Two-color QCD as a laboratory to explore cold and dense medium: Numerical experiments and effective models

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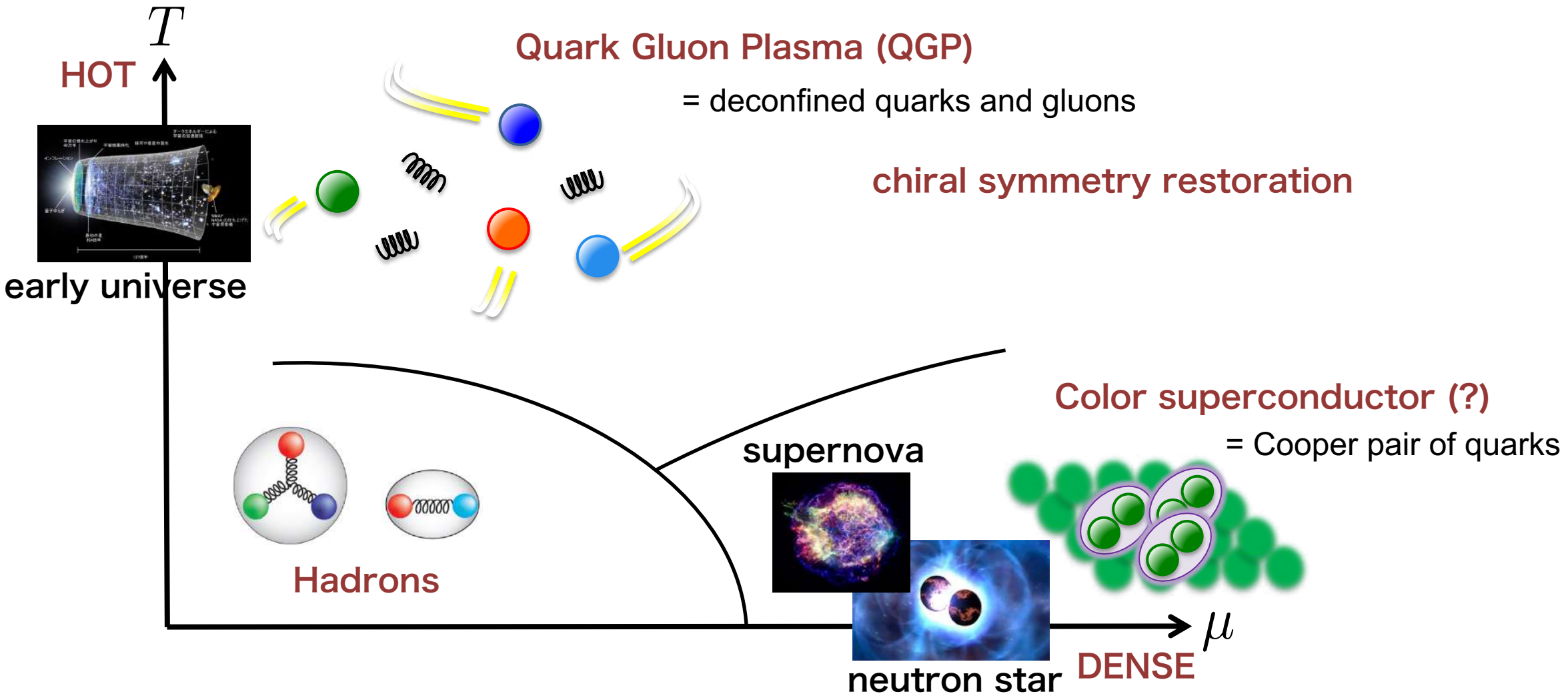
Introduction

- QCD phase diagram



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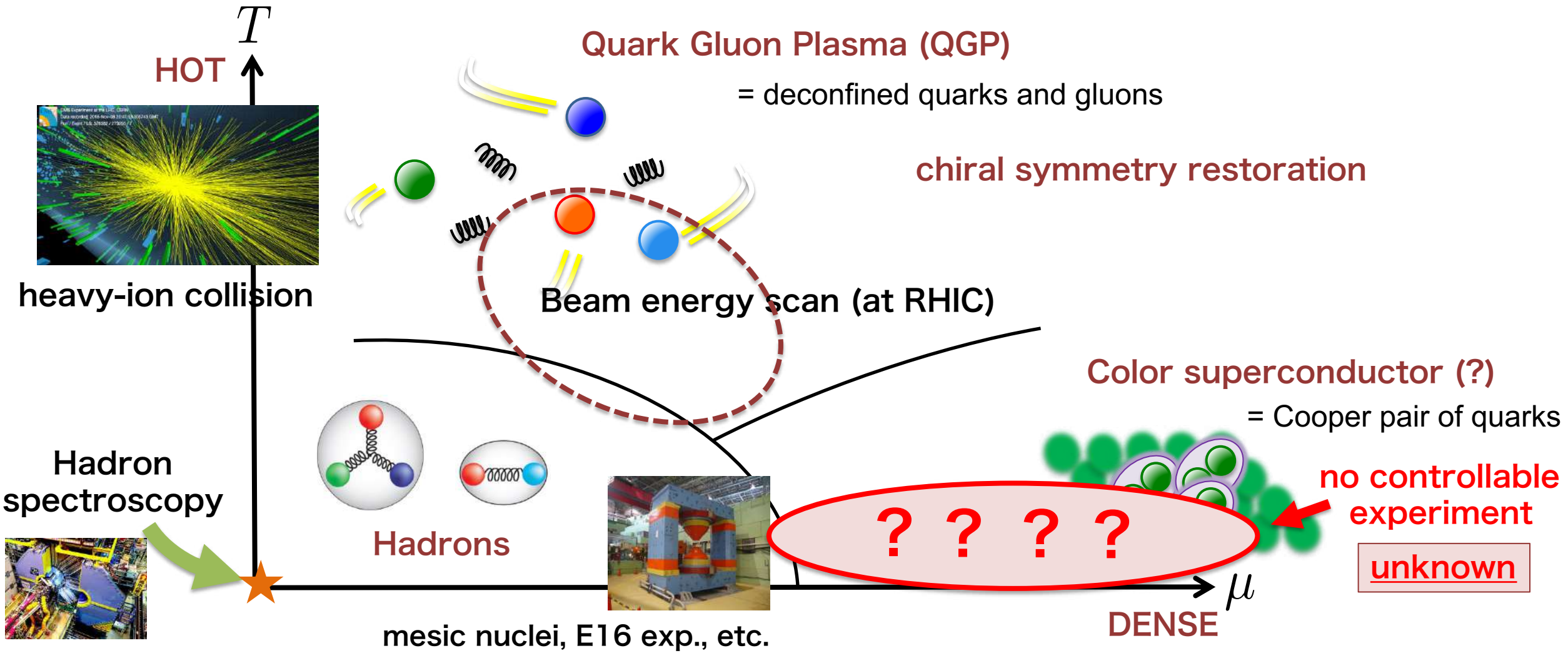

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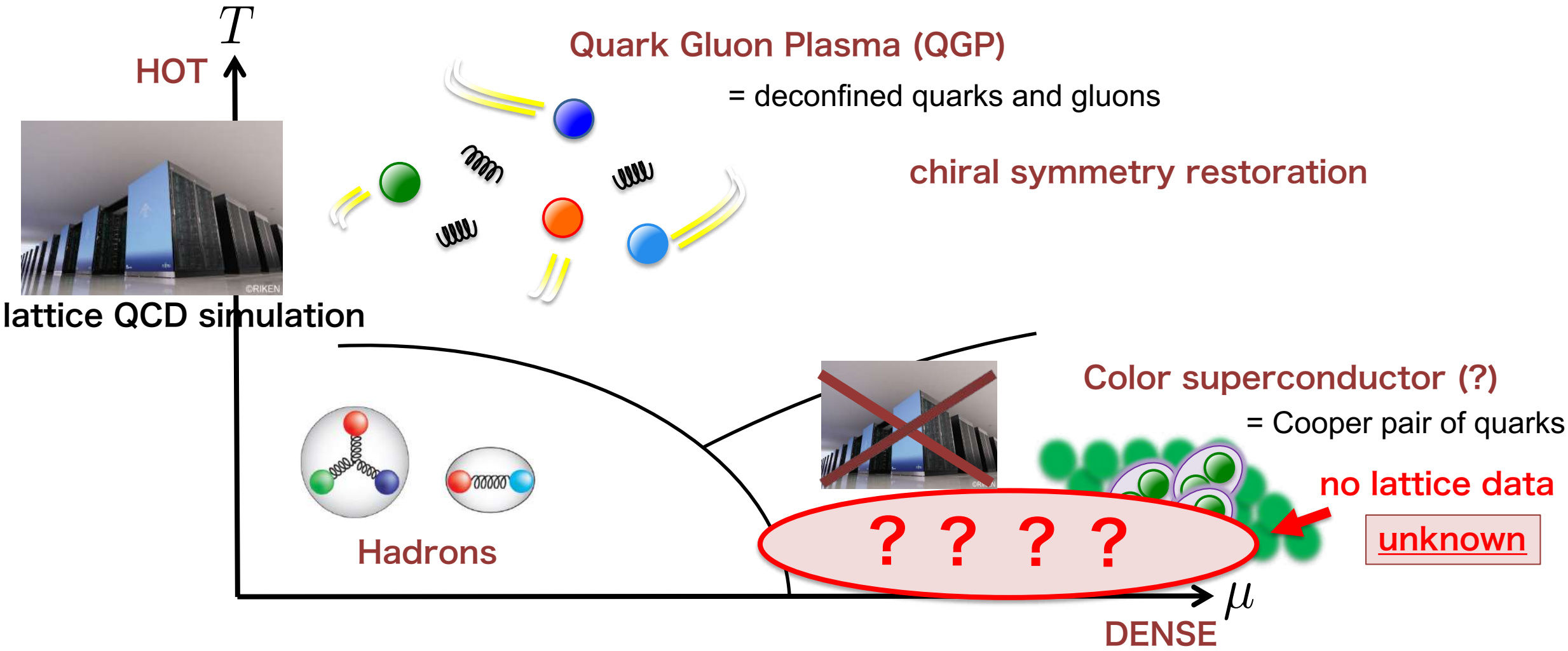
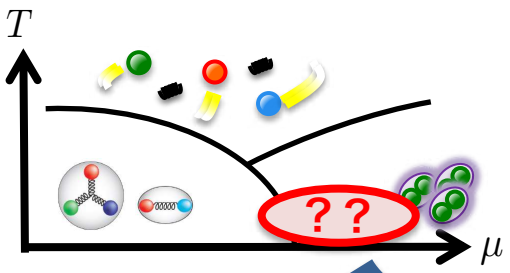


Introduction

- **Lattice study in dense QCD?**



three-color QCD (our world)

- Lattice sim. at density is not easy

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m) \in \mathbb{C}$$ ✕

sign problem

with $N_f = (\mathrm{even})$

two-color QCD (imaginary world)

- Lattice sim. at density is possible!

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m)^{N_f} \geq 0$$ ◎

sign problem disappears thanks to $SU(2)_c$ pseudoreality

Introduction

- **Lattice study in dense QCD?**

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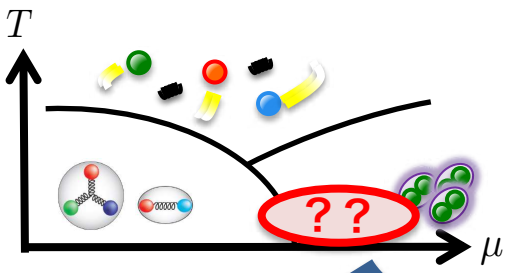

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sign problem disappears thanks to $SU(2)_c$ pseudoreality

- Baryon is made of three quarks

nucleon

some differences

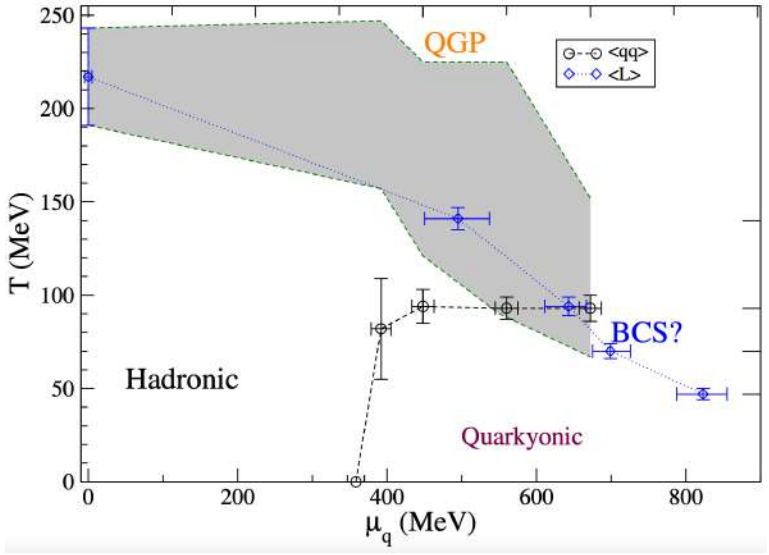
- Baryon is made of two quarks

diquark baryon

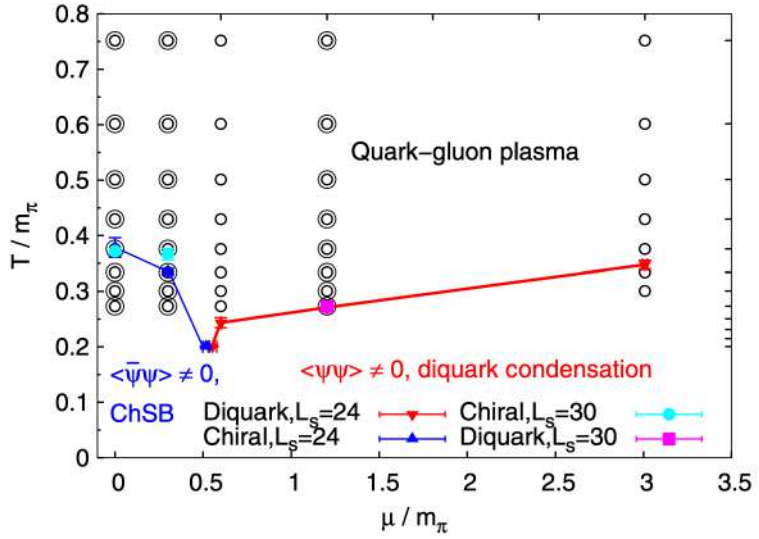
- **Two-color QCD** is a useful laboratory to explore cold-dense medium with lattice simulations

Introduction

- **Phase diagram in two-color QCD (QC_2D)**
  - Examples of simulation results of phase diagram in QC_2D

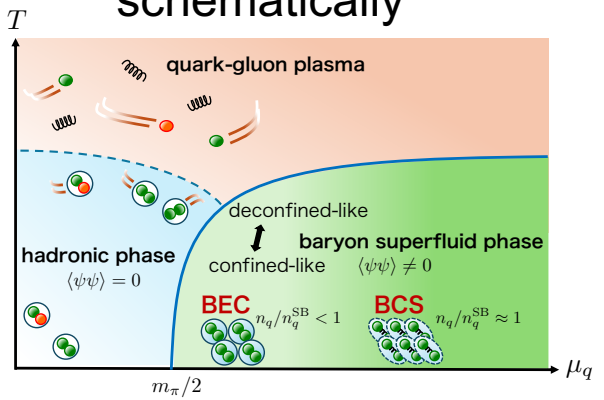


Boz-Cotter-Fister-Mehta-Skullerud (2013)



Buividovich-Smith-Smekal (2020)

schematically



...

