The Abs-Normal Form and Beyond: Rethinking Piecewise Linear Models

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Algorithmic Piecewise Differentiation

- **Algorithmic Piecewise Differentiation (APD)** is an extension of standard AD.
- APD computes **gen. derivatives for abs-factorable** functions $y = F(x)\colon \mathbb{R}^n \to \mathbb{R}^m$.
- Abs-factorable functions are piecewise smooth functions represented by an **evaluation procedure**
- Eval. procedure includes **absolute value as switching function** besides the other elementary functions.
- APD uses **an extended eval. procedure** to compute directional derivatives Δv_i of intermediates v_i.
- Key for APD is **a modified propagation rule** for the elementary functions φ_i in the eval. procedure:

$$\Delta \mathbf{v}_i = \begin{cases} |\mathbf{v}_j + \Delta \mathbf{v}_j| - |\mathbf{v}_j| & \text{if } \varphi_i(\mathbf{v}_j)_{j \prec i} \equiv |\mathbf{v}_j|, \\ \sum_{j \prec i} \left(\dfrac{\partial \varphi_i}{\partial v_j}(v_j)_{j \prec i} \right) \Delta \mathbf{v}_j & \text{else.} \end{cases}$$

- Propagation rule yields an extended evaluation procedure representing a **piecewise linear approximation** $\Delta y = \Delta F(x; \Delta x)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ of F for fixed $x \in \mathbb{R}^n$ in direction $\Delta x \in \mathbb{R}^n$.

Abs-Normal Form

- P.l. approximation ΔF can be represented algebraically by a **system of piecewise linear equations**.
- One representation for $\Delta y = \Delta F(x; \Delta x): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is the **Abs-Normal Form (ANF)**:

$$\begin{bmatrix} \Delta \mathbf{z} \\ \Delta \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} + \begin{bmatrix} \mathbf{Z} & \mathbf{L} \\ \mathbf{J} & \mathbf{Y} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ |\Delta \mathbf{z}| \end{bmatrix}.$$

- ANF uses a vector $\Delta z \in \mathbb{R}^s$ of **switching variables** and its elementwise absolute value $|\Delta z| \in \mathbb{R}^s$.
- **Dimension** $s \in \mathbb{N}$ of $\Delta z \in \mathbb{R}^s$ depends on number of absolute-value calls in evaluation procedure.
- The **block entries of the ANF** are certain derivatives of the function F at $x \in \mathbb{R}^n$.

$$\mathbf{a} = \mathbf{a}(\mathbf{x}) \in \mathbb{R}^s, \qquad \mathbf{Z} = \mathbf{Z}(\mathbf{x}) \in \mathbb{R}^{s \times n}, \qquad \mathbf{L} = \mathbf{L}(\mathbf{x}) \in \mathbb{R}^{s \times s},$$
$$\mathbf{b} = \mathbf{b}(\mathbf{x}) \in \mathbb{R}^m, \qquad \mathbf{J} = \mathbf{J}(\mathbf{x}) \in \mathbb{R}^{m \times n}, \qquad \mathbf{Y} = \mathbf{Y}(\mathbf{x}) \in \mathbb{R}^{m \times s}$$

Halfpipe Example

- Standard non-smooth, non-linear, non-convex example:

$$\mathbf{y} = F(\mathbf{x}) := \max(0, \mathbf{x}_2^2 - \max(0, \mathbf{x}_1)).$$

- Reformulation in terms of the absolute-value function:

