Disentangling Singularities with Non-linear Mappings

A new method for NNLO computations

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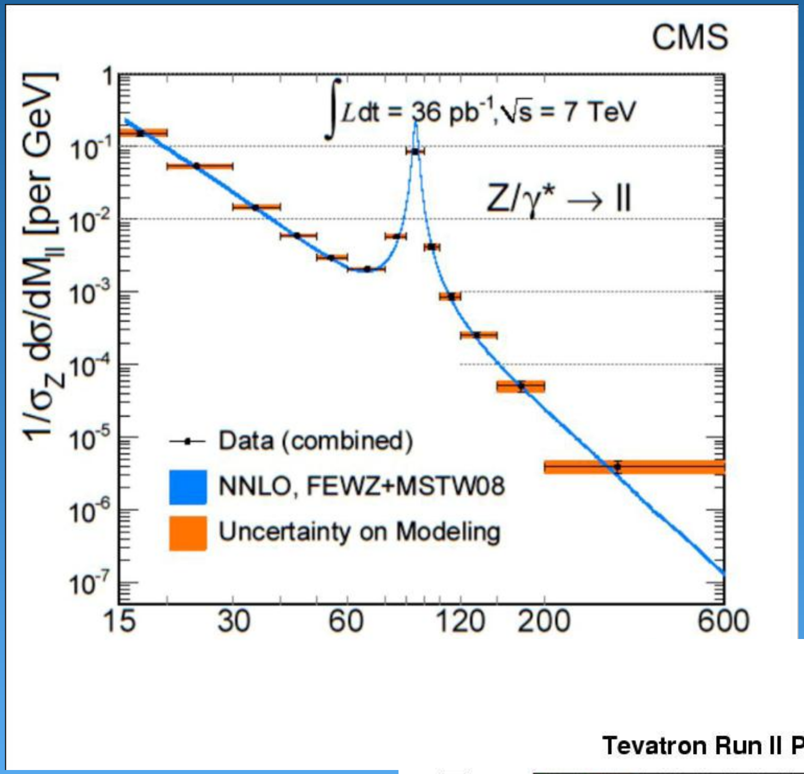
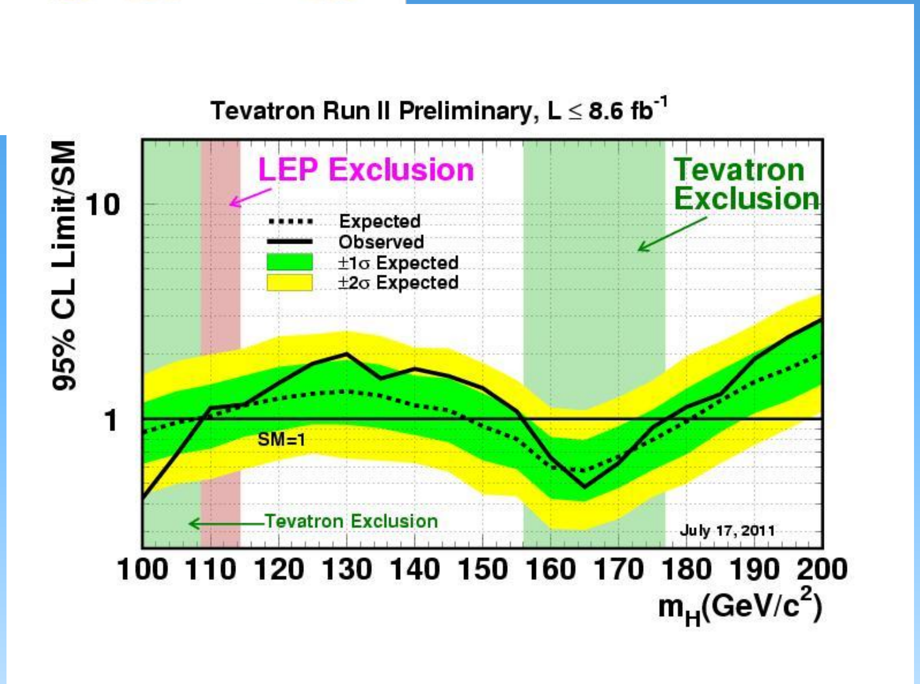
Differential NNLO: a young and promising field in the LHC era

- less than a decade old
- starting from simple decays and single production processes
- moved to complicated jet-production processes and pair production at colliders
- already a significant impact on precision phenomenology at collider experiments





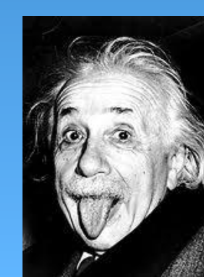
$$\alpha_s(M_Z) = 0.1175 \pm 0.0020\,(\mathrm{exp}) \pm 0.0015\,(\mathrm{theo})$$

Basic mathematical problem

- **Divergent** loop and phase-space **integrals** , regulated by $\epsilon = (D-4)/2$

$$\mathrm{FiniteObservable} = \sum_j \int_{\mathrm{boundaries}} \prod_i dp_i \mathrm{DivergentAmplitude}_j(\{p_k\})$$

- **Fixed** integration **boundaries** for loops and ***inclusive*** phase-space integrations.