- Currently at least four lattice simulation groups are active
  - Ireland/UK group (Hands, Skullerud, …)
  - UK group (Buividovich, ...)
  - Russian group (Bornyakov, ...)
  - Japanese group (Iida-san, Itou-san, …)

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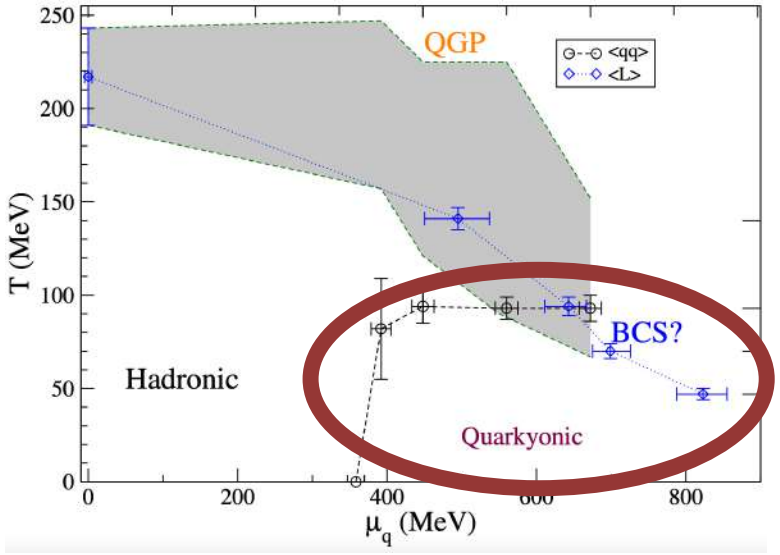

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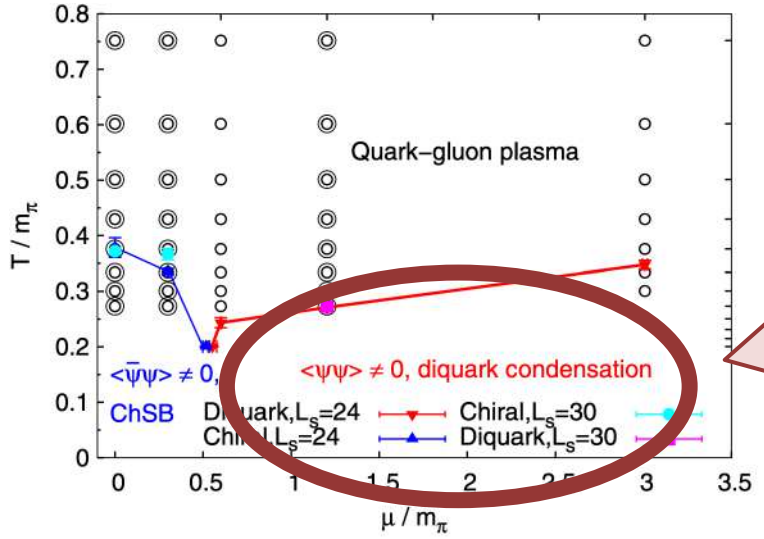

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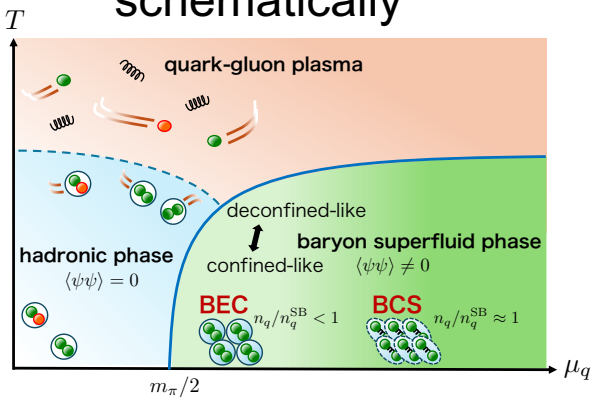

...

diquark condensed phase
(baryon superfluid phase)

$$\langle \psi\psi \rangle \neq 0$$

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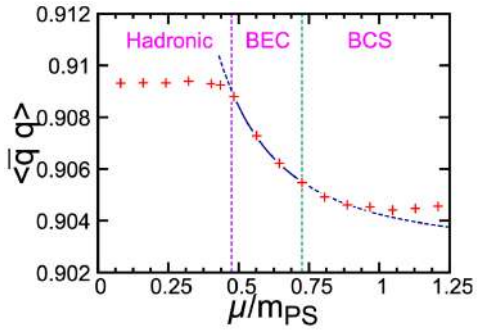
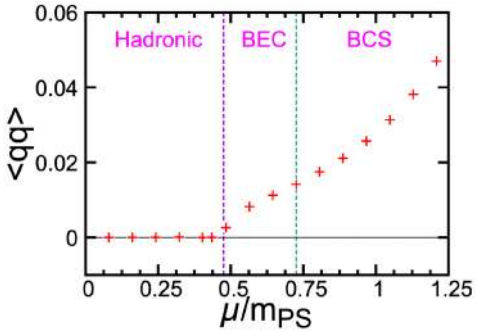
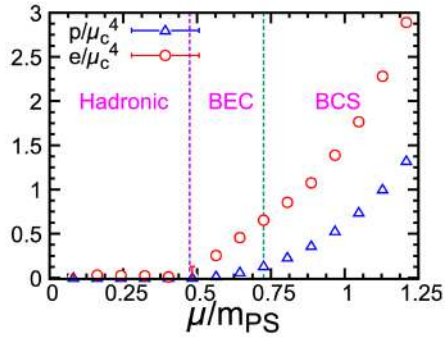
Introduction

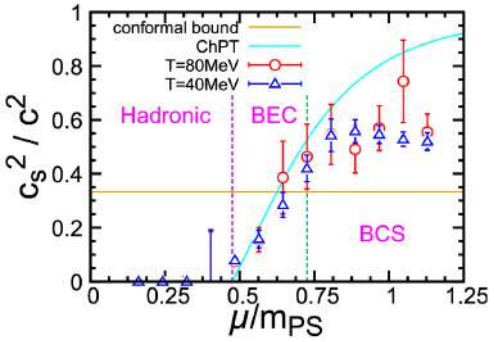
- **Lattice results**
  - In addition to phase diagram, hadron mass spectrum, gluon propagator, transport coefficient, EoS, sound velocity, $\langle\bar{\psi}\psi\rangle, \langle\psi\psi\rangle, \langle L\rangle$, etc. have been simulated

Japanese group









. . .

My approach

- (i) Regard QC_2D lattice simulations as useful "numerical experiments" of cold and dense QCD, and (ii) give interpretation from symmetry viewpoints based on effective models

Suenaga (2025) (Review paper)

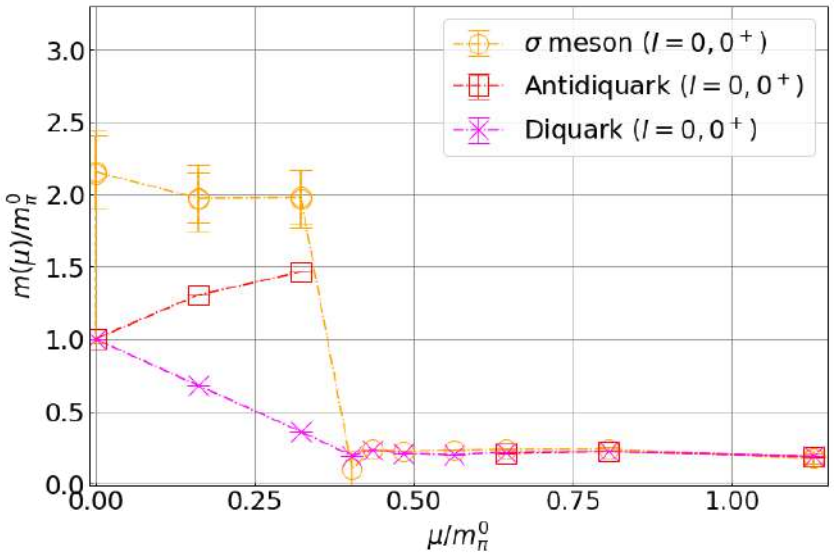
My publications on QC_2D

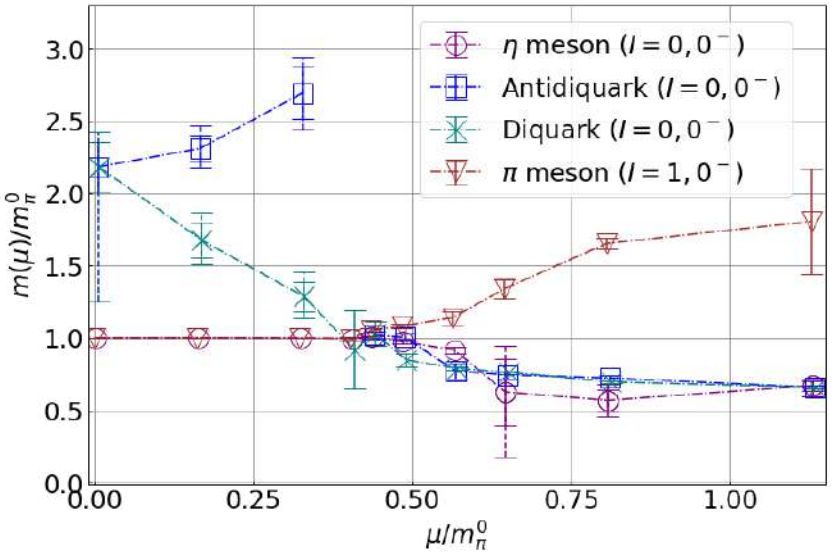
Gluon propagator: **Suenaga**-Kojo(2019), Kojo-**Suenaga**(2021), CSE effect: **Suenaga**-Kojo(2021), Sound velocity: Kojo-**Suenaga**(2022), Kawaguchi-**Suenaga**(2024), Topological susceptibility: Kawaguchi-**Suenaga**(2023), Hadron mass: **Suenaga**-Murakami-Itou-Iida (2023, 2024), FRG analysis: Fejos-**Suenaga**(2025, 2025), in preparations

Introduction

Lattice results on hadron masses

Murakami et al (2023)





← pion $(I=1,0^-)$

$I=0,0^-$ hadron **lighter than pion!**

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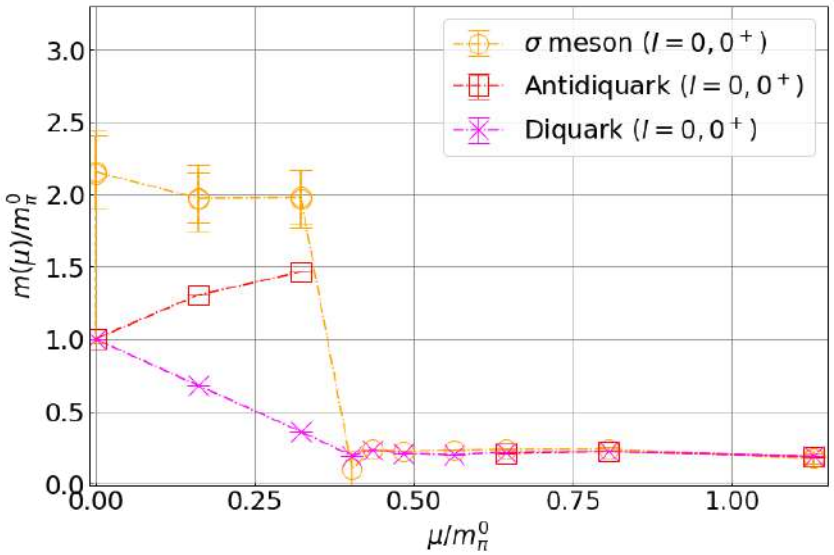
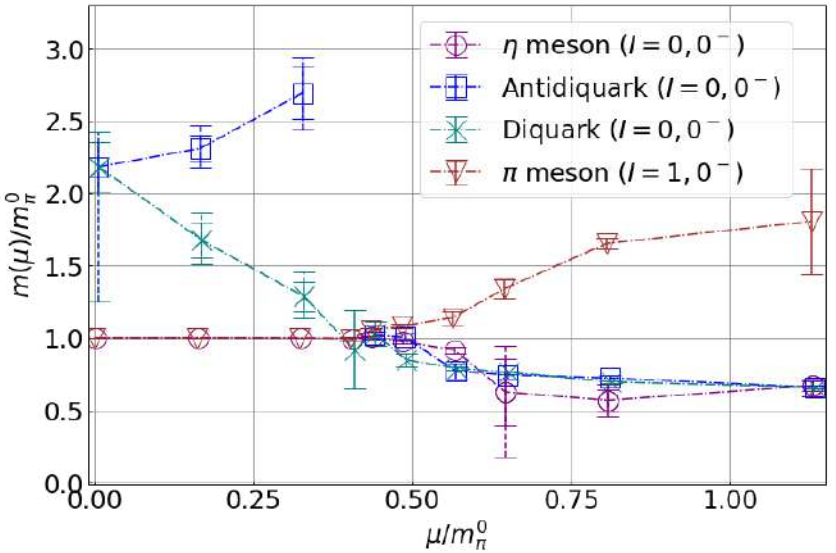




← pion $(I=1,0^-)$

$I=0,0^-$ hadron **lighter than pion!**

- Pion is no longer light in superfluid phase $(\text{for } m_\pi^0/2 \lesssim \mu)$!

- ChPT is powerful but cannot treat $I=0,0^-$ hadron ⇨ ChPT is no longer the useful "low-energy EFT" in superfluid phase of QC_2D!

Kogut et al (2000)

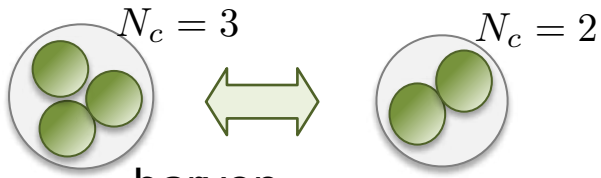
⇨ I constructed another model (linear sigma model) as a reasonable EFT in (dense) QC_2D

Q: What can we learn from QC_2D?

- It is too naïve to bring all obtained information to Nc=3 world

  → detailed dynamics is different

eg



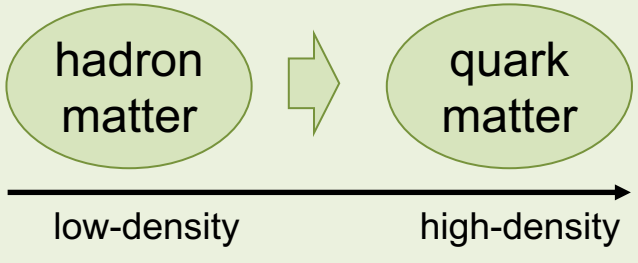
Lattice numerical experiment

- But we can get information on

  - to what extent simple hadron model description can apply
  - how to incorporate quark d.o.f. into hadron model (→ unified model)
  - deep understanding of quark matter in high-dense regime

  ⋮

universal regardless of Nc



Model

- **Pauli-Gursey SU(4) symmetry**

  - Pseudo reality of $SU(2)_c$ allows us to rewrite QC_2D Lagrangian with massless quarks as

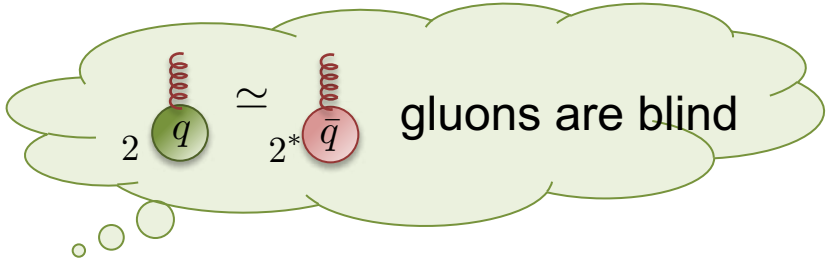
$$\mathcal{L}_{\mathrm{QC_2D}} = \bar{\psi} i \not{\partial} \psi - g_s \bar{\psi} \not{A}^a T_c^a \psi = \Psi^\dagger i \partial_\mu \sigma^\mu \Psi - g_s \Psi^\dagger A_\mu^a T_c^a \sigma^\mu \Psi$$

pseudoreality: $\sigma^2 \sigma^a \sigma^2 = -(\sigma^a)^*$

In two-flavor: $\Psi = (\psi_R, \tilde{\psi}_L)^T = (u_R, d_R, \tilde{u}_L, \tilde{d}_L)^T$ with $\tilde{\psi}_L = \sigma^2 \tau_c^2 \psi_L^*$

Four-dimensional Pauli matrix: $\sigma^\mu = (1, \sigma^i)$

  - $\mathcal{L}_{\mathrm{QC_2D}}$ is obviously invariant under $\Psi \to g\Psi$ $[g \in SU(4)]$



$SU(2)_L \times SU(2)_R$ chiral symmetry

[breaking: $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$]

enlarged

Pauli-Gursey $SU(4)$ symmetry

[breaking: $SU(4) \to Sp(4)$]