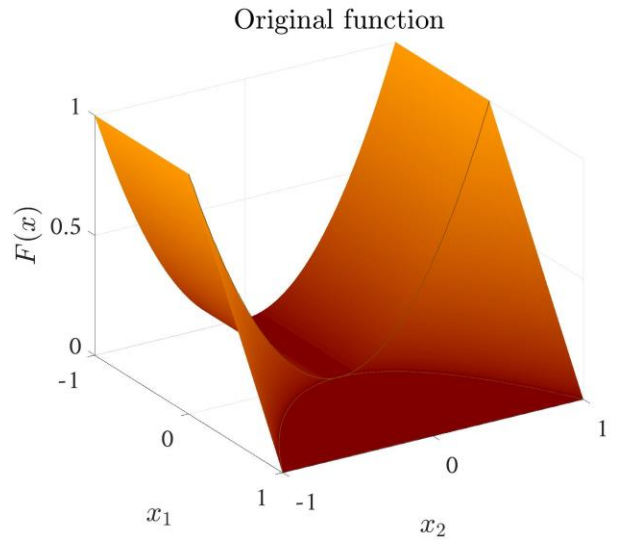
$$\mathbf{y} = \tfrac{1}{2}\left(\mathbf{x}_2^2 - \tfrac{1}{2}(\mathbf{x}_1 + |x_1|) + \left|\mathbf{x}_2^2 - \tfrac{1}{2}(\mathbf{x}_1 + |x_1|)\right|\right).$$

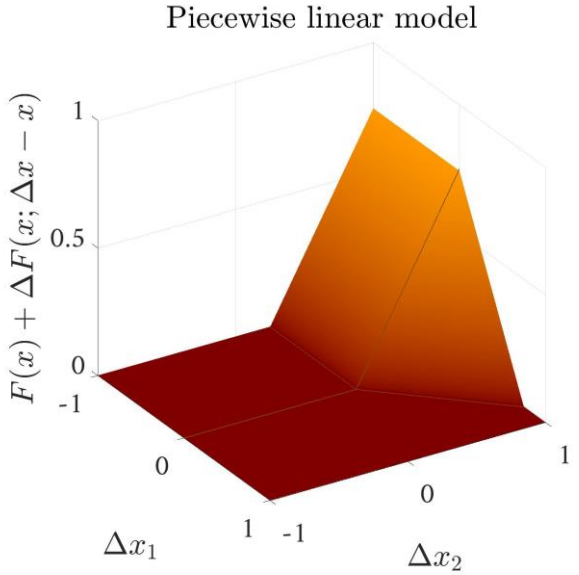
- Piecewise linear approximation $\Delta y = \Delta F(x; \Delta x)$ of F:

$$\begin{aligned}\Delta\mathbf{y} = &\tfrac{1}{2}\left(\mathbf{x}_2^2 + 2\mathbf{x}_2\Delta\mathbf{x}_2 - \tfrac{1}{2}(\mathbf{x}_1 + \Delta\mathbf{x}_1 + |\mathbf{x}_1 + \Delta\mathbf{x}_1|)\right.\\ &\left.+\left|\mathbf{x}_2^2 + 2\mathbf{x}_2\Delta\mathbf{x}_2 - \tfrac{1}{2}(\mathbf{x}_1 + \Delta\mathbf{x}_1 + |\mathbf{x}_1 + \Delta\mathbf{x}_1|)\right|\right) - F(\mathbf{x}).\end{aligned}$$

- ANF of p.l. approximation ΔF of F at the point x:

$$\left[\begin{array}{c}\Delta\mathbf{z}_1\\ \Delta\mathbf{z}_2\\ \hline \Delta\mathbf{y}\end{array}\right] = \left[\begin{array}{c}\mathbf{x}_1\\ \mathbf{x}_2^2 - \frac{\mathbf{x}_1}{2}\\ \hline \frac{\mathbf{x}_2^2}{2} - \frac{\mathbf{x}_1}{4} - F(\mathbf{x})\end{array}\right] + \left[\begin{array}{cc|cc}1 & 0 & 0 & 0\\ -\frac{1}{2} & 2\mathbf{x}_2 & -\frac{1}{2} & 0\\ \hline -\frac{1}{4} & \mathbf{x}_2 & -\frac{1}{4} & \frac{1}{2}\end{array}\right] \left[\begin{array}{c}\Delta\mathbf{x}_1\\ \Delta\mathbf{x}_2\\ \hline |\Delta\mathbf{z}_1|\\ |\Delta\mathbf{z}_2|\end{array}\right]$$