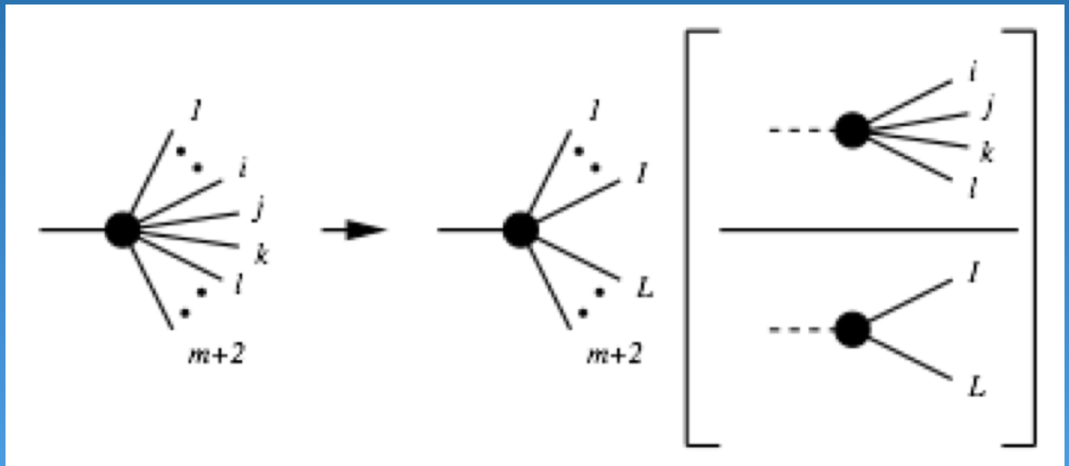


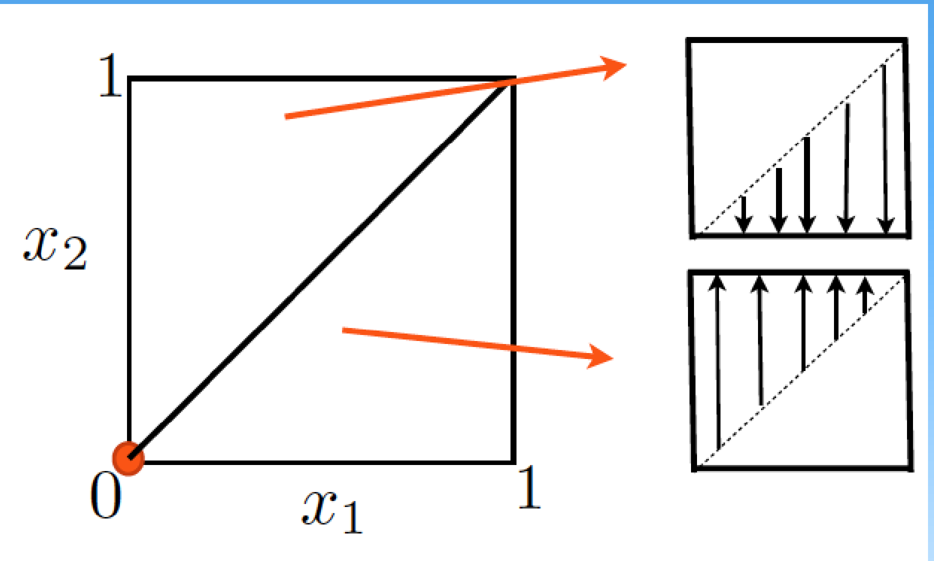
- Infrared safe but otherwise **arbitrary boundaries** of phase-space for acceptance cuts and ***differential*** distributions.

OVERLAPPING SINGULARITIES

Differential Methods

- NNLO needed **new** methods for phase-space integrations with arbitrary cuts and experimental observables.
- subtraction, antennae, kt-subtraction, sector decomposition, slicing, physical sectors, ...
- Many conceptual problems remain. Room and need for fresh ideas!





Subtracting factorised singularities

- Just use

$$\int_0^1 dx x^\epsilon \frac{1}{x} = \frac{1}{\epsilon}$$

- To obtain

$$\int_0^1 dx x^\epsilon \frac{f(x)}{x} = \frac{f(0)}{\epsilon} + \int_0^1 dx x^\epsilon \frac{f(x) - f(0)}{x}$$

Infinite in d=4 **Finite**

How to deal with Overlapping singularities?