Pauli (1957), Gursey (1958)

5 NG bosons

Model

- **Linear sigma model (LSM)**

$$\Psi = (\psi_R, \tilde{\psi}_L)^T = (u_R, d_R, \tilde{u}_L, \tilde{d}_L)^T$$

- LSM is based on linear rep. of a quark bilinear: $\Psi_j \sigma^2 \tau_c^2 \Psi_i$

→ would be reasonable in medium where { quark d.o.f. will get seen; broken strength of $SU(4) \to Sp(4)$ is changed

- The main building block is

cf, $\Sigma = \sigma + i\pi^a \tau^a$ for $N_c = 3$

$$\Sigma_{ij} = \frac{1}{2} \begin{pmatrix} 0 & -\frac{B'-iB}{2\sqrt{2}} & \frac{\sigma - i\eta + a_0^0 - i\pi^0}{4} & \frac{a_0^+ - i\pi^+}{2\sqrt{2}} \\ \frac{B'-iB}{2\sqrt{2}} & 0 & \frac{a_0^- - i\pi^-}{2\sqrt{2}} & \frac{\sigma - i\eta - a_0^0 + i\pi^0}{4} \\ -\frac{\sigma - i\eta + a_0^0 - i\pi^0}{4} & -\frac{a_0^- - i\pi^-}{2\sqrt{2}} & 0 & -\frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} \\ -\frac{a_0^+ - i\pi^+}{2\sqrt{2}} & -\frac{\sigma - i\eta - a_0^0 + i\pi^0}{4} & \frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} & 0 \end{pmatrix}_{ij} \sim \Psi_j \sigma^2 \tau_c^2 \Psi_i$$

- Hadron assignments are

$$B \sim -\frac{i}{\sqrt{2}} \psi^T C \gamma_5 \tau_c^2 \tau_f^2 \psi \quad B' \sim -\frac{1}{\sqrt{2}} \psi^T C \tau_c^2 \tau_f^2 \psi \quad \sigma \sim \bar{\psi}\psi$$
$$a_0^a \sim \bar{\psi} \tau_f^a \psi \quad \eta \sim \bar{\psi} i\gamma_5 \psi \quad \pi^a \sim \bar{\psi} i\gamma_5 \tau_f^a \psi$$

Hadron	J^P	Quark number	Isospin	
σ	0^+	0	0	mesons
a_0	0^+	0	1	mesons
η	0^-	0	0	mesons
π	0^-	0	1	mesons
B ($\bar{B}$)	0^+	+2(−2)	0	baryons
B' ($\bar{B}'$)	0^-	+2(−2)	0	baryons

Model

- **Lagrangian of linear sigma model (LSM)**

  - (approximately) $SU(4)$-invariant LSM Lagrangian is given by

$$\mathcal{L} = \mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - m_0^2 \mathrm{tr}[\Sigma^\dagger \Sigma] - \lambda_1 \left(\mathrm{tr}[\Sigma^\dagger \Sigma]\right)^2 - \lambda_2 \mathrm{tr}[(\Sigma^\dagger \Sigma)^2] + \mathrm{tr}[H^\dagger \Sigma + \Sigma^\dagger H] + c(\det\Sigma + \det\Sigma^\dagger)$$

$$D_\mu \Sigma = \partial_\mu \Sigma - i\mu_q \delta_{\mu 0}\{J, \Sigma\} \quad \text{with} \quad J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

chemical potential effect

$$H = h_q E \quad \text{with} \quad E = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

current-quark mass effect

$U(1)_A$ anomaly

where $\Sigma = \frac{1}{2} \begin{pmatrix} 0 & -\frac{B'-iB}{2\sqrt{2}} & \frac{\sigma - i\eta + a_0^0 - i\pi^0}{4} & \frac{a_0^+ - i\pi^+}{2\sqrt{2}} \\ \frac{B'-iB}{2\sqrt{2}} & 0 & \frac{a_0^- - i\pi^-}{2\sqrt{2}} & \frac{\sigma - i\eta - a_0^0 + i\pi^0}{4} \\ -\frac{\sigma - i\eta + a_0^0 - i\pi^0}{4} & -\frac{a_0^- - i\pi^-}{2\sqrt{2}} & 0 & -\frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} \\ -\frac{a_0^+ - i\pi^+}{2\sqrt{2}} & -\frac{\sigma - i\eta - a_0^0 + i\pi^0}{4} & \frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} & 0 \end{pmatrix}$

$$\Sigma \to g\Sigma g^T \quad [g \in SU(4)]$$

Hadron	J^P	Quark number	Isospin	
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B $(\bar{B})$	0^+	$+2(-2)$	0	baryons
B' $(\bar{B}')$	0^-	$+2(-2)$	0	baryons

- Advantage of LSM
  - treatment of $I = 0, 0^-$ hadrons (mandatory from lattice result!)
  - we can see relations between parity partners

parity partner

$\eta, \pi \leftrightarrow \sigma, a_0$

$B(\bar{B}) \leftrightarrow B'(\bar{B}')$

Mean fields

- **Mean field**

- The mean fields are $\sigma_0 \equiv \langle\sigma\rangle$ and $\Delta \equiv \left\langle \frac{B+\bar{B}}{\sqrt{2}} \right\rangle$

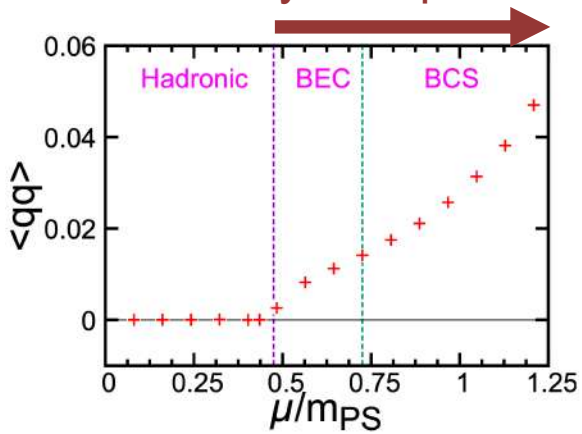
$\sigma_0 \sim \langle\bar{\psi}\psi\rangle$: chiral condensate

$\Delta \sim -\frac{i}{2}\langle\psi^T C\gamma_5\tau_c^2\tau_f^2\psi\rangle + \text{h.c.}$: diquark condensate

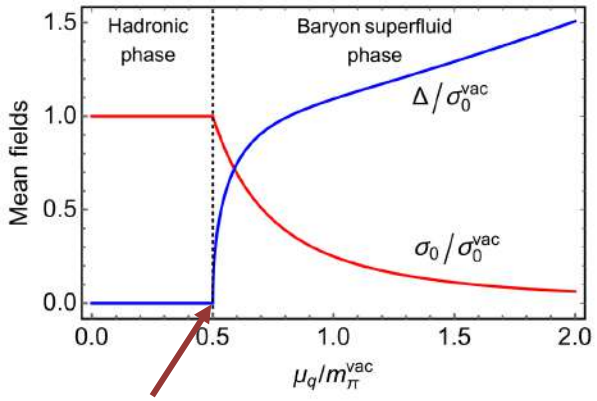
diquark condensate by lattice

Iida et al (2024)

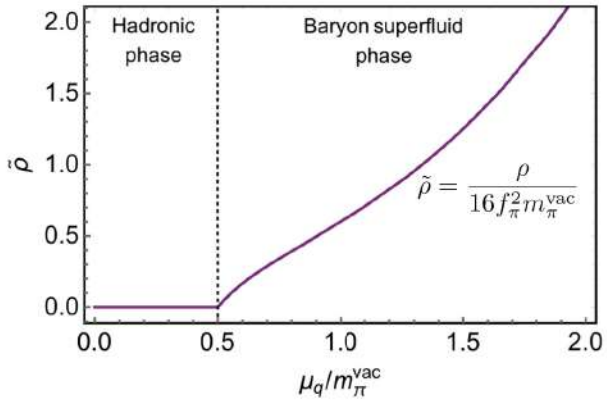
baryon superfluidity



σ_0 and Δ vs μ_q



density ρ vs μ_q



2nd order phase transition at $\mu_q = m_\pi^{\rm vac}/2$

Input here

$\sigma_0^{\rm vac} = 250\,\text{MeV}$ (put by hand)

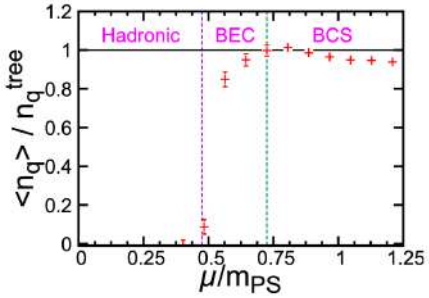
$\lambda_1 = c = 0$ (large Nc)

$m_\pi^{\rm vac} = 738\,\text{MeV}$

$m_{a_0}^{\rm vac}/m_\pi^{\rm vac} = 2.18$

} lattice Murakami et al

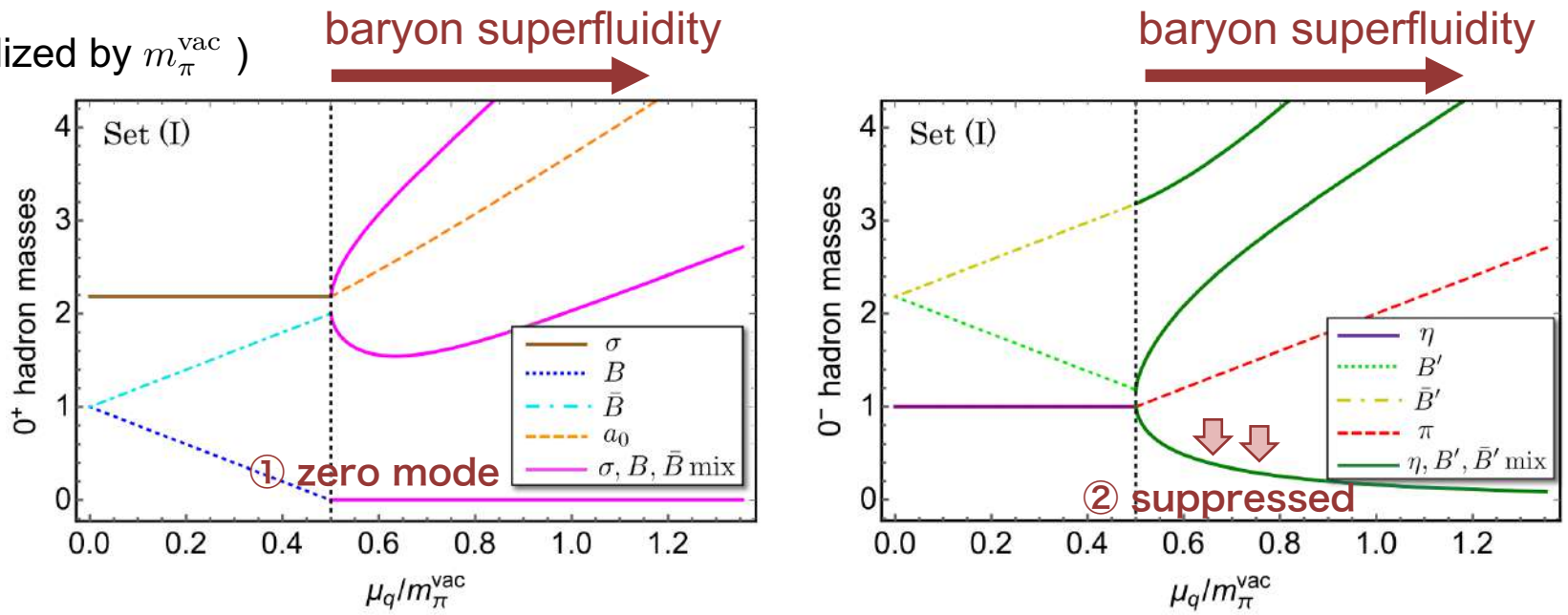
$$\begin{cases} \text{BEC}: \langle n_q\rangle/n_q^{\rm tree} \lesssim 1 \\ \text{BCS}: \langle n_q\rangle/n_q^{\rm tree} \approx 1 \end{cases}$$



Hadron masses

- **Results on hadron mass at finite μ_q (T=0)**

(normalized by $m_\pi^{\rm vac}$)



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$m_{a_0}^{\rm vac}/m_\pi^{\rm vac} = 2.18$

} lattice Murakami et al

- Baryon number violation in superfluid phase

$\sigma \leftrightarrow B \leftrightarrow \bar{B}$ mixing (0+ sector)

$\eta \leftrightarrow B' \leftrightarrow \bar{B}'$ mixing (0^- sector)

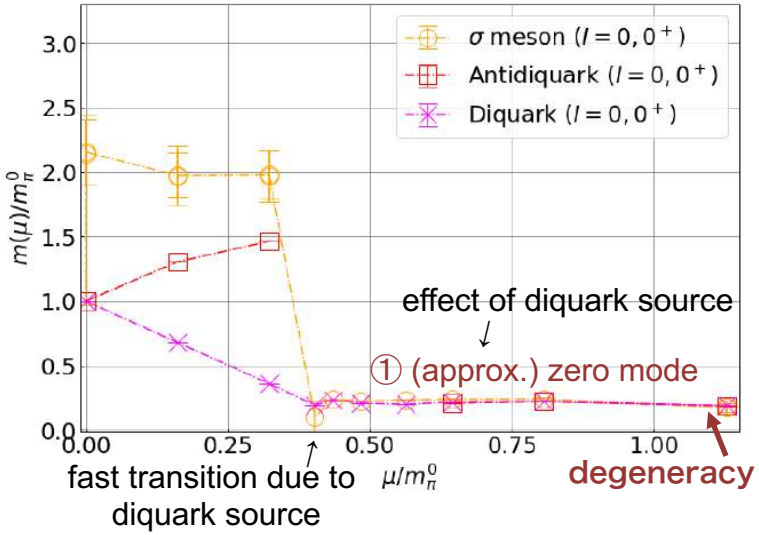
① zero mode (NG mode of $U(1)$ baryon-number breaking)

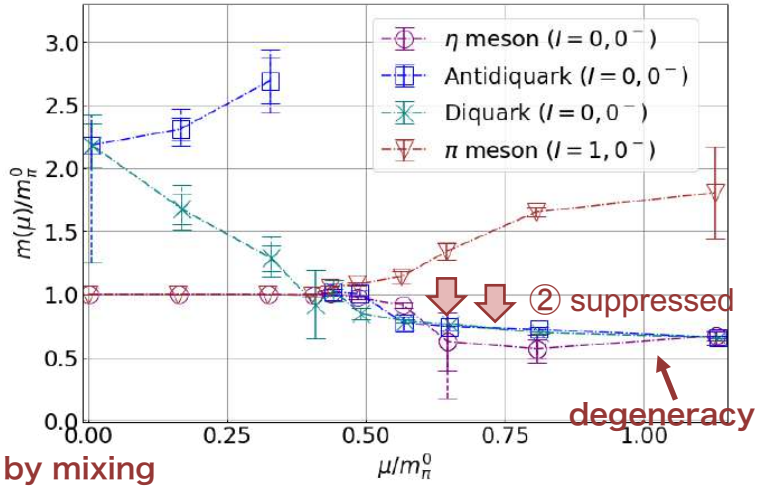
② nonlinear suppression of "η" mass due to the mixing