ReLU Normal-Form

- Instead of using the absolute value function, APD can also be formulated using the **ReLU function**:

$$\llbracket \mathbf{v} \rrbracket := \max(0, \mathbf{v}) = \tfrac{1}{2}(\mathbf{v} + |\mathbf{v}|), \llbracket\ \rrbracket : \mathbb{R} \to \mathbb{R}.$$

- Analogous to the propagation rule for the absolute-value, one finds the **propagation rule for ReLU**:

$$\Delta \mathbf{v}_i = \begin{cases} \llbracket \mathbf{v}_j + \Delta \mathbf{v}_j \rrbracket - \llbracket \mathbf{v}_j \rrbracket & \text{if } \varphi_i(\mathbf{v}_j)_{j \prec i} \equiv \llbracket \mathbf{v}_j \rrbracket, \\ \sum_{j \prec i} \left(\dfrac{\partial \varphi_i}{\partial v_j}(v_j)_{j \prec i} \right) \Delta \mathbf{v}_j & \text{else.} \end{cases}$$

- Analogous to the ANF, one can derive a **ReLU Normal Form** (RNF) based on the ReLU function:

$$\begin{bmatrix} \Delta \mathbf{z} \\ \Delta \mathbf{y} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{Z}} & \tilde{\mathbf{L}} \\ \tilde{\mathbf{J}} & \tilde{\mathbf{Y}} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \llbracket \Delta \mathbf{z} \rrbracket \end{bmatrix}.$$

- Block-entries have same size as in ANF and are again certain derivatives of the function F at x.

Halfpipe Example

- Formulation of the **Halfpipe function using the ReLU** as switching function:

$$\mathbf{y} = F(\mathbf{x}_1, \mathbf{x}_2) = \max(0, \mathbf{x}_2^2 - \max(0, \mathbf{x}_1)) = \|\mathbf{x}_2^2 - \|\mathbf{x}_1\|\|.$$

- Accordingly, the piecewise linear approximation ΔF can be stated in terms of the ReLU:

$$\Delta F(\mathbf{x}; \Delta\mathbf{x}) = \|\mathbf{x}_2^2 + 2\mathbf{x}_2\Delta\mathbf{x}_2 - \|\mathbf{x}_1 + \Delta\mathbf{x}_1\|\| - F(\mathbf{x}).$$

- The corresponding RNF for the piecewise approximation ΔF is now given by:

$$\left[\begin{array}{c} \Delta\mathbf{z}_1 \\ \Delta\mathbf{z}_2 \\ \hline \Delta\mathbf{y} \end{array}\right] = \left[\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2^2 \\ \hline -F(\mathbf{x}) \end{array}\right] + \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 2\mathbf{x}_2 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \Delta\mathbf{x}_1 \\ \Delta\mathbf{x}_2 \\ \hline \|\Delta\mathbf{z}_1\| \\ \|\Delta\mathbf{z}_2\| \end{array}\right].$$

Connection ANF and RNF

The relationship between the absolute value and ReLU function provides a **direct connection** between ANF and RNF.

$$\begin{bmatrix}\Delta \mathbf{z}\\ \Delta \mathbf{y}\end{bmatrix} = \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} + \begin{bmatrix}\mathbf{Z} & \mathbf{L}\\ \mathbf{J} & \mathbf{Y}\end{bmatrix}\begin{bmatrix}\Delta \mathbf{x}\\ |\Delta \mathbf{z}|\end{bmatrix} \quad \longleftrightarrow \quad \begin{bmatrix}\Delta \mathbf{z}\\ \Delta \mathbf{y}\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{a}}\\ \tilde{\mathbf{b}}\end{bmatrix} + \begin{bmatrix}\tilde{\mathbf{Z}} & \tilde{\mathbf{L}}\\ \tilde{\mathbf{J}} & \tilde{\mathbf{Y}}\end{bmatrix}\begin{bmatrix}\Delta \mathbf{x}\\ \|\Delta \mathbf{z}\|\end{bmatrix}$$