- FIXED BOUNDARIES
  Mellin-Barnes, differential equations, successive Feynman parameter integrations,...
- ARBITRARY BOUNDARIES (I)
  Subtraction method based on infrared safety and QCD factorization to divide the integration into a singularity free numerical integral and integrals with fixed boundaries.
- ARBITRARY BOUNDARIES (II)
  Sector decomposition
  NEW: Non-linear mappings

A toy example with sector decomposition

Binoth,Heinrich; Denner,Roth; Hepp

$$I = \int_0^1 dx dy \frac{x^\epsilon}{x(ax+y)}$$

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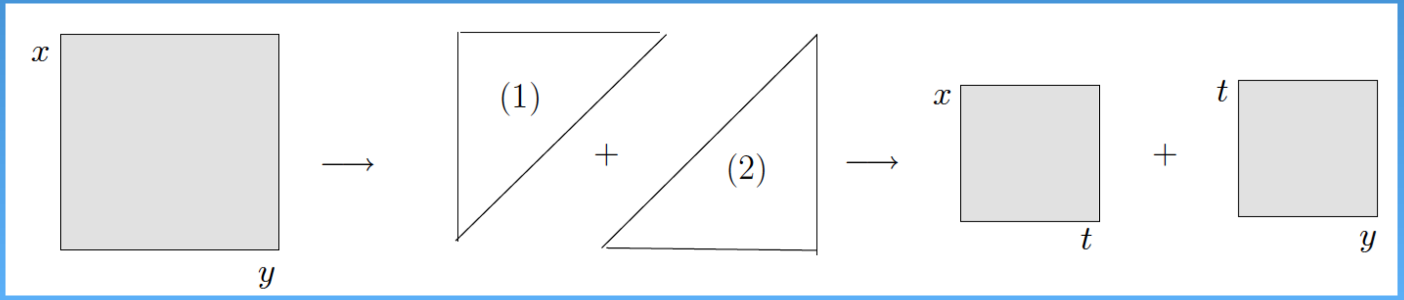
Slice phase-space



$$dxdy = dxdy \left[\Theta(x \geq y) + \Theta(y \geq x)\right]$$

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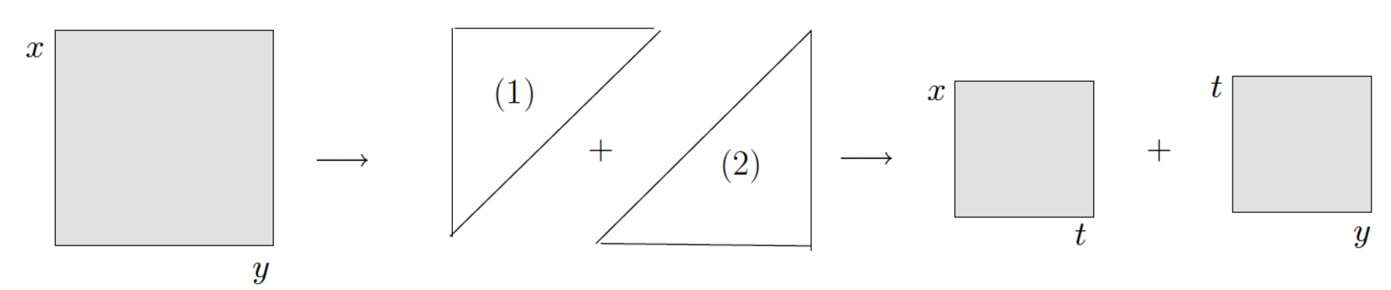

$$dxdy = dxdy\left[\Theta(x \geq y) + \Theta(y \geq x)\right]$$

Restore boundaries

$$I = \int_0^1 dx dt \frac{(x)^\epsilon}{x(a+t)} + \int_0^1 dt dy \frac{(yt)^\epsilon}{yt(at+1)}$$

$$y = tx \qquad x = ty$$

Singularities are factorized!

Cost: integral proliferation

Non-linear mappings

- Factorizes overlapping singularities
- trivializes extraction of poles
- Local

...like sector decomposition

...but

- Easier to implement
- Does not proliferate integrations
- Transparent and more physical factorisation of singularities

A toy example with nonlinear mapping:

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factorizes the singularity

spoils integration boundaries

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factorizes the singularity — spoils integration boundaries

$$x \mapsto \frac{x(y/a)}{1-x+(y/a)} \qquad \mapsto \int_0^1 dx dy \frac{(xy)^\epsilon}{xy} \left(a(1-x)+y\right)^{-\epsilon}$$

factorizes the singularity — preserves integration boundaries

A systematic method of non-linear mappings at NNLO

- Most divergent (massless) two-loop integrals

  arXiv:1011.4867

- Double real-radiation integrals which emerge in hadron collider processes (Higgs, top-pair,...)