Hadron masses

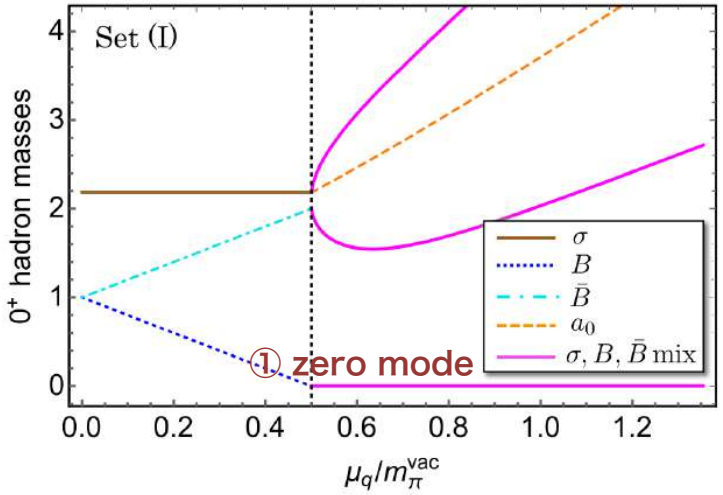
- **Comparison with lattice**

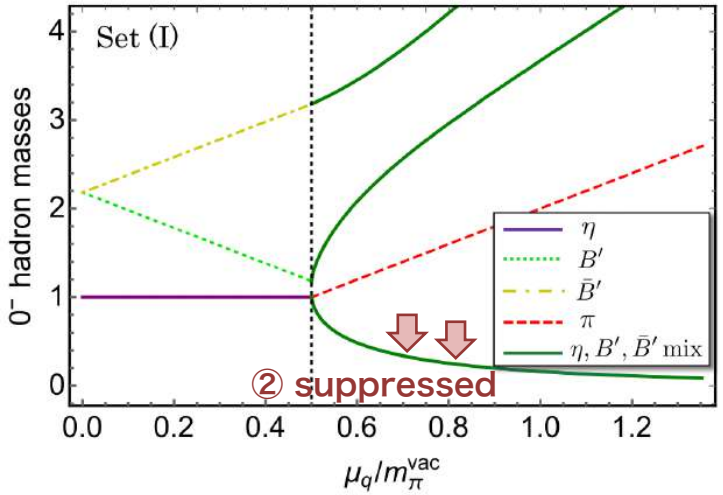
Lattice (Murakami et al)





My LSM

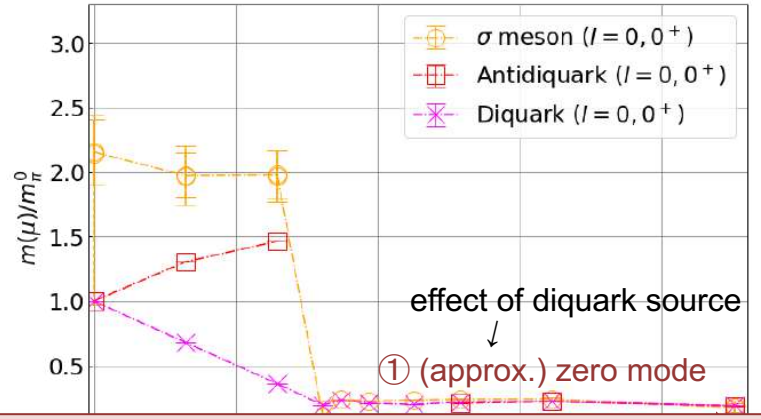


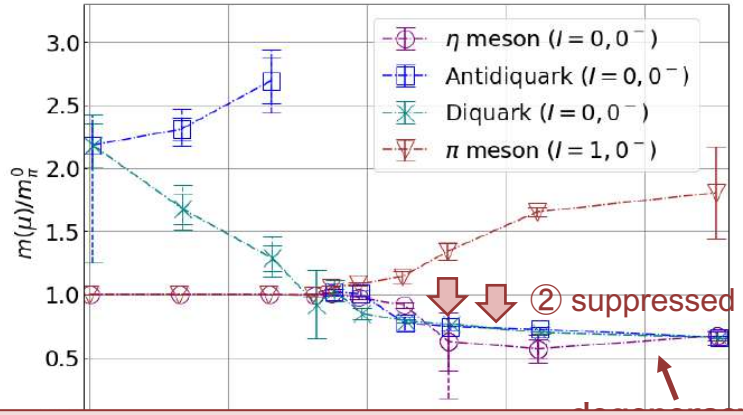


Hadron masses

- **Comparison with lattice**

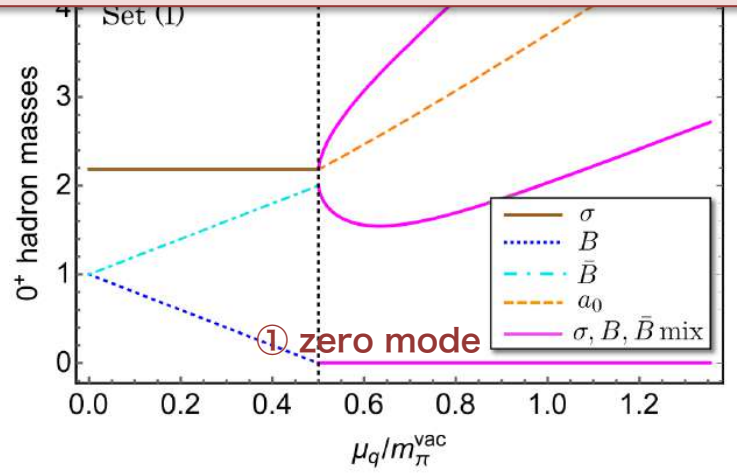
Lattice (Murakami et al)

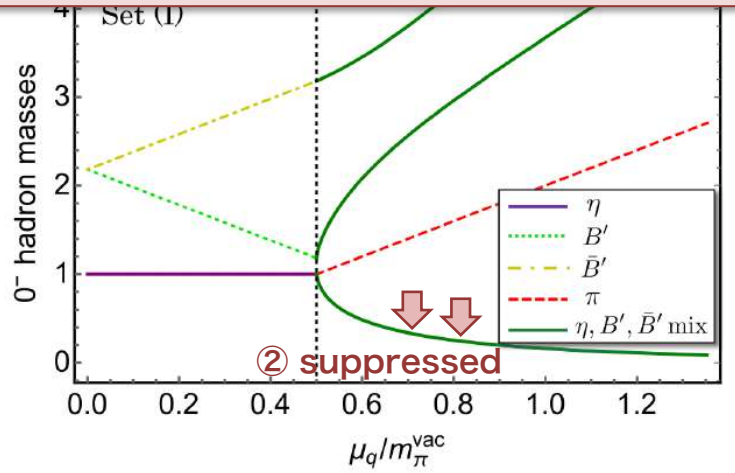




The lowest excitations are qualitatively described by my LSM well!

My LSM





NOTE

ChPT cannot treat $\eta, B', \bar{B}'$

⇔ Kogut et al (2000)

linear rep. is essential!

Discussion

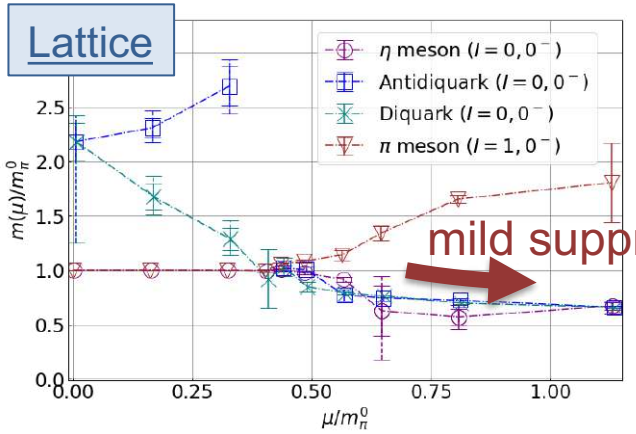
- **Anomaly enhancement in superfluid phase?**

  - In hadronic phase $(\mu < m_\pi^{\rm vac}/2)$, the $U(1)_A$ anomaly effect seems to be small

  - In superfluid phase $(m_\pi^{\rm vac}/2 \le \mu)$, the $U(1)_A$ anomaly effect could be enhanced?

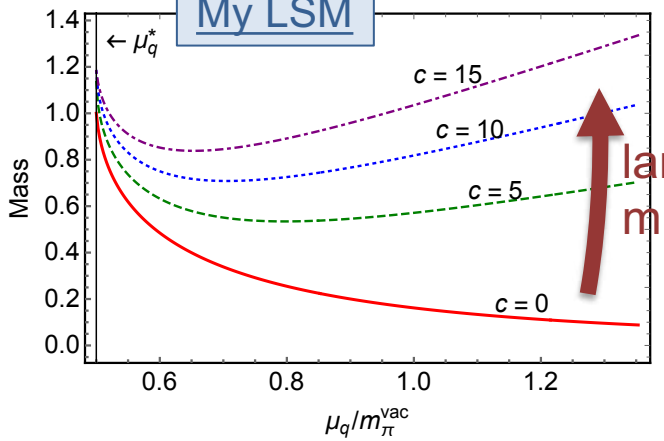
C : anomaly coefficient

Lattice



mild suppression

My LSM



large anomaly yields milder suppression

- Need to see μ dep. of anomaly coefficients to confirm anomaly enhancement

  ⇨ eg, FRG analysis Work in progress with G. Fejos

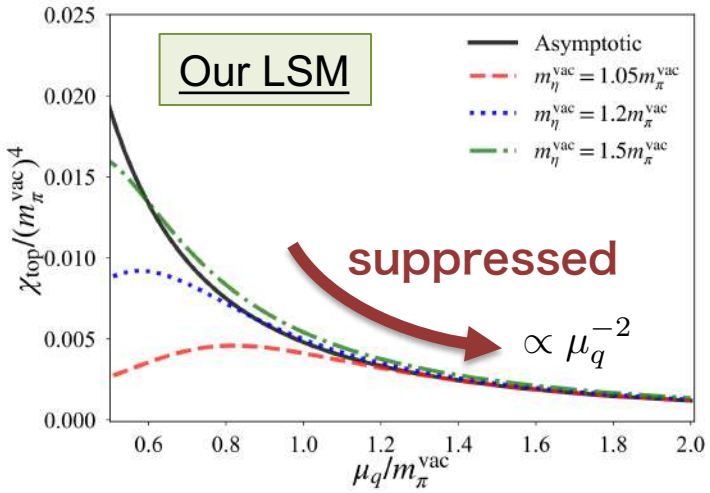
Topological susceptibility $\chi_{\rm top}$

- $\chi_{\rm top}$ **in superfluid phase**

$$\chi_{\rm top} \equiv -i\int d^4x\langle 0|{\rm T}Q(x)Q(0)|0\rangle = \frac{1}{2}(m_\pi^{\rm vac})^4(f_\pi^{\rm vac})^2\Big(D_\pi(0)-D_\eta(0)\Big)$$

chiral Ward-Takahashi identity

with $\begin{cases} Q=(g_s^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma} \\ D_\phi(p)={\rm F.T.}\,\langle 0|{\rm T}\phi(x)\phi(0)|0\rangle \end{cases}$



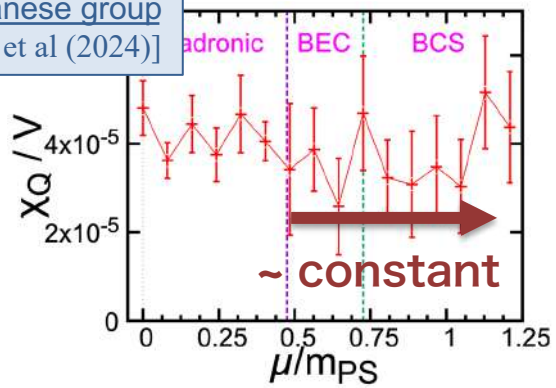
- Suppression of topological susceptibility for large μ_q

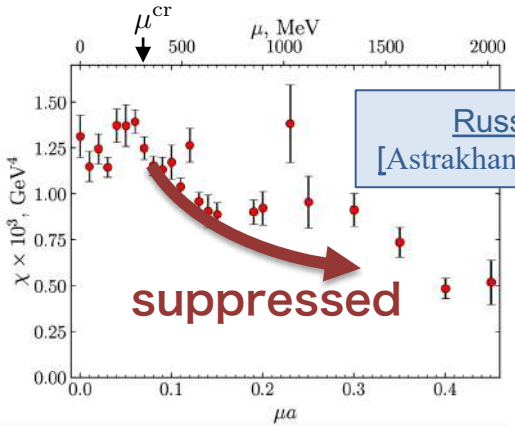
Asymptotic: $\chi_{\rm top} \to \dfrac{(f_\pi^{\rm vac})^2(m_\pi^{\rm vac})^4}{12}\mu_q^{-2}$ Kawaguchi-**Suenaga** (2023)

essentially from the chiral restoration $\sigma_0 \propto \mu_q^{-2}$

comparison with lattice

Japanese group
[Iida et al (2024)]





Russian group
[Astrakhantsev et al (2020)]

- Inconsistent lattice results

⇔ No conclusive statement yet

- Needs more refined simulations

Sound velocity

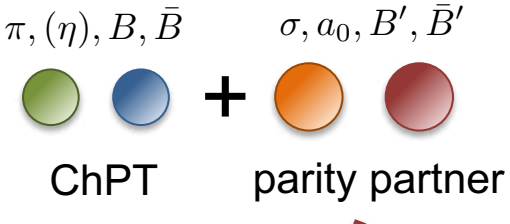
- **Sound velocity at mean-field level within the LSM**

pressure: $$p = \underbrace{f_\pi^2 m_\pi^2 \left(\bar{\mu}^2 + \frac{1}{\bar{\mu}^2}\right)}_{\text{ChPT result}} + f_\pi^2 m_\pi^2 \left[\frac{4}{\delta \bar{m}_{\sigma-\pi}^2}(\bar{\mu}^2 - 1)^2\right]$$

energy: $$\epsilon = \underbrace{f_\pi^2 m_\pi^2 \left[\frac{(\bar{\mu}^2+3)(\bar{\mu}^2-1)}{\bar{\mu}^2}\right]}_{\text{ChPT result}} + f_\pi^2 m_\pi^2 \left[\frac{4}{\delta \bar{m}_{\sigma-\pi}^2}(3\bar{\mu}^2+1)(\bar{\mu}^2-1)\right]$$

sound velocity: $$c_s^2 = \frac{(1 - 1/\bar{\mu}^4) + 8(\bar{\mu}^2 - 1)/\delta \bar{m}_{\sigma-\pi}^2}{(1 + 3/\bar{\mu}^4) + 8(3\bar{\mu}^2 - 1)/\delta \bar{m}_{\sigma-\pi}^2}$$

$$\bar{\mu} = \mu/\mu_{\rm cr} = 2\mu/m_\pi$$
$$\delta \bar{m}_{\sigma-\pi}^2 = (\tilde{m}_\sigma^2 - m_\pi^2)/\mu_{\rm cr}^2$$



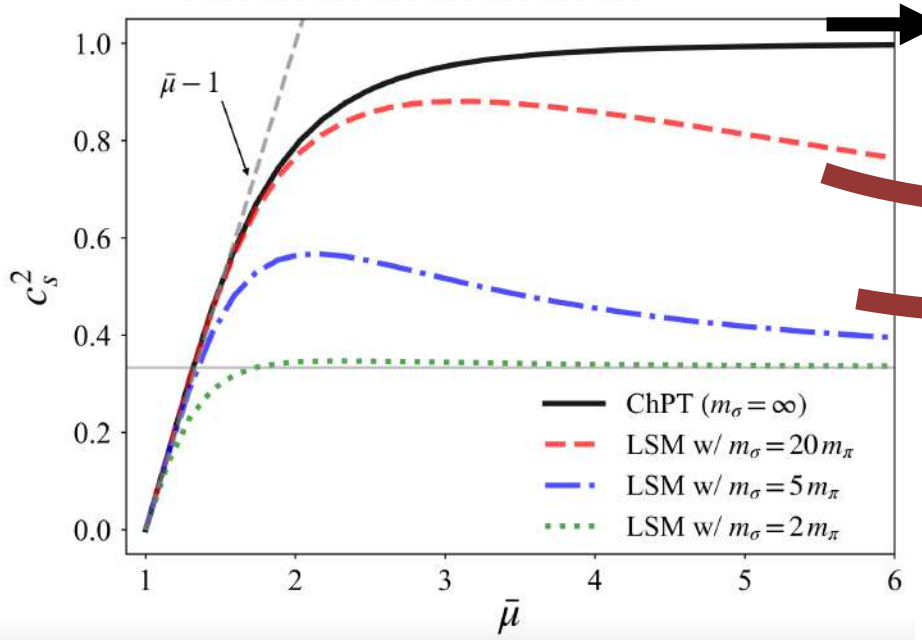
Universal structure: $(\text{LSM result}) = (\text{ChPT result}) + (1/\delta \bar{m}_{\sigma-\pi}^2 \text{ contribution})$

- Integrating out the parity partners $(m_\sigma \to \infty)$ yields the ChPT results $(1/\delta \bar{m}_{\sigma-\pi}^2 \to 0)$