$$\begin{aligned}
\mathbf{a} &= (I - \tfrac{1}{2}\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{a}}, & \tilde{\mathbf{a}} &= (I + \mathbf{L})^{-1}\mathbf{a},\\
\mathbf{Z} &= (I - \tfrac{1}{2}\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{Z}}, & \tilde{\mathbf{Z}} &= (I + \mathbf{L})^{-1}\mathbf{Z},\\
\mathbf{L} &= \tfrac{1}{2}(I - \tfrac{1}{2}\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{L}} & \tilde{\mathbf{L}} &= 2(I + \mathbf{L})^{-1}\mathbf{L}\\
\mathbf{b} &= \tilde{\mathbf{b}} + \tfrac{1}{2}\tilde{\mathbf{Y}}(I - \tfrac{1}{2}\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{a}}, & \tilde{\mathbf{b}} &= \mathbf{b} - \mathbf{Y}(I + \mathbf{L})^{-1}\mathbf{a},\\
\mathbf{J} &= \tilde{\mathbf{J}} + \tfrac{1}{2}\tilde{\mathbf{Y}}(I - \tfrac{1}{2}\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{Z}}, & \tilde{\mathbf{J}} &= \mathbf{J} - \mathbf{Y}(I + \mathbf{L})^{-1}\mathbf{Z},\\
\mathbf{Y} &= \tfrac{1}{2}\tilde{\mathbf{Y}}(I - \tfrac{1}{2}\tilde{\mathbf{L}})^{-1} & \tilde{\mathbf{Y}} &= 2\mathbf{Y}(I + \mathbf{L})^{-1}
\end{aligned}$$

(Most of the) **theoretical results for ANFs (evaluation, solution, gradient,…) directly translate to RNFs !!!**

ANF vs. RNF for the Halfpipe example

(ANF)

$$\left[\begin{array}{c} \Delta \mathbf{z}_1 \\ \Delta \mathbf{z}_2 \\ \hline \Delta \mathbf{y} \end{array}\right] = \left[\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2^2 - \frac{\mathbf{x}_1}{2} \\ \hline \frac{\mathbf{x}_2^2}{2} - \frac{\mathbf{x}_1}{4} - F(\mathbf{x}) \end{array}\right] + \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 2\mathbf{x}_2 & -\frac{1}{2} & 0 \\ \hline -\frac{1}{4} & \mathbf{x}_2 & -\frac{1}{4} & \frac{1}{2} \end{array}\right] \left[\begin{array}{c} \Delta \mathbf{x}_1 \\ \Delta \mathbf{x}_2 \\ \hline |\Delta \mathbf{z}_1| \\ |\Delta \mathbf{z}_2| \end{array}\right]$$

vs.

(RNF)