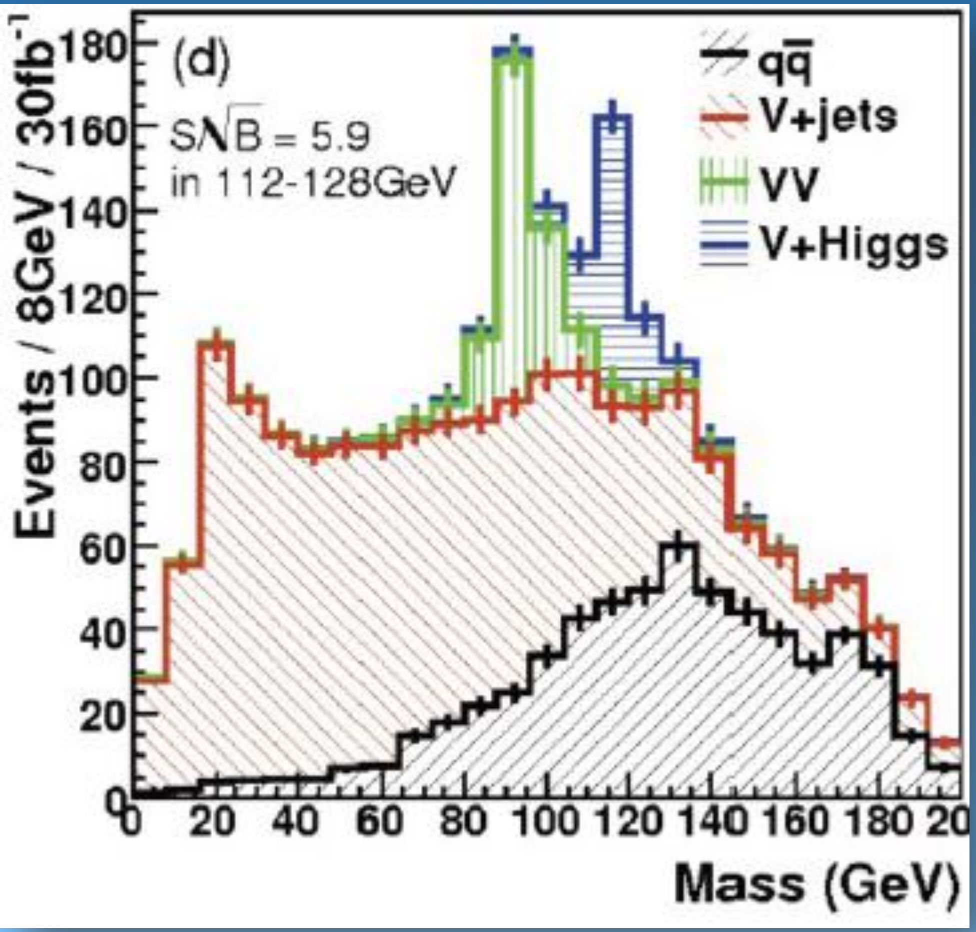
  arXiv:1011.4867

- Real-virtual. (this work)

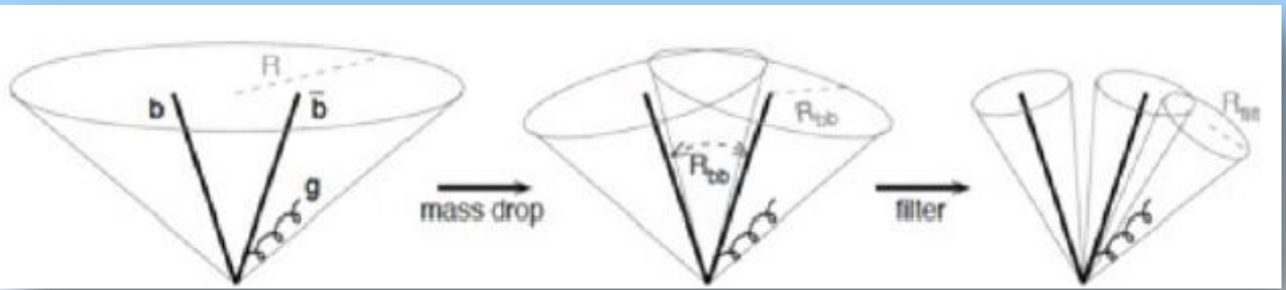
- Double real-radiation for decays. (this work)