Sound velocity

- **Sound velocity peak** Kawaguchi-**Suenaga** (2024)

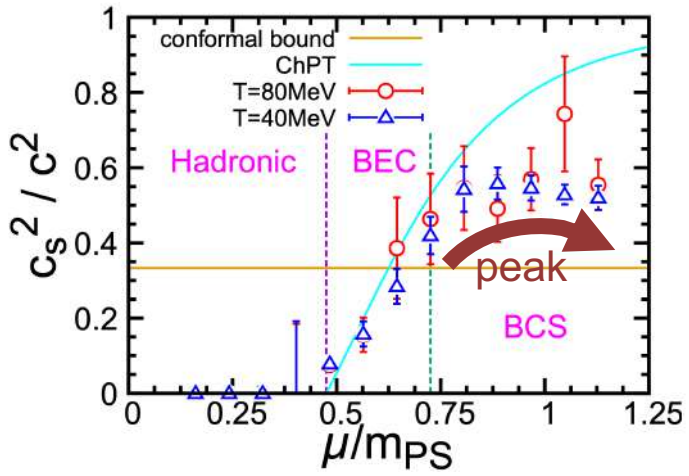
$\bar{\mu} = \mu/\mu_{\rm cr} = 2\mu/m_\pi$

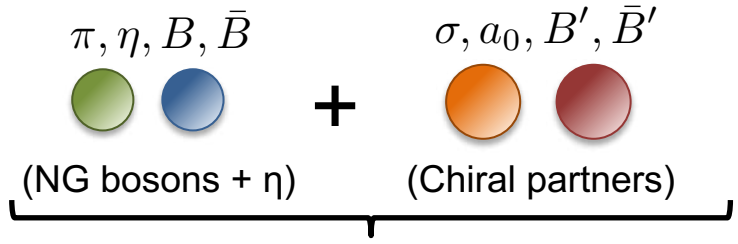


ChPT result $(c_s^2 \to 1)$: no peak

peak when $m_\sigma < \infty$
$(c_s^2 \to 1/3)$

Lattice: Iida et al (2024)





- The peak structure is driven by contributions from parity partners

- Fluctuation and spin-1 hadron effect are needed for more quantitative comparison

- Any connection with crossover to quark matter ? cf, talk by Kojo

FRG for anomaly effect

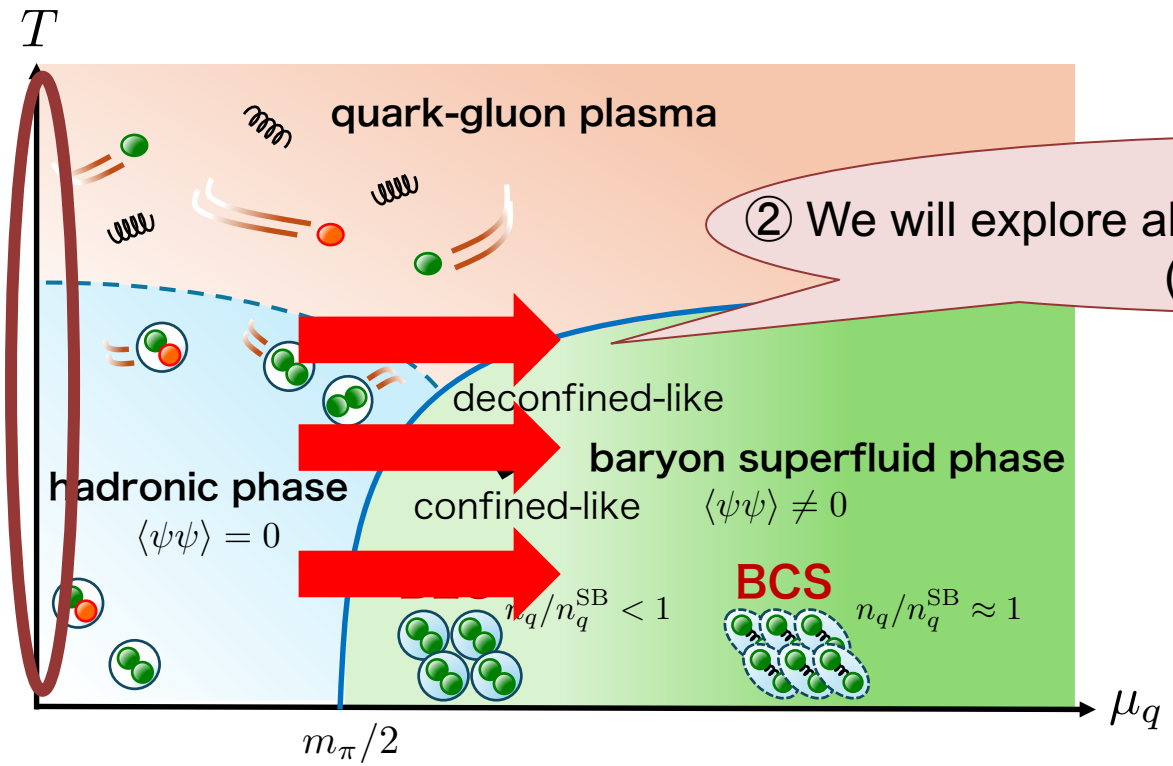
- **FRG analysis in QC_2D**

  - Functional renormalization group (FRG) method to confirm anomaly enhancement in superfluid phase

  - FRG in superfluid phase is not easy due to $\begin{cases} C \text{ violation due to finite } \mu \\ U(1)_B \text{ violation due to diquark condensate } \langle\psi\psi\rangle \end{cases}$

① First we studied in hot medium (this talk)

Fejos-**Suenaga** (2025, 2025)



② We will explore all regime including superfluid phase (on-going work)

FRG for anomaly effect

- **FRG analysis in hot QC_2D**

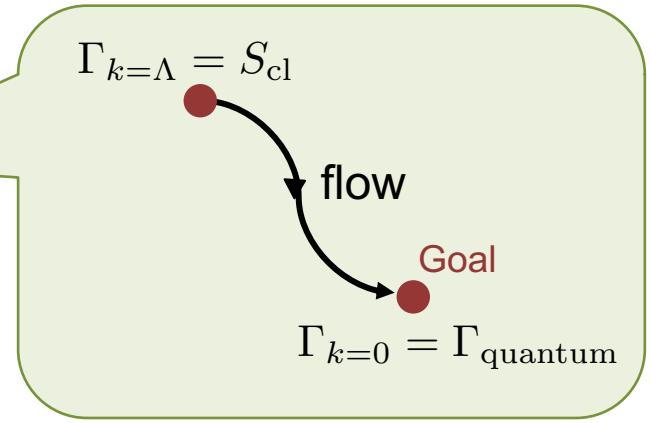
  - The flow equation is $\partial_k \Gamma_k = \frac{1}{2}\tilde{\partial}_k \mathrm{Tr}\,\mathrm{Log}\,[\Gamma_k'' + R_k]$, with $\tilde{\partial}_k \equiv \partial_k R_k \frac{\partial}{\partial R_k}$

    → Regulator R_k allows us to include fluctuations of $p \gtrsim k$

$$\begin{cases} R_k \to \infty \ (k \to \Lambda) \\ R_k \to 0 \ (k \to 0) \end{cases} \Rightarrow$$

3D Litim regulator

$$R_k(\boldsymbol{p}) = (k^2 - \boldsymbol{p}^2)\theta(k^2 - \boldsymbol{p}^2)$$



  - k-dependent effective action: $\Gamma_k = \int_x \left(\mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + h\sigma \ - V_k[\Sigma] \right)$

    with assumption: $V_k[\Sigma] = m_k^2 I_1 + \tilde{\lambda}_{1,k} I_1^2 + \tilde{\lambda}_{2,k} I_2 + \underbrace{a_k I_A + c_{1,k} I_A^2 + c_{2,k} I_1 I_A}_{\text{anomaly}}$

ansatz
the same structure
as classical one

invariants: $I_1 = \mathrm{Tr}[\Sigma^\dagger \Sigma] \quad I_2 = \mathrm{Tr}[(\Sigma^\dagger \Sigma - \frac{1}{4}\mathrm{Tr}[\Sigma^\dagger \Sigma]\mathbf{1})^2] \quad I_A = \frac{1}{2}\mathrm{Tr}[\tilde{\Sigma}\Sigma + \tilde{\Sigma}^\dagger \Sigma^\dagger] \quad \left(\tilde{\Sigma}_{ij} = \frac{1}{2}\epsilon_{ijkl}\Sigma_{kl}\right)$

I_A: anomalous

$$\Rightarrow \quad \Gamma_k'' = \omega_n^2 + \boldsymbol{p}^2 + V_k'' \quad \text{with} \quad V_k'' = \begin{pmatrix} \frac{\partial^2 V_k}{\partial\sigma\partial\sigma} & \cdots \\ \vdots & \ddots \end{pmatrix}_{12\times 12}$$

(12 = # of hadrons)

FRG for anomaly effect

- **Numerical results**

$\mathring{X}$ means vacuum value of X

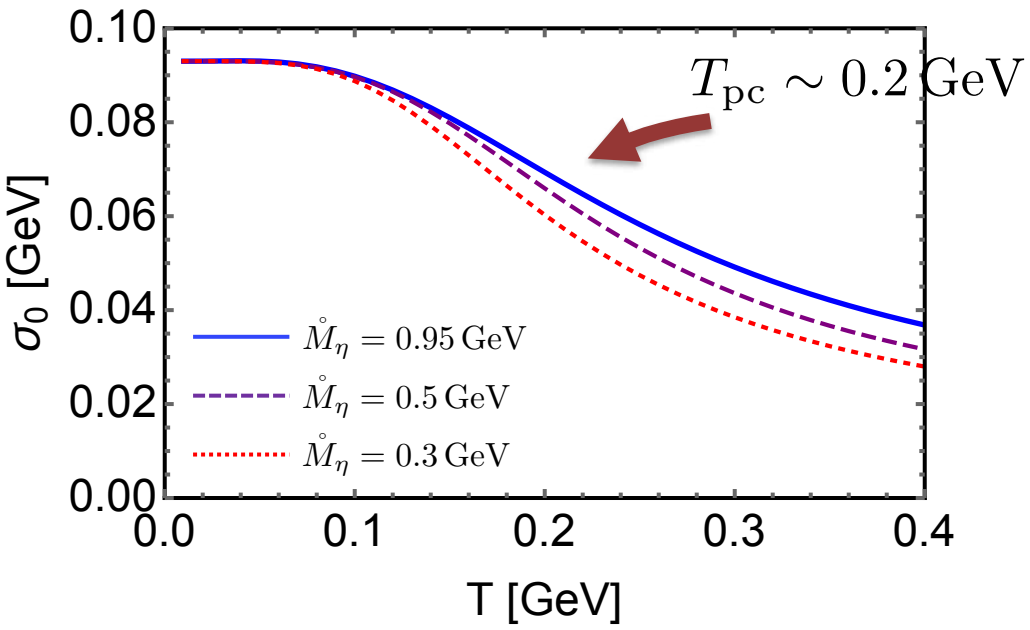
initial conditions	physical inputs
$\tilde{\lambda}_{2,\Lambda} = 4\tilde{\lambda}_{1,\Lambda}$ (large Nc)	$\mathring{M}_\pi = 0.14\,\mathrm{GeV}$
$c_{1,\Lambda} = c_{2,\Lambda} = 0$	$\mathring{f}_\pi = 0.093\,\mathrm{GeV}$

- $\mathring{M}_\eta$ is treated as a free parameter (anomaly effect is our main interest)

M_η [GeV]	m_Λ^2 [GeV^2]	a_Λ [GeV]	$\lambda_{1,\Lambda}$	$\lambda_{2,\Lambda}$	$c_{1,\Lambda}$	$c_{2,\Lambda}$
0.3	−0.0426	−0.032	1.25	5	0	0
0.5	0.0465	−0.108	1.25	5	0	0
0.95	0.385	−0.428	1.25	5	0	0

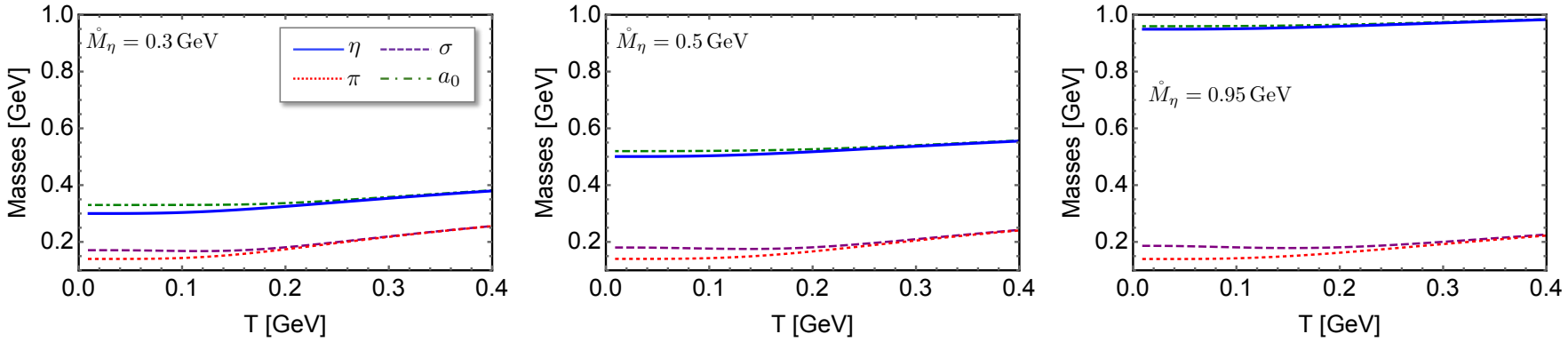
- $\lambda_{2,\Lambda} \lesssim 5$ to yield chiral restoration in hot medium

chiral restoration at finite T



FRG for anomaly effect

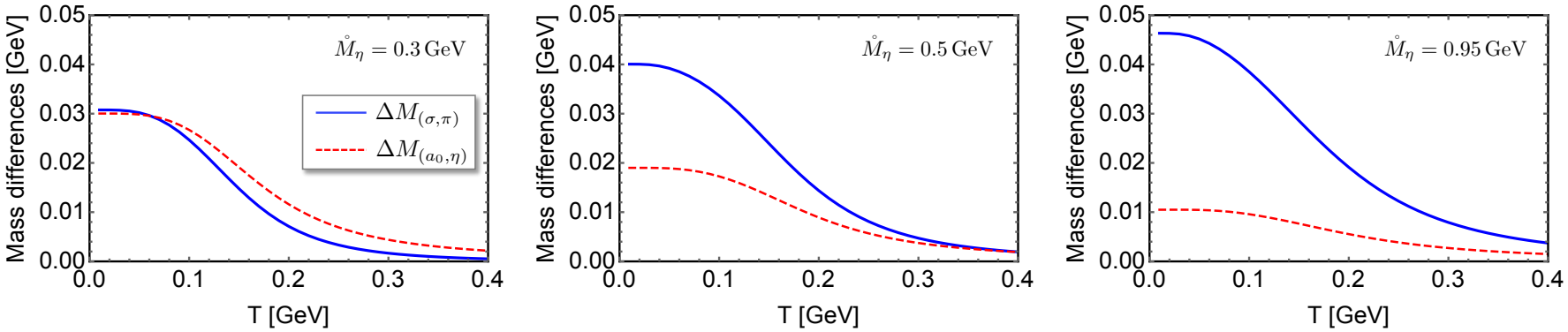
- Hadron masses at finite T



$$M_\sigma^2, M_\pi^2 \sim m^2 + a$$
$$M_{a_0}^2, M_\eta^2 \sim m^2 - a$$
$$(a < 0)$$