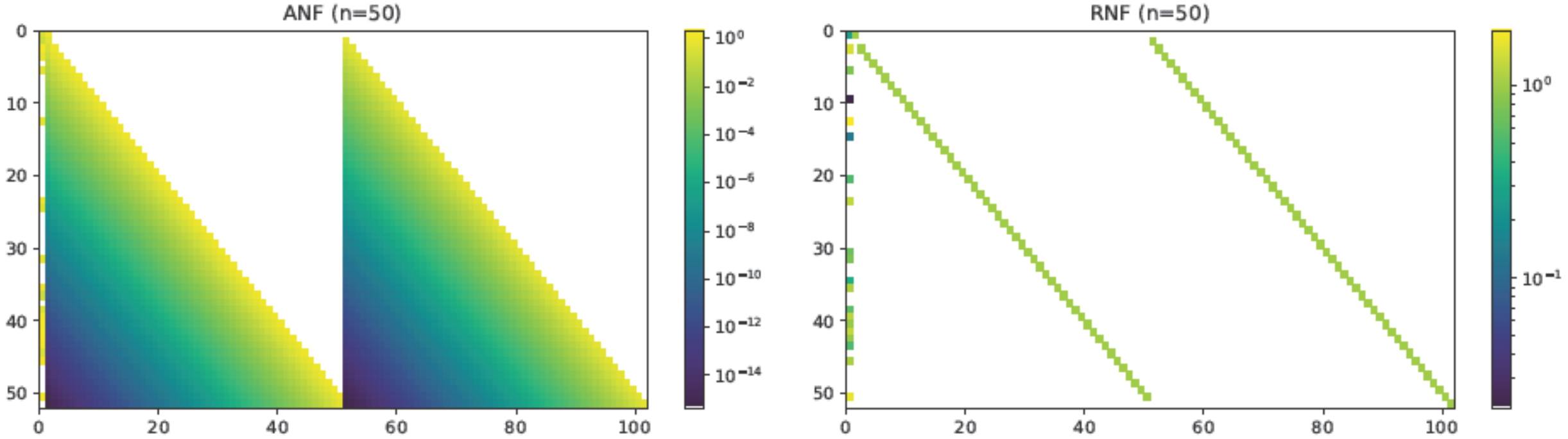
$$\left[\begin{array}{c} \Delta \mathbf{z}_1 \\ \Delta \mathbf{z}_2 \\ \hline \Delta \mathbf{y} \end{array}\right] = \left[\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2^2 \\ \hline -F(\mathbf{x}) \end{array}\right] + \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 2\mathbf{x}_2 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \Delta \mathbf{x}_1 \\ \Delta \mathbf{x}_2 \\ \hline \|\Delta \mathbf{z}_1\| \\ \|\Delta \mathbf{z}_2\| \end{array}\right].$$

IMPORTANT: Note that the two representations are equivalent but differ in their sparsity structure !!!

“FNN” - Example

$$\mathbf{y} = F(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \|\mathbf{x}_n + \|\mathbf{x}_{n-1} + \cdots + \|\mathbf{x}_1 + \|\mathbf{x}_0\|\|\|\|.$$

$$\Delta\mathbf{y} = \|\mathbf{x}_n + \Delta\mathbf{x}_n + \|\mathbf{x}_{n-1} + \Delta\mathbf{x}_{n-1} + \cdots + \|\mathbf{x}_1 + \Delta\mathbf{x}_1 + \|\mathbf{x}_0 + \Delta\mathbf{x}_0\|\|\|\| - F(\mathbf{x}).$$



ANF, RNF and beyond

- Similar to the RNF for the ReLU, one can use **other switching functions** ψ such as NegativeRelu, Clamp,…
- Each (suitable) switching give rise to another **corresponding normal-form**:

$$\begin{bmatrix} \Delta\mathbf{z} \\ \Delta\mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} + \begin{bmatrix} \mathbf{Z} & \mathbf{L} \\ \mathbf{J} & \mathbf{Y} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x} \\ \psi(\Delta\mathbf{z}) \end{bmatrix}$$

- Depending on the switching function, **theoretical results can be adapted** to these normal forms.
- In fact, one does not restrain to only one but consider a **mix of switching functions**.
- The latter results in a **piecewise normal form**, where $\psi\colon \mathbb{R}^s \to \mathbb{R}^s$ is mix of switching functions.
- *Warning:* Translating the results for the mixed case, it is obviously a bit harder and not straight forward ;-)

Conclusion

- Modified propagation rule for **absolute value function was a good start for APD.**
- BUT reducing all non-smoothness to the **absolute value function is suboptimal**.
- Alternatively, one can **use other switching functions** (ReLU, NReLU, Clamp,…).
- For most of these switching functions, the **propagation rule is straight forward.**
- Switching functions yields different normal forms (ReLU-RNF, CLAMP->CNF,…).
- (Most of the) **theoretical results for ANFs translate** to RNFs, CNFs, PNFs., …
- **Piecewise normal form** seems to be a promising very general representation.