1st physical application

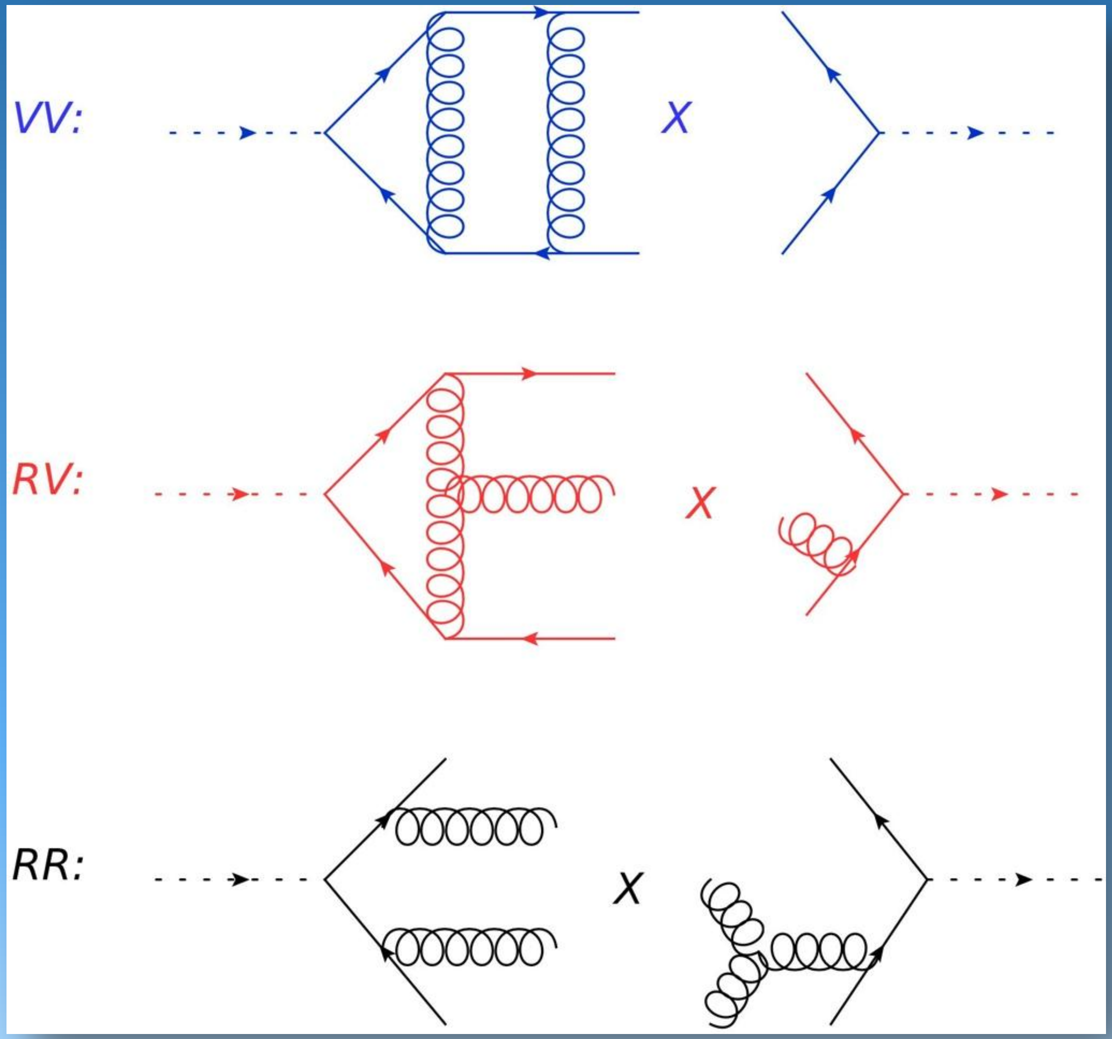
- The decay of a **Higgs boson to bottom-quarks** is dominant for a light Higgs boson.
- A viable discovery with associated Higgs.
- Gluon radiation off the bottom-pair system is important for fat-jet analyses.
- Nice proof of principle of our method.



Butterworth,Davison,Rubin,Salam

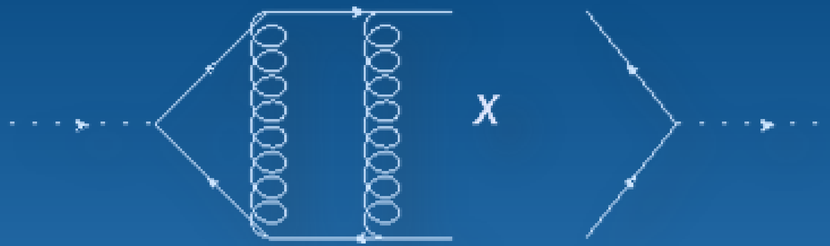


Feynman diagrams



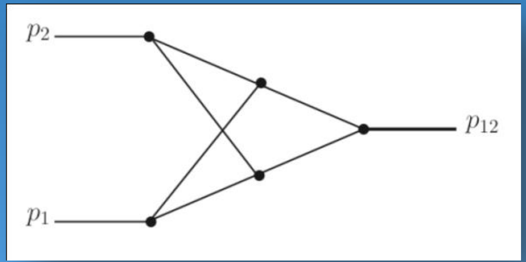
- Easily done analytically... Highly complicated application of non-linear mappings.
- Non-trivial! Overlapping loop and phase-space singularities.
- **Difficult**, but well suited problem for our method.