- Quartic couplings are suppressed at any k

- Mass differences of parity partners at finite T



- Mass degeneracies are predicted

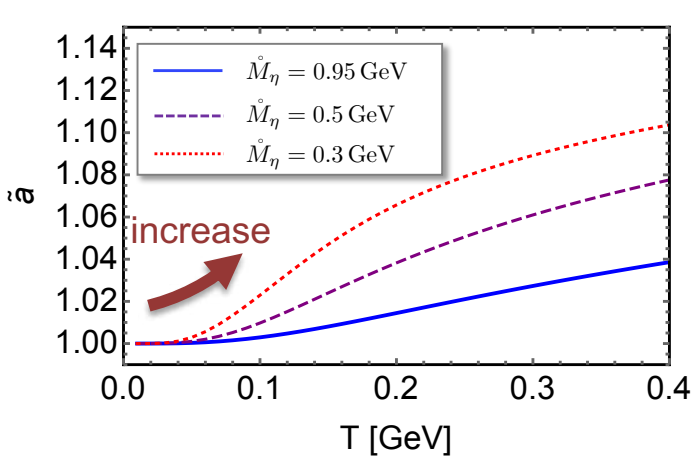
FRG for anomaly effect

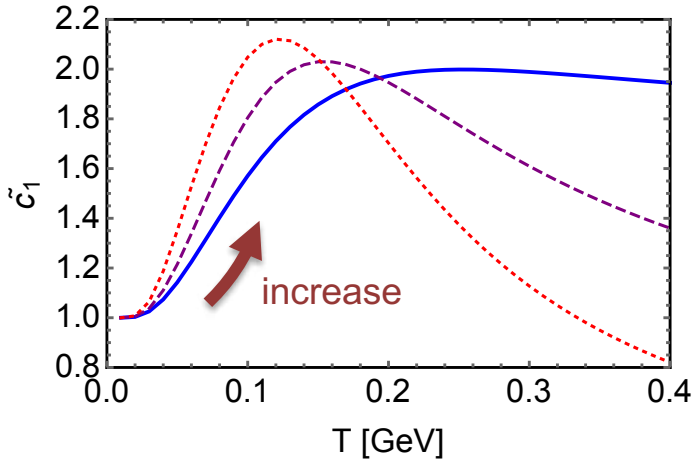
- **Anomaly coefficients at finite T**

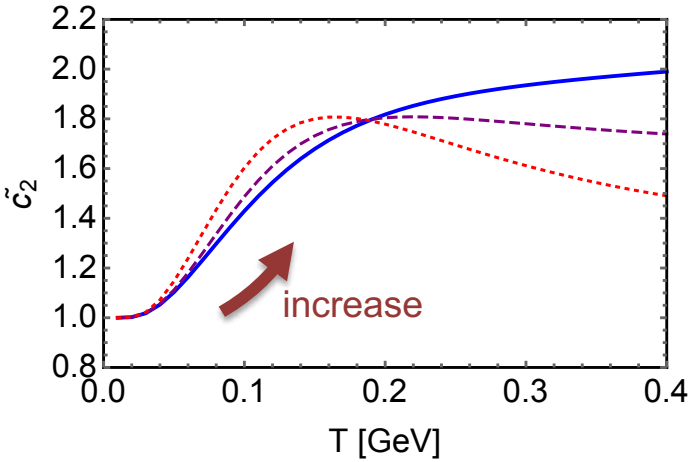
  - $a < 0,\ c_1 < 0,\ c_2 > 0$ at any k and T

$$\tilde{X} \equiv \frac{X}{X_{T=0}}$$

NOTE: $T_{\rm pc} \sim 0.2\,{\rm GeV}$







- All the anomaly coefficients are **enhanced at finite T**

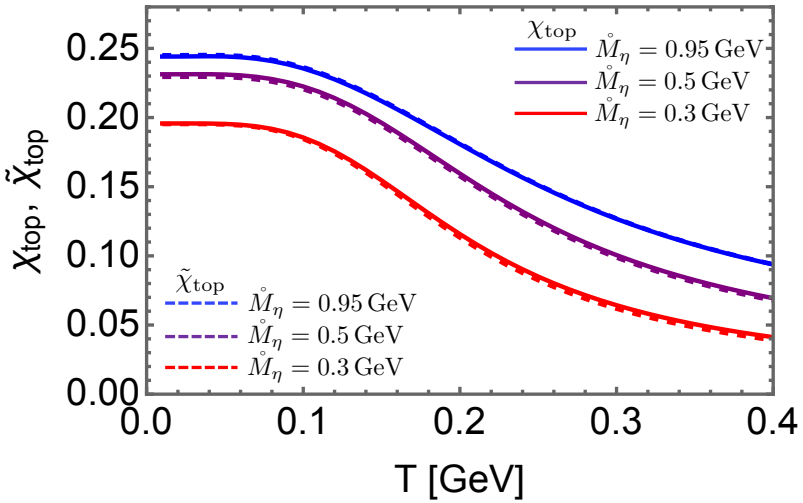
  ⇔ consistent with $N_f = 2+1,\ N_c = 3$ work by FRG Fejos-Hosaka (2016)

FRG for anomaly effect

- **Topological susceptibility at finite T**

Definition: $\bar{\chi}_{\rm top} \equiv -i\int d^4x\langle 0|{\rm T}Q(x)Q(0)|0\rangle = \frac{\mathring{M}_\pi^4\mathring{\sigma}_0^2}{4}\left(\frac{1}{M_\pi^2}-\frac{1}{M_\eta^2}\right)$ with $Q=(g_s^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$

chiral Ward-Takahashi identity



$$\chi_{\rm top} \equiv \frac{\bar{\chi}_{\rm top}}{\mathring{M}_\pi^2\mathring{\sigma}_0^2} = \frac{\mathring{M}_\pi^2}{4}\left(\frac{1}{M_\pi^2}-\frac{1}{M_\pi^2+(M_\eta^2-M_\pi^2)}\right) \quad (\leftarrow \text{full result})$$

$$\tilde{\chi}_{\rm top} \equiv \frac{\mathring{M}_\pi^2}{4}\left(\frac{1}{M_\pi^2}-\frac{1}{M_\pi^2+(\mathring{M}_\eta^2-\mathring{M}_\pi^2)}\right) \quad (\leftarrow \text{reference})$$

anomaly effect is constant = no anomaly change at finite T

- $\chi_{\rm top}$ is suppressed at higher temperature followed by chiral restoration: $\chi_{\rm top} \sim \frac{\mathring{M}_\pi^2}{4}\frac{1}{M_\pi^2} \propto \sigma_0$
- No sizable difference between $\chi_{\rm top}$ and $\tilde{\chi}_{\rm top}$

  → topological susceptibility is not suitable probe to see corrections of anomaly effects

2+2 flavor LSM

- **LSM for $N_f = 2+2$ in QC$_2$D (no sign problem)**

Sakai-**Suenaga**, submitted to PRD

- Useful to see roles of heavy flavors in dense medium

- Quark bilinear $\Sigma \sim \Psi_j^T \sigma^2 \tau_c^2 \Psi_i$ is now 8×8 matrix

$$\Psi = (q_{1R}, q_{2R}, Q_{1R}, Q_{2R}, \tilde{q}_{1L}, \tilde{q}_{2L}, \tilde{Q}_{1L}, \tilde{Q}_{2L})^T$$
with $\tilde{\psi}_L = \sigma^2 \tau_c^2 \psi_L^*$

Building block: $\Sigma = (\mathcal{S}^a - i\mathcal{P}^a) X^a E$

$\begin{cases} X^a : \text{generators of } SU(8)/Sp(8) \\ E : 8 \times 8 \text{ symplectic matrix} \end{cases}$

simple extension of $N_f = 2$ one

Hadron	Field parametrization	J^P	N_q	N_Q	I_q	I_Q
η_{qq}	$\frac{1}{\sqrt{2}}\mathcal{P}^0 + \frac{1}{\sqrt{3}}\mathcal{P}^8 + \frac{1}{\sqrt{6}}\mathcal{P}^{15}$	0^-	0	0	0	0
η_{QQ}	$-\frac{1}{\sqrt{3}}\mathcal{P}^8 + \sqrt{\frac{2}{3}}\mathcal{P}^{15}$	0^-	0	0	0	0
$\pi_{0,qq}$	$\mathcal{P}^3$, $\frac{1}{\sqrt{2}}(\mathcal{P}^1 \pm i\mathcal{P}^2)$	0^-	0	0	1	0
$\pi_{0,QQ}$	$\frac{1}{\sqrt{2}}\mathcal{P}^0 - \frac{1}{\sqrt{3}}\mathcal{P}^8 - \frac{1}{\sqrt{6}}\mathcal{P}^{15}$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{13} \pm i\mathcal{P}^{14})$	0^-	0	0	0	1
K_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{P}^4 - i\mathcal{P}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^6 - i\mathcal{P}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^9 - i\mathcal{P}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{11} - i\mathcal{P}^{12})$	0^-	+1	−1	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{K}_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^4 + i\mathcal{P}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^6 + i\mathcal{P}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^9 + i\mathcal{P}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{11} + i\mathcal{P}^{12})$	0^-	−1	+1	$\frac{1}{2}$	$\frac{1}{2}$
B_{qq}	$\frac{1}{\sqrt{2}}(\mathcal{P}^{17} - i\mathcal{P}^{16})$	0^+	+2	0	0	0
$\bar{B}_{qq}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^{17} + i\mathcal{P}^{16})$	0^+	−2	0	0	0
B_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{P}^{19} - i\mathcal{P}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{21} - i\mathcal{P}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{23} - i\mathcal{P}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{25} - i\mathcal{P}^{24})$	0^+	+1	+1	0	0
$\bar{B}_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^{19} + i\mathcal{P}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{21} + i\mathcal{P}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{23} + i\mathcal{P}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{25} + i\mathcal{P}^{24})$	0^+	−1	−1	0	0
B_{QQ}	$\frac{1}{\sqrt{2}}(\mathcal{P}^{27} - i\mathcal{P}^{26})$	0^+	0	+2	0	0
$\bar{B}_{QQ}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^{27} + i\mathcal{P}^{26})$	0^+	0	−2	0	0

Hadron	Field parametrization	J^P	N_q	N_Q	I_q	I_Q
σ_{qq}	$\frac{1}{\sqrt{2}}\mathcal{S}^0 + \frac{1}{\sqrt{3}}\mathcal{S}^8 + \frac{1}{\sqrt{6}}\mathcal{S}^{15}$	0^+	0	0	0	0
σ_{QQ}	$-\frac{1}{\sqrt{3}}\mathcal{S}^8 + \sqrt{\frac{2}{3}}\mathcal{S}^{15}$	0^+	0	0	0	0
$a_{0,qq}$	$\mathcal{S}^3$, $\frac{1}{\sqrt{2}}(\mathcal{S}^1 \pm i\mathcal{S}^2)$	0^+	0	0	1	0
$a_{0,QQ}$	$\frac{1}{\sqrt{2}}\mathcal{S}^0 - \frac{1}{\sqrt{3}}\mathcal{S}^8 - \frac{1}{\sqrt{6}}\mathcal{S}^{15}$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{13} \pm i\mathcal{S}^{14})$	0^+	0	0	0	1
κ_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{S}^4 - i\mathcal{S}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^6 - i\mathcal{S}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^9 - i\mathcal{S}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{11} - i\mathcal{S}^{12})$	0^+	+1	−1	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{\kappa}_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^4 + i\mathcal{S}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^6 + i\mathcal{S}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^9 + i\mathcal{S}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{11} + i\mathcal{S}^{12})$	0^+	−1	+1	$\frac{1}{2}$	$\frac{1}{2}$
B'_{qq}	$\frac{1}{\sqrt{2}}(\mathcal{S}^{17} - i\mathcal{S}^{16})$	0^-	+2	0	0	0
$\bar{B}'_{qq}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^{17} + i\mathcal{S}^{16})$	0^-	−2	0	0	0
B'_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{S}^{19} - i\mathcal{S}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{21} - i\mathcal{S}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{23} - i\mathcal{S}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{25} - i\mathcal{S}^{24})$	0^-	+1	+1	0	0
$\bar{B}'_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^{19} + i\mathcal{S}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{21} + i\mathcal{S}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{23} + i\mathcal{S}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{25} + i\mathcal{S}^{24})$	0^-	−1	−1	0	0
B'_{QQ}	$\frac{1}{\sqrt{2}}(\mathcal{S}^{27} - i\mathcal{S}^{26})$	0^-	0	+2	0	0
$\bar{B}'_{QQ}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^{27} + i\mathcal{S}^{26})$	0^-	0	−2	0	0

2+2 flavor LSM

- **Lagrangian and mean fields**

  - (approximately) $SU(8)$-invariant LSM

$U(1)_A$ anomaly

$$\mathcal{L} = \mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - m_0^2 \mathrm{tr}[\Sigma^\dagger \Sigma] - \lambda_1 \left(\mathrm{tr}[\Sigma^\dagger \Sigma]\right)^2 - \lambda_2 \mathrm{tr}[(\Sigma^\dagger \Sigma)^2] + \bar{c}\,\mathrm{tr}[\zeta^\dagger \Sigma + \Sigma^\dagger \zeta] + c\,\epsilon_{ijklmnop}(\Sigma_{ij}\Sigma_{kl}\Sigma_{mn}\Sigma_{op} + \mathrm{h.c.})$$

$$D_\mu \Sigma = \partial_\mu \Sigma - i\mu_q \delta_{\mu 0}\{J, \Sigma\} \quad \text{with} \quad J = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix}_{8\times 8}$$

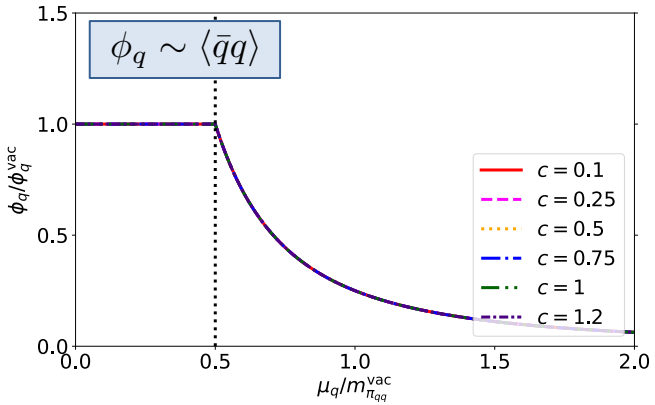
chemical potential effect $(\mu_q = \mu_Q)$

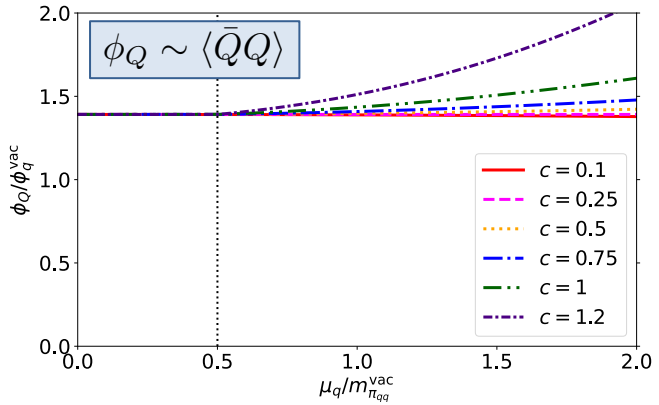
$$\zeta = \frac{1}{2}(m_q E_q + m_Q E_Q) \quad \text{with} \quad E_q = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ -\mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{8\times 8} \quad E_Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & -\mathbf{1} & 0 \end{pmatrix}_{8\times 8}$$

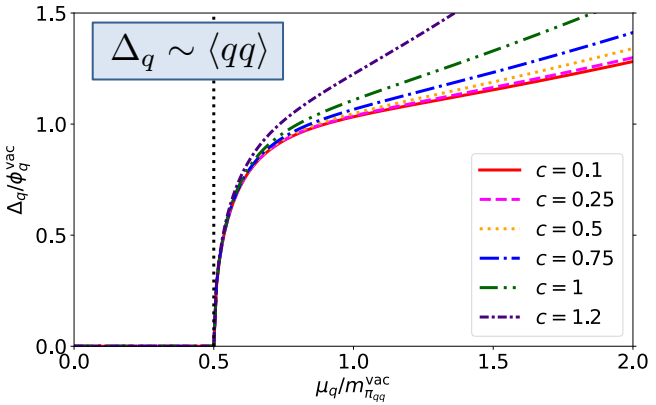
current-quark mass effect

- μ_q dependences of the mean fields for several c with $\lambda_1 = 6$ + input

$$\begin{cases} m_{\pi_{qq}}^{\mathrm{vac}} = 138\,\mathrm{MeV} & m_K^{\mathrm{vac}} = 494\,\mathrm{MeV} \\ f_\pi^{\mathrm{vac}} = 92\,\mathrm{MeV} & f_K^{\mathrm{vac}} = 110\,\mathrm{MeV} \end{cases}$$