Outlook

- Examples/Applications/Implementation/Numerical results/…
- Connection to vertex-/edge-elimination rules on computational (Hyper)graphs,…
- …

Request

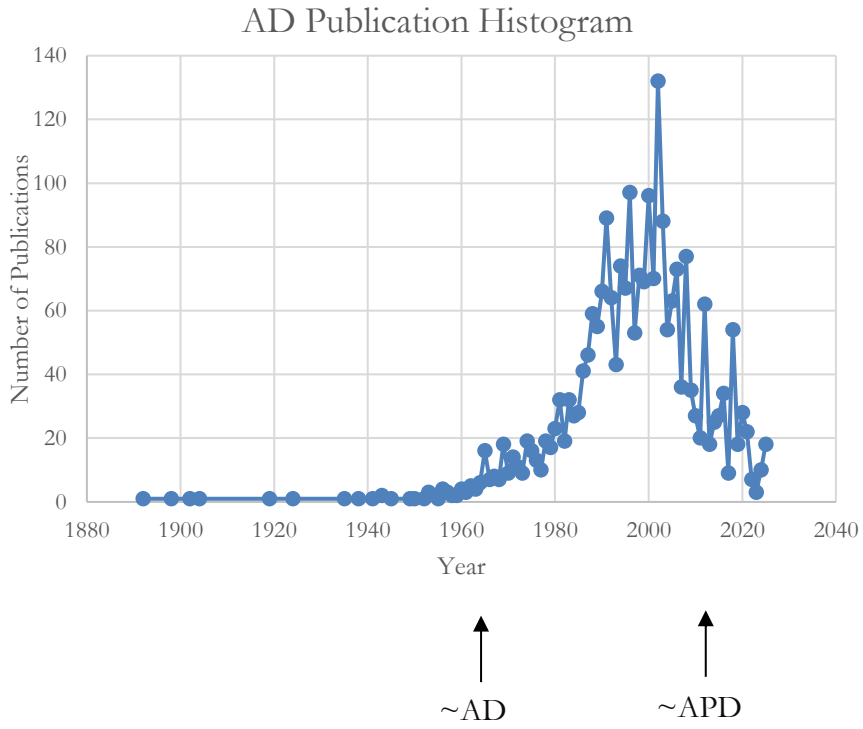
- Please use:

$\llbracket \mathbf{v} \rrbracket := \mathrm{ReLU}(\mathbf{v})$	$\llparenthesis \mathbf{v} \rrparenthesis := \mathrm{InverseReLU}(\mathbf{v})$	$\llbracket \mathbf{v} \rrbracket_{u,w} := \mathrm{Clamp}(\mathbf{v}, \mathbf{u}, \mathbf{w})$
`\llfloor v \rrfloor`	`\llceil v \rrceil`	`\llbracket v \rrbracket`

References

1. **R. Wengert**, *A simple automatic derivative evaluation program*. Commun. ACM 7, 8. https://doi.org/10.1145/355586.364791A**,** 1964.
2. **Griewank and A. Walther,** *Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation*, SIAM, Philadelphia, PA, US, 2nd ed., 2008
3. **A. Griewank.** *On stable piecewise linearization and generalized algorithmic differentiation.* Optimization Methods and Software, 28(6). https://doi.org/10.1080/10556788.2013.796683, 2013
4. **A. Griewank et al.,** *Solving piecewise linear systems in abs-normal form.* Linear Algebra and its Applications, 471, 500-530, https://doi.org/10.1016/j.laa.2014.12.017, 2015
5. **T. B. & K. Narayanan.** *Study of the numerical efficiency of structured abs-normal forms.* Optimization Methods and Software, 36(5). https://doi.org/10.1080/10556788.2019.1613654, 2021
6. **T. B.** *(Almost) matrix-free solver for piecewise linear functions in abs-normal form.* Numerical Linear Algebra with Applications. 26:e2258. https://doi.org/10.1002/nla.2258, 2019
7. **T.B. and ???**, *~PNFs and their connection to APD* , to appear in ???, >2025

8. …other publications, whose contributions are equally valued though not individually cited. If you need further direction, just ask. And please don't feel left out if your work isn't mentioned.



Quo vaditis, AD et APD?

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