Double Virtual



arXiv:1011.4867

- Used nonlinear mappings to do the cross triangle, ending with 7 integrals, contrasted to 64 with sector decomposition.

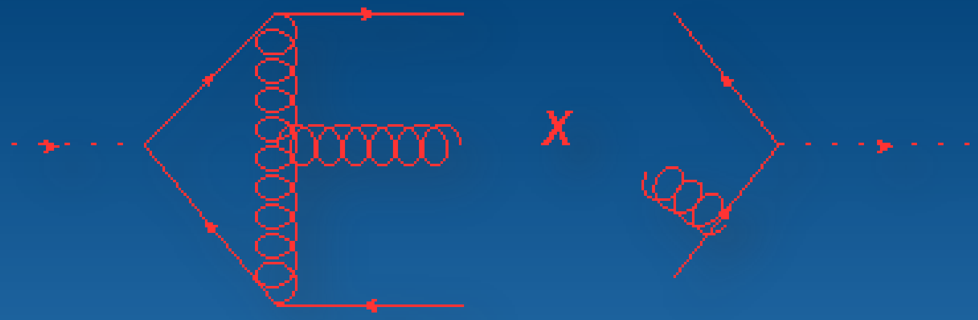


$$= 4^{2+2\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{z y^{1+\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-1-\epsilon}}{[x(1-x)+yz(x-x_1)(x-x_2)]^{2+2\epsilon}}$$

- Also analytically with reducing (Laporta algorithm, AIR) to master integrals (known since 1987).

- Use the analytic result in our Monte-Carlo program for the decay width.

- Our method can be **useful** for two-loop amplitudes which are not yet known analytically (more masses, off-shell legs, ...)

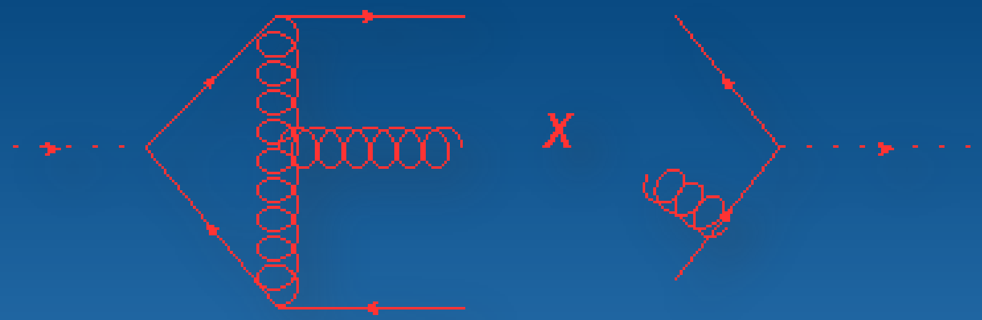


Real-Virtual

- Use Laporta algorithm (AIR) to reduce to master integrals (box and bubble)
- Need to integrate the one-loop box over singular phase-space (non-smooth off-shell to on-shell leg limit)

$$\int d\mathrm{PS}_3 \frac{{}_2F_1(1, 1-\epsilon, -\epsilon, -\frac{u}{t})}{ut}$$

Real-Virtual



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$$\int d\mathrm{PS}_3 \frac{{}_2F_1(1, 1-\epsilon, -\epsilon, -\frac{u}{t})}{ut}$$

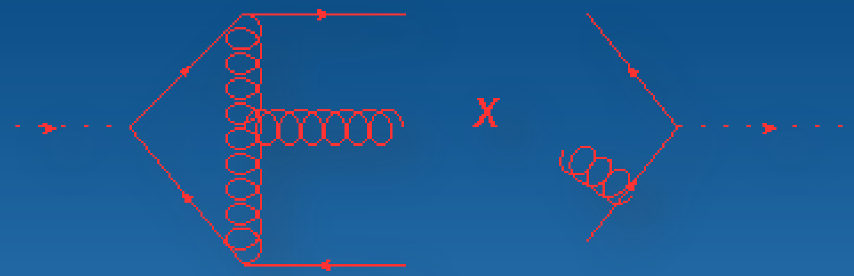
- Use Euler representation of hypergeometric function

$${}_2F_1\left(1, 1-\epsilon, -\epsilon, -\frac{u}{t}\right) = -\epsilon t \int_0^1 dx_3 \frac{x_3^{-1-\epsilon}}{t + ux_3}$$

- Apply non-linear mapping

$$x_3 \mapsto \frac{x_3 t/u}{1 - x_3 + t/u}$$

Real-Virtual (II)