2+2 flavor LSM

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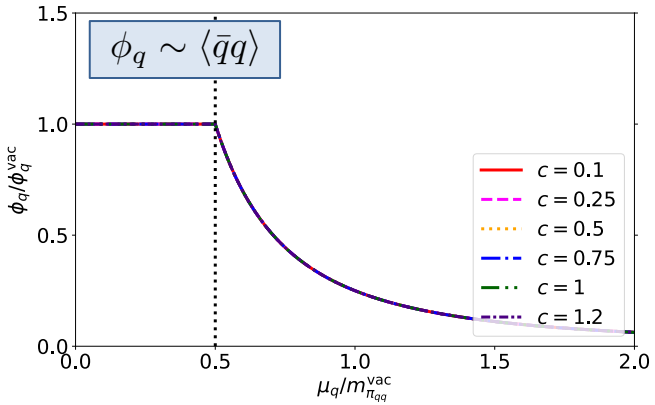
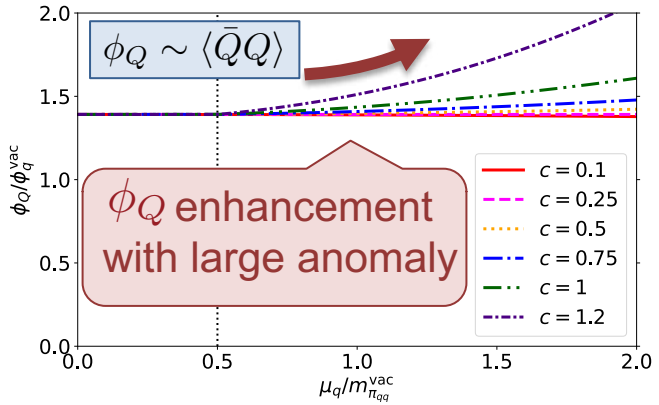
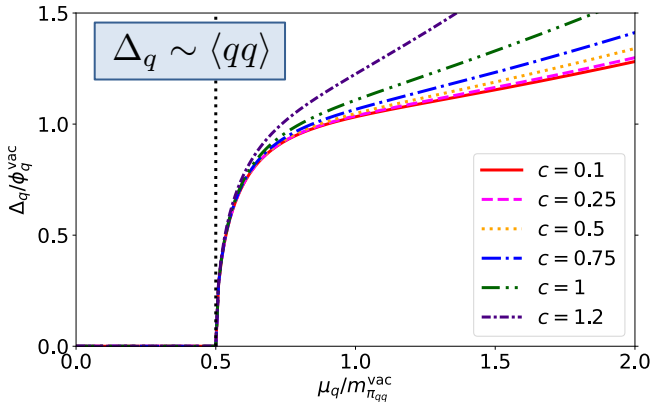


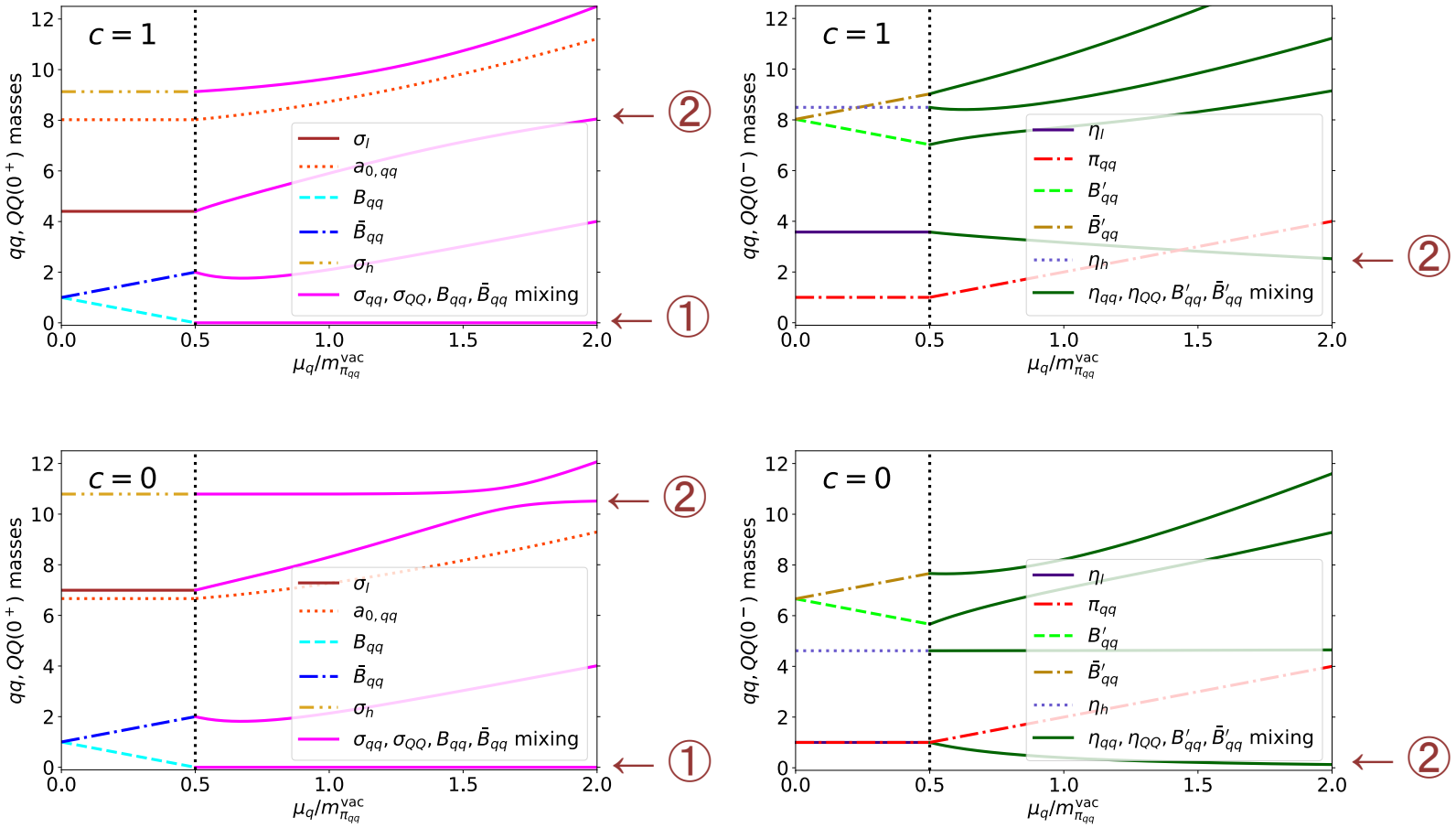




Light-light hadron masses

w/ anomaly →

w/o anomaly →



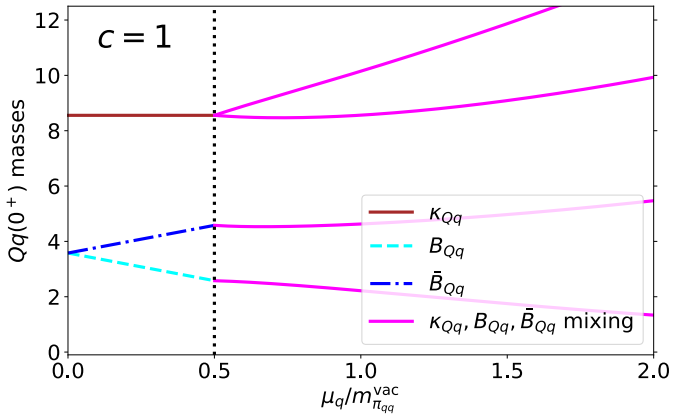
① zero mode (NG mode of $U(1)$ light-quark number breaking)

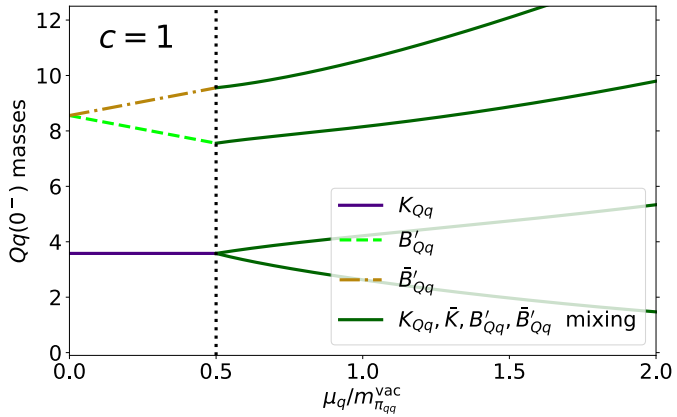
② large anomaly dependences in " σ " and " η " modes

- mixings among $(\sigma_{qq}, \sigma_{QQ}, B_{qq}, \bar{B}_{qq})$ and $(\eta_{qq}, \eta_{QQ}, B'_{qq}, \bar{B}'_{qq})$ occur

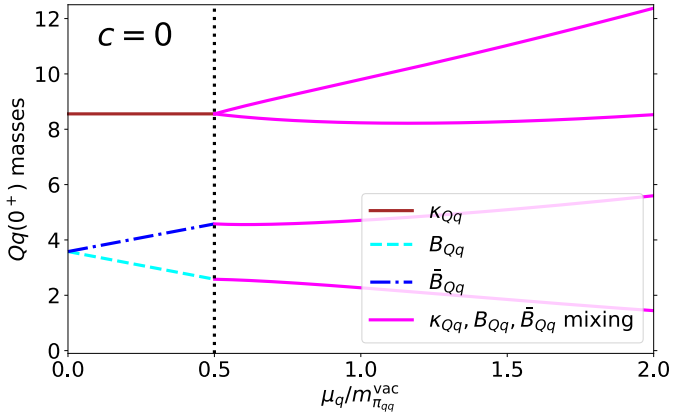
Heavy-light hadron masses

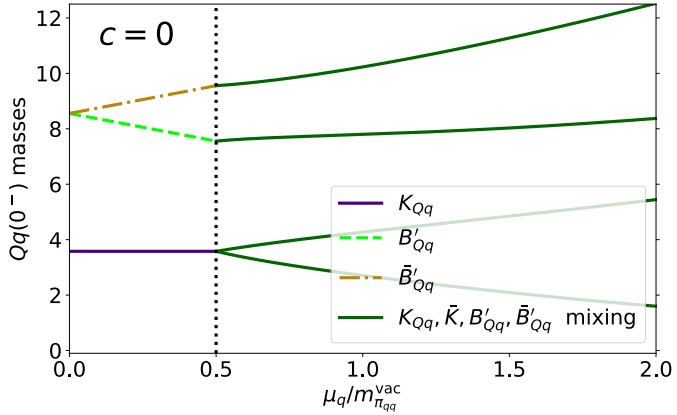
w/ anomaly →





w/o anomaly →

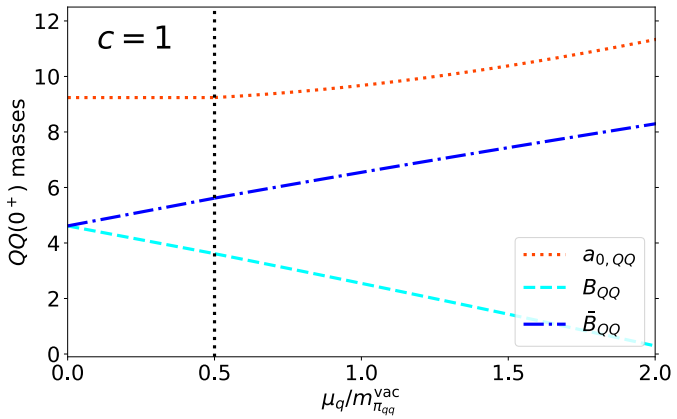


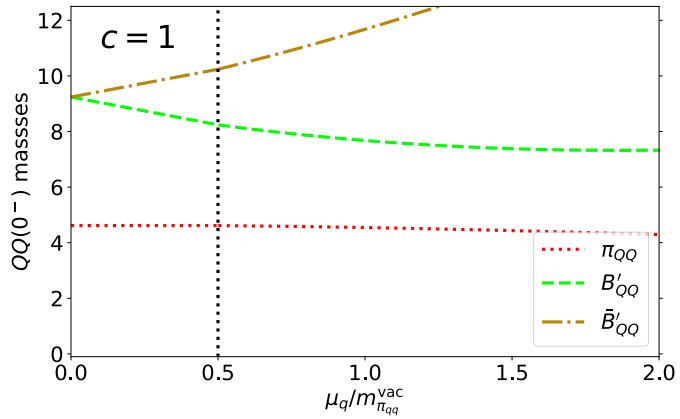


- mixings among $(\kappa_{Qq}, \bar{\kappa}_{Qq}, B_{Qq}, \bar{B}_{Qq})$ and $(K_{Qq}, \bar{K}_{Qq}, B'_{Qq}, \bar{B}'_{Qq})$ occur
- tiny anomaly dependences

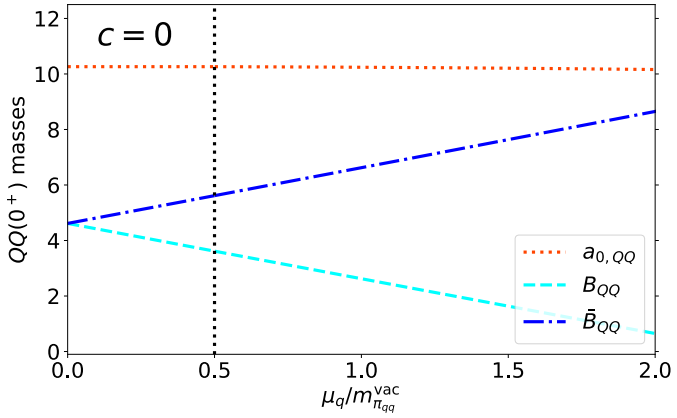
Heavy-heavy hadron masses

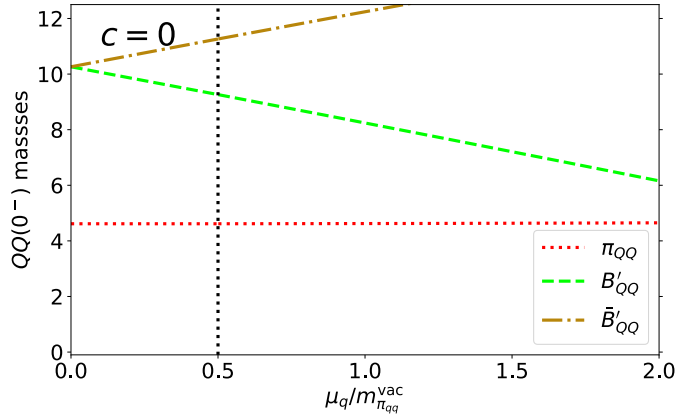
w/ anomaly →





w/o anomaly →

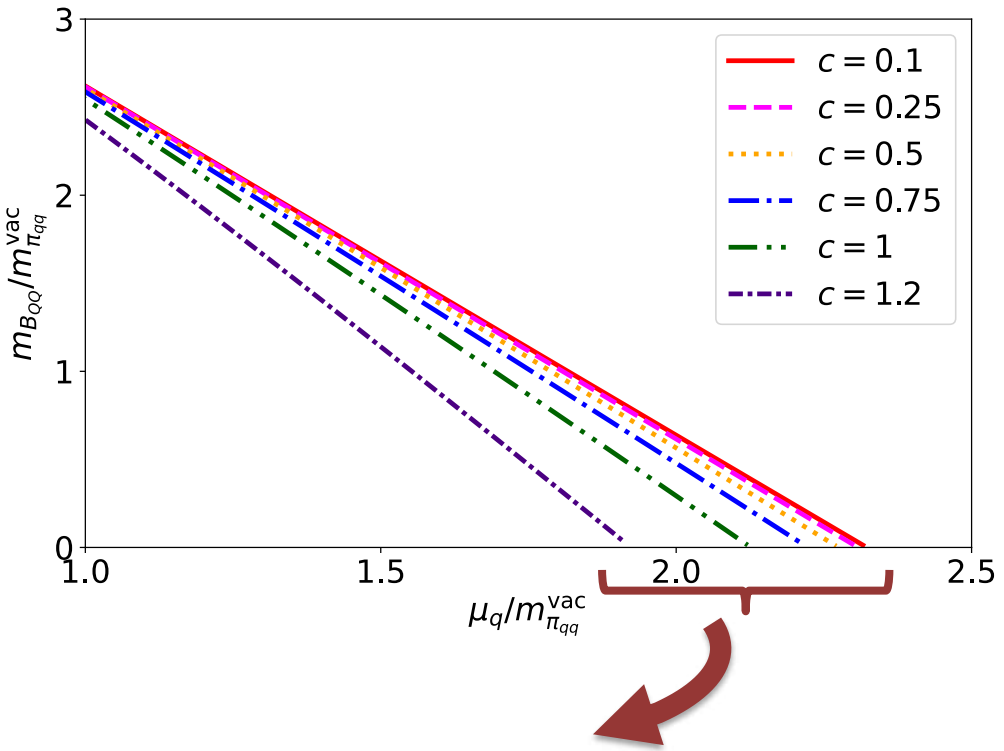




- no mixings occur ($U(1)$ heavy-quark number symmetry is preserved)
- small but non-negligible anomaly dependences

Heavy-heavy hadron masses

- **Possibility of novel diquark condensate $\langle QQ \rangle$ ($B_{QQ} \sim QQ$)**



what happened with quark d.o.f?

