrm{s}}}\right) \simeq \Delta \exp\left(-\dfrac{2\sqrt{2}\pi}{11}\dfrac{\mu}{g\Delta}\right)$

# Physics at the edge

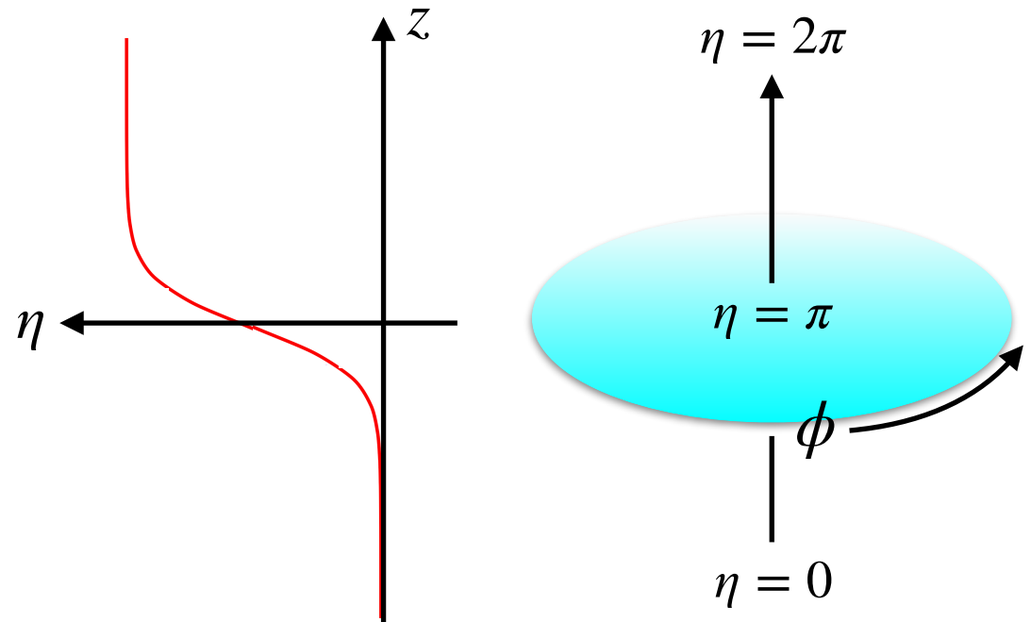
see, e.g., X. G. Wen, "Quantum Field Theory of Many-Body Systems"

- Gauge fixing $a_t + \omega a_\theta = 0$ in $(r, \theta)$ coordinates
- $\tilde{\theta} := \theta - \omega t,\ \tilde{t} := t,\ \tilde{r} := r \to \tilde{a}_{\tilde{t}} := a_t + \omega a_\theta = 0,\ \tilde{a}_{\tilde{\theta}} = a_\theta,\ \tilde{a}_{\tilde{r}} = a_r$

$$S_{\mathrm{CS}}[a] = \frac{2}{4\pi} \int \mathrm{d}^3 x \epsilon^{\tilde{\mu}\tilde{\nu}\tilde{\rho}} \tilde{a}_{\tilde{\mu}} \partial_{\tilde{\nu}} \tilde{a}_{\tilde{\rho}}$$

- EOM for $\tilde{a}_{\tilde{t}}$: $\tilde{f}_{\tilde{\theta}\tilde{r}} = 0 \to \tilde{a}_{\tilde{i}} = \partial_{\tilde{i}} \phi$ $(\tilde{i} = \tilde{\theta}, \tilde{r})$ as a constraint
- Edge theory = 1+1D CFT (chiral Tomonaga-Luttinger liquid):

$$\begin{aligned} S_{\mathrm{edge}} &= \frac{2}{4\pi} \int \mathrm{d}\tilde{t} \mathrm{d}\tilde{\theta} \partial_{\tilde{t}} \phi \partial_{\tilde{\theta}} \phi \\ &= \frac{2}{4\pi} \int \mathrm{d}t \mathrm{d}\theta (\partial_t + \omega \partial_\theta) \phi \partial_\theta \phi \end{aligned}$$



# 2D CFT analysis

- For generic $O$ under $z \to w(z)$ in complex plane $z = x_1 + \mathrm{i}x_2$,

$$O'(w,\bar{w}) = \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^{h} \left(\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{w}}\right)^{\bar{h}} O(z,\bar{z})$$

$$\longrightarrow O'(\lambda z, \lambda\bar{z}) = \lambda^{-\underbrace{(h+\bar{h})}_{\text{scaling}}} O(z,\bar{z}), \quad O'(\mathrm{e}^{\mathrm{i}\theta} z, \mathrm{e}^{-\mathrm{i}\theta}\bar{z}) = \mathrm{e}^{-\mathrm{i}\underbrace{(h-\bar{h})}_{\text{spin}}\theta} O(z,\bar{z})$$

- Two-point function for the vertex operator $V_N = \mathrm{e}^{\mathrm{i}N\phi}$ ($N = 2$):

$$\langle V_N(z) V_M(0)\rangle \sim \mathrm{e}^{-NM\langle\phi(z)\phi(0)\rangle} \sim \frac{\delta_{N+M,0}}{z^N} \qquad \because \langle\phi(z)\phi(0)\rangle = -\tfrac{\#}{N}\ln|z|$$

- Scaling dimension & spin $N/2 = 1$, energy ~ $C/R$