- Our mapping simply “re”-derives a known identity

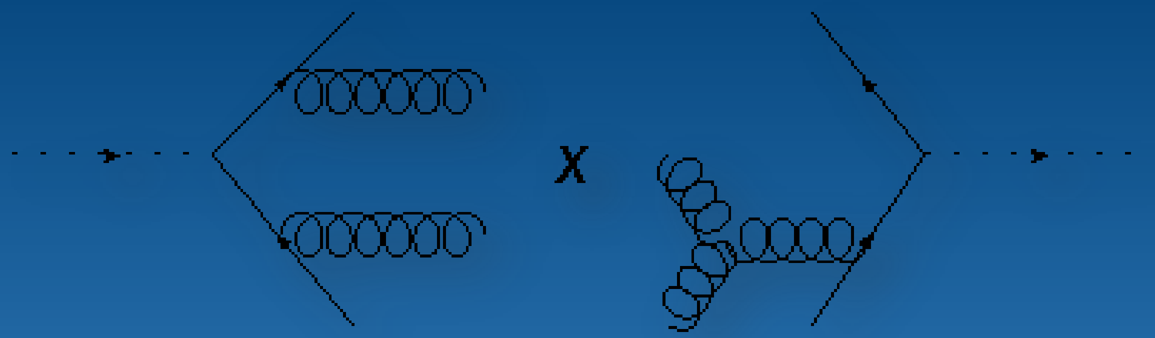
$$ {}_2F_1(a,b,c;z) = (1-z)^{-b}\, {}_2F_1\left(c-a,b,c;\frac{z}{z-1}\right) $$

Full regulator dependence must be kept and combined with phase-space measure

carefully expanded in epsilon and subtracted in soft/collinear limits

- Implemented both analytic and semi-analytic (non-linear mapping) methods. Surprisingly, no difference in evaluation time

Double Real



- Overlapping singularities "thrive" in Feynman diagrams with double real emissions

$$\bar{\lambda} = 1 - \lambda$$

$$\begin{aligned} s_{234} &= \lambda_1 \\ s_{34} &= \lambda_1 \lambda_2 \\ s_{23} &= \lambda_1 \bar{\lambda}_2 \lambda_4 \\ s_{24} &= \lambda_1 \bar{\lambda}_2 \bar{\lambda}_4 \\ s_{12} &= \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 \\ s_{134} &= \lambda_2 + \lambda_3 \bar{\lambda}_1 \bar{\lambda}_2 \end{aligned}$$

$$\begin{aligned} s_{13} &= \bar{\lambda}_1 \left[\lambda_4 \lambda_3 + \lambda_2 \bar{\lambda}_3 \bar{\lambda}_4 + 2\cos(\lambda_5 \pi) \sqrt{\lambda_2 \lambda_3 \bar{\lambda}_3 \lambda_4 \bar{\lambda}_4} \right] \\ s_{14} &= \bar{\lambda}_1 \left[\lambda_3 \bar{\lambda}_4 + \lambda_2 \bar{\lambda}_3 \lambda_4 - 2\cos(\lambda_5 \pi) \sqrt{\lambda_2 \lambda_3 \bar{\lambda}_3 \lambda_4 \bar{\lambda}_4} \right]. \end{aligned}$$

- We have factorized ALL overlapping singularities with partial fractioning and just three mapping at most! e.g.

$$\alpha(x, A) := \frac{xA}{xA + \bar{x}}$$

$$I_2 = \int d\Phi_4 \frac{J(p_1, p_2, p_3, p_4)}{s_{13} s_{23} s_{134} s_{234}}$$

$$\begin{aligned} \lambda_2 &\mapsto \alpha(\lambda_2, \lambda_3) \\ \lambda_4 &\mapsto \alpha(\lambda_4, \lambda_2 \bar{\lambda}_3) \\ \lambda_2 &\mapsto \alpha(\lambda_2, \bar{\lambda}_1) \end{aligned}$$

The inclusive check

- Numerically

$$\Gamma^{NNLO}_{H\to b\bar{b}} = \Gamma^{LO}_{H\to b\bar{b}}\left[1 + \left(\frac{\alpha_s}{\pi}\right) 5.6666(4) + \left(\frac{\alpha_s}{\pi}\right)^2 29.14(2) + \mathcal{O}(\alpha_s^3)\right]$$

- Analytically

$$\Gamma^{NNLO}_{H\to b\bar{b}} = \Gamma^{LO}_{H\to b\bar{b}}\left[1 + \left(\frac{\alpha_s}{\pi}\right) 5.6666666.. + \left(\frac{\alpha_s}{\pi}\right)^2 29.146714.. + \mathcal{O}(\alpha_s^3)\right]$$