- $m_{B_{QQ}} \to 0$ signals the appearance of $\Delta_Q \sim \langle QQ \rangle$
- Larger anomaly effects make the onset μ_q smaller (= anomaly is a “catalyst”)

Conclusions

- Constructed Linear sigma model (LSM) in QC_2D to study spin-0 hadrons including *parity partners* at μ_q

⇩ Succeeded in explaining lattice mass spectrum qualitatively!

Then
- Applied to topological susceptibility, sound velocity
+ FRG analysis (in hot medium), 2+2 flavor LSM study
+ Extension with spin-1 hadrons (similarly to eLSM by Frankfurt group)
:

Suenaga et al (2024)

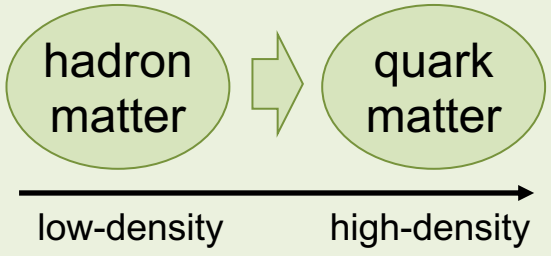
⇔ Comparison with (future) lattice

- On-going: Check of anomaly enhancement in superfluid phase with FRG

From QC_2D study we can learn

- to what extent simple hadron model description can apply
- how to incorporate quark d.o.f. into hadron model (→ unified model)
- deep understanding of quark matter in high-dense regime
:

universal regardless of Nc



LSM with spin-1 hadrons

Spin-1 hadron matrix

$$\Phi^\mu = \frac{1}{2}\begin{pmatrix} \frac{\omega+\rho^0-(f_1+a_1^0)}{\sqrt{2}} & \rho^+ - a_1^+ & \sqrt{2}B_S^{I_z=+1} & B_S^{I_z=0} - B_{AS} \\ \rho^- - a_1^- & \frac{\omega-\rho^0-(f_1-a_1^0)}{\sqrt{2}} & B_S^{I_z=0} + B_{AS} & \sqrt{2}B_S^{I_z=-1} \\ \sqrt{2}\bar{B}_S^{I_z=-1} & \bar{B}_S^{I_z=0} + \bar{B}_{AS} & -\frac{\omega+\rho^0+f_1+a_1^0}{\sqrt{2}} & -(\rho^- + a_1^-) \\ \bar{B}_S^{I_z=0} - \bar{B}_{AS} & \sqrt{2}\bar{B}_S^{I_z=+1} & -(\rho^+ + a_1^+) & -\frac{\omega-\rho^0+f_1-a_1^0}{\sqrt{2}} \end{pmatrix}^\mu$$

	Hadron	J^P	Quark number	Isospin
mesons	ω	1^-	0	0
	ρ	1^-	0	1
	f_1	1^+	0	0
	a_1	1^+	0	1
diquark baryons	B_S $(\bar{B}_S)$	1^+	+2 (−2)	1
	B_{AS} $(\bar{B}_{AS})$	1^-	+2 (−2)	0

SU(4) transformation laws: $\Phi^\mu \to U\Phi^\mu U^\dagger \quad [U \in SU(4)]$

explicit breaking $U(1)_A$ anomaly

$$\begin{aligned}\mathcal{L}_{\rm eLSM} &= \mathrm{tr}[D_\mu\Sigma^\dagger D^\mu\Sigma] - m_0^2\mathrm{tr}[\Sigma^\dagger\Sigma] - \lambda_1\left(\mathrm{tr}[\Sigma^\dagger\Sigma]\right)^2 - \lambda_2\mathrm{tr}[(\Sigma^\dagger\Sigma)^2] + \mathrm{tr}[H^\dagger\Sigma + \Sigma^\dagger H] + c(\det\Sigma + \det\Sigma^\dagger) \\ &- \frac{1}{2}\mathrm{tr}[\Phi_{\mu\nu}\Phi^{\mu\nu}] + m_1^2\mathrm{tr}[\Phi_\mu\Phi^\mu] + ig_3\mathrm{tr}\left[\Phi_{\mu\nu}[\Phi^\mu,\Phi^\nu]\right] + h_1\mathrm{tr}[\Sigma^\dagger\Sigma]\mathrm{tr}[\Phi_\mu\Phi^\mu] + h_2\mathrm{tr}[\Sigma\Sigma^\dagger\Phi_\mu\Phi^\mu] \\ &+ h_3\mathrm{tr}[\Phi_\mu^T\Sigma^\dagger\Phi^\mu\Sigma] + g_4\mathrm{tr}[\Phi_\mu\Phi_\nu\Phi^\mu\Phi^\nu] + g_5\mathrm{tr}[\Phi_\mu\Phi^\mu\Phi_\nu\Phi^\nu] + g_6\mathrm{tr}[\Phi_\mu\Phi^\mu]\mathrm{tr}[\Phi_\nu\Phi^\nu] + g_7\mathrm{tr}[\Phi_\mu\Phi_\nu]\mathrm{tr}[\Phi^\mu\Phi^\nu]\end{aligned}$$

$$\begin{aligned}\Phi_{\mu\nu} &\equiv D_\mu\Phi_\nu - D_\nu\Phi_\mu \\ D_\mu\Sigma &\equiv \partial_\mu\Sigma - iG_\mu\Sigma - i\Sigma G_\mu^T - ig_1\Phi_\mu\Sigma - ig_2\Sigma\Phi_\mu^T \\ D_\mu\Phi_\nu &\equiv \partial_\mu\Phi_\nu - i[G_\mu,\Phi_\nu]\end{aligned}$$

$H = h_q E$ with $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\pi - a_1$ mixings, etc.

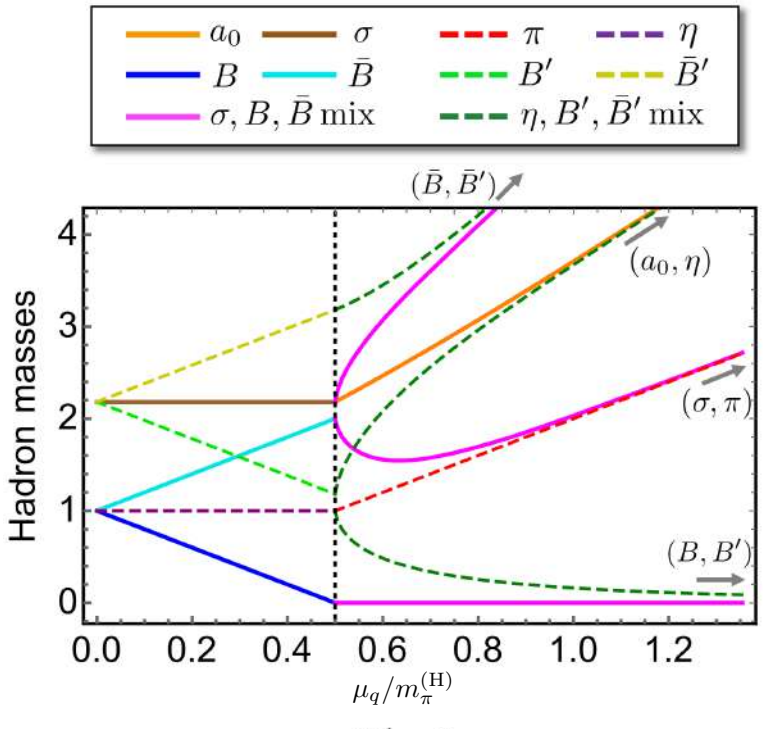
MF in superfluid phase

mesons: $\sigma_0 = \langle\sigma\rangle \qquad \bar{\omega} = \langle\omega^{\mu=0}\rangle$

diquarks: $\Delta = \left\langle \frac{B+\bar{B}}{\sqrt{2}} \right\rangle \qquad \bar{V} = \left\langle \frac{\bar{B}_{AS}^{\mu=0} - B_{AS}^{\mu=0}}{\sqrt{2}i} \right\rangle$

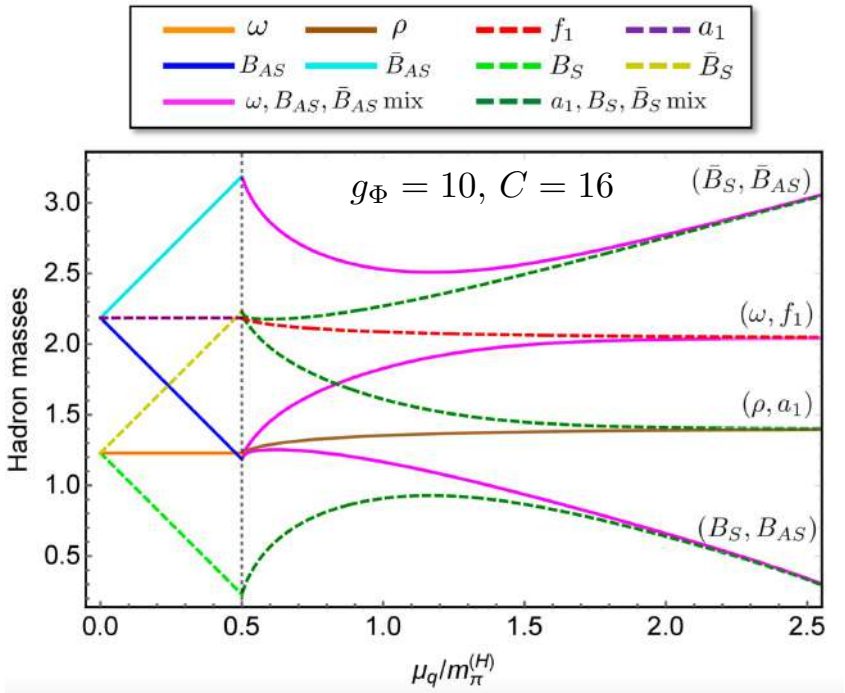
LSM with spin-1 hadrons

Spin-0 hadrons



Spin-1 hadrons

Suenaga et al (2024)

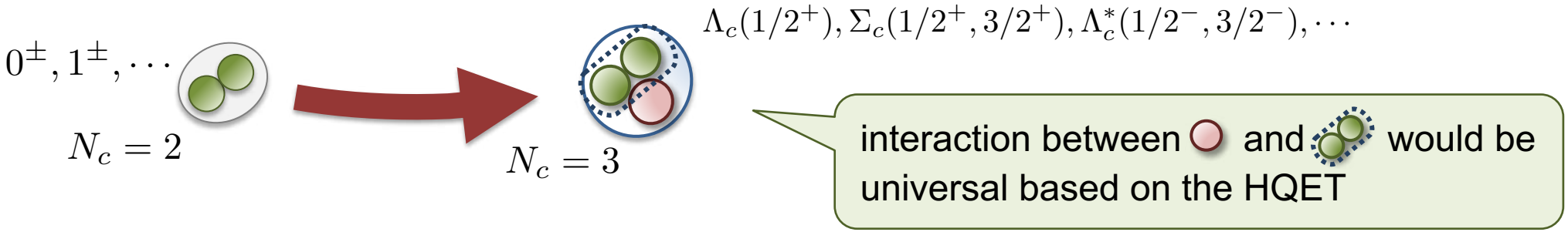


- Mass degeneracy of chiral partners at high μ_q is clearly predicted
  → challenging issue for future lattice simulation

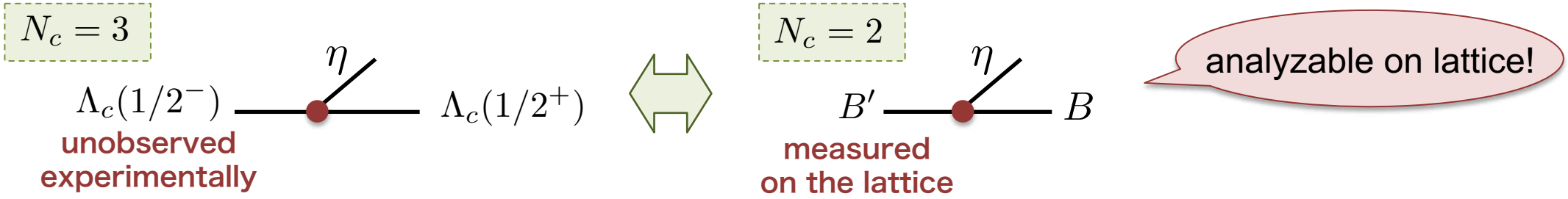
Other directions

- **Analysis with Nf=2+1 at vanishing μ_q**

- QC_2D is useful to extract information of singly heavy baryon (SHB) spectrum



- Hint for unobserved (HQS-singlet) $\Lambda_c(1/2^-)$ from B' study in QC_2D?



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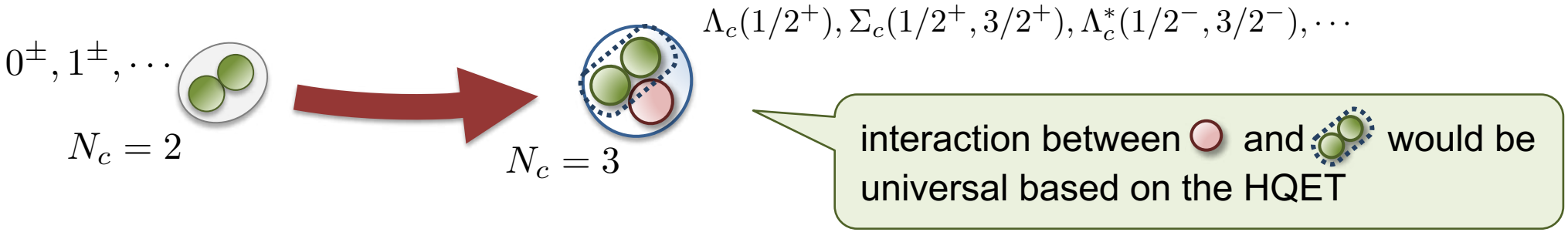
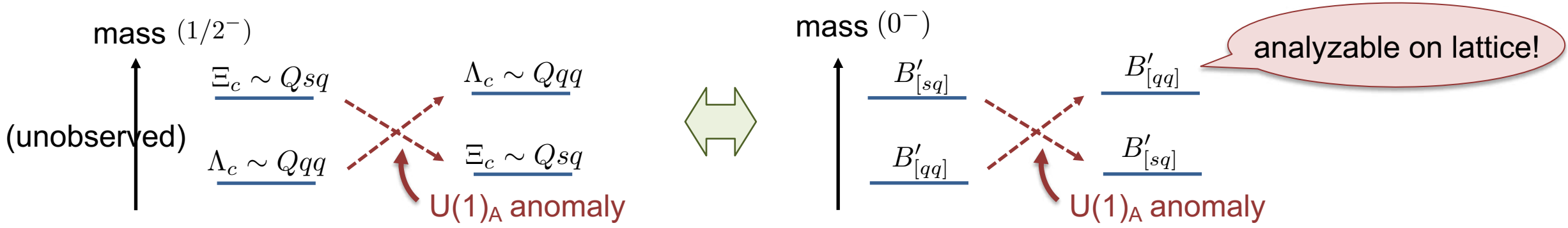


- Insights into **inverse mass hierarchy?**



theoretical prediction, Harada et al (2020)