Jet rates with JADE algorithm

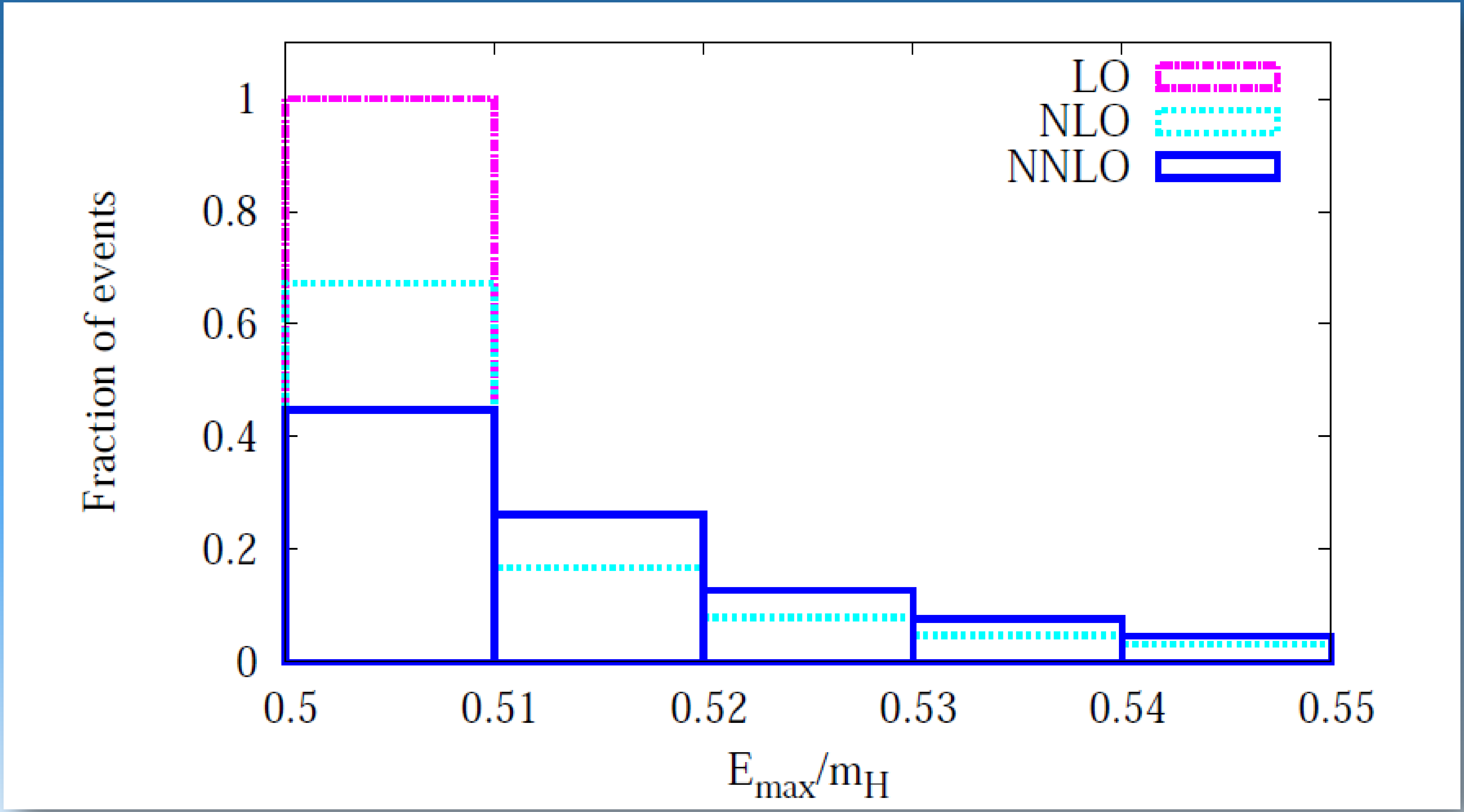
$$y_{cut} = 0.01$$

$$\Gamma^{NNLO}_{H\to b\bar{b}}(\mathrm{2JetRate}) = \Gamma^{LO}_{H\to b\bar{b}}\left[1 - \left(\frac{\alpha_s}{\pi}\right) 13.591(6) - \left(\frac{\alpha_s}{\pi}\right)^2 307(2) + \mathcal{O}(\alpha_s^3)\right]$$

$$\Gamma^{NLO}_{H\to b\bar{b}}(\mathrm{3JetRate}) = \Gamma^{LO}_{H\to b\bar{b}}\left[+ \left(\frac{\alpha_s}{\pi}\right) 19.258(4) + \left(\frac{\alpha_s}{\pi}\right)^2 241(2) + \mathcal{O}(\alpha_s^3)\right]$$

$$\Gamma^{LO}_{H\to b\bar{b}}(\mathrm{4JetRate}) = \Gamma^{LO}_{H\to b\bar{b}}\left[+ \left(\frac{\alpha_s}{\pi}\right)^2 94.1(1) + \mathcal{O}(\alpha_s^3)\right]$$

A fully differential observable: Maximum Energy in 2 Jet rate



Conclusions

- Established a new method for fully differential NNLO computations
- Studied a wide spectrum of cases, initial state radiation, final state radiation, two-loop integrals, real-virtual
- First physics application: the differential decay rate of the Higgs boson to bottom quarks
- More to follow….