Large-x resummation in semi-inclusive e^+e^- annihilation

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29 September 2011

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Presented at: RADCOR 2011, Mamallapuram, India

Introduction: semi-inclusive e^+e^- annihilation (SIA)

Fragmentation functions $F_a^h(x, Q^2)$ in $e^+ e^- \to \gamma, Z \to h + X$

$$\frac{1}{\sigma_{tot}} \frac{d^2\sigma}{dx\, d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\, F_T^h + \frac{3}{4}\sin^2\theta\, F_L^h + \frac{3}{4}\cos\theta\, F_A^h$$

$$x = \frac{2\, p \cdot q}{Q^2}, \qquad \text{where} \qquad Q^2 \equiv q^2 > 0$$

$\theta \longrightarrow$ angle in the centre-of-mass frame between $e^{-(+)}$ and the hadron $h(p)$.

Factorisation formula (terms $\mathcal{O}(1/Q)$ neglected)

$$F_a^h(x, Q^2) = \sum_{f=q,\bar{q},g} \int_x^1 \frac{dz}{z} c_{a,f}^T\left(z, \alpha(Q^2)\right) D_f^h\left(\frac{x}{z}, Q^2\right)$$

$c_{a,f}^T$ have been calculated up to order α_s^2 [Rijken, van Neerven ('96, '97)]

Time-like splitting functions $P^T(x, \alpha_s(Q^2))$ in evolution equation

$$\frac{d}{d\ln Q^2} D_a^h(x, Q^2) = \int_x^1 \frac{dz}{z} P_{ba}^T(z, \alpha_s(Q^2))\, D_b^h\left(\frac{x}{z}, Q^2\right) .$$

T-like and S-like cases related by Analytic cont. [Blümlein,Ravindran,vanNeerven(2000)]
$P^{(0)T}(x)$ identical to their space-like couterparts [Gribov,Lipatov(1972),...]
$P^{(1)T}(x)$ [Curci,Furmanski,Petronzio(80), Floratos,Kounnas,Lacaze(81), Stratmann,Vogelsang(97),...]
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Soft Gluon Exponentiation (SGE): resums dominant $(1-x)_+^{-1}$ large-x contributions to $c_{T,q}(x,\alpha_s)$ and $c^T_{\phi,g}(x,\alpha_s)$: NNNLL, 7 logs [Moch,Vogt(2009)].

Recent studies address also resummation for $(1-x)^0$ terms with SGE [Grunberg(07),Laenen,Magnea,Stavenga(08), Grunberg,Ravindran(09), Laenen,Stavenga,White(09)]

Physical Kernel methods allow resummation of the highest three $(1-x)^0$ logarithms (flavour non-singlet case) [Moch,Vogt(2009)]

Our approach: functional form together with KLN

All-order results for the highest three large-x logarithms of time-like splitting and coefficient functions in Higgs- (in heavy top limit, with eff. $\phi G_{\mu\nu} G^{\mu\nu}$ coupling) and gauge-boson exchange SIA are presented.

These results have been derived by studying the unfactorised partonic fragmentation function in terms of constraints imposed by the functional forms together with their Kinoshita-Lee-Nauenberg (KLN) cancellations required by the *mass factorisation* theorem.

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Introduction: SIA in the large-x limit

We look at the large-x structure of the time-like coefficient funcion

$$c^T_{a,k}(x,\alpha_s) = \sum_{n=1}^{\infty} a_s^n c^{T(n)}_{a,k}(x) \qquad \text{with} \qquad a_s \equiv \frac{\alpha_s}{4\pi}$$

'Off-diagonal' coeff's fnct's: double-log higher-order enhancement as $x \to 1$

$$c^{T(n)}_{a,k}(x) \;=\; \sum_{l=0}^{2n-2} D^{T(n,l)}_{a,k} \ln^{2n-1-l}(1-x) + \mathcal{O}(1) \quad \textit{for} \quad a,\, k \,=\, T,\, g \;\; \textit{or} \;\; \phi,\, q$$

$$c^{T(n)}_{L,k}(x) \;=\; (1-x)^{\delta_{kg}} \left(\sum_{l=0}^{2n-3} D^{T(n,l)}_{a,k} \ln^{2n-2-l}(1-x) + \mathcal{O}(1) \right)$$

Writing the expansion of the time-like splitting functions

$$P^T_{ik}(x,\alpha_s) = \sum_{n=0}^{\infty} a_s^{n+1}\, P^{T(n)}_{ik}(x)$$

Diagonal splitting functions (in MSbar) stable under higher-order corrections

$$P^{T(n-1)}_{kk}(x) = A^{T(n)}_k (1-x)^{-1}_+ + B^{T(n)}_k \delta(1-x) + C^{T(n)}_k \ln(1-x) + \mathcal{O}(1)$$

[Korchemsky(89), Moch,Vermaseren,Vogt(04), Dokshitzer,Marchesini,Salam(05)]

Off-diagonal splitting functions show double-logarithmic enhancement

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Threshold logarithms before factorisation (I)

Unfactorized partonic structure functions in $D = 4 - 2\epsilon$ dimensions

$$T_{a,j} = \tilde{C}_{a,i} Z_{ij}, \quad -\gamma \equiv P = \frac{dZ}{d\ln Q^2} Z^{-1}, \quad \frac{da_s}{d\ln Q^2} = -\epsilon a_s + \beta_{D=4}$$

a_s^n : $\epsilon^{-n} \dots \epsilon^{-2}$: lower-order terms, ϵ^{-1}: n-loop splitting functions$+ \dots$,

ϵ^0: n-loop coefficient fct's $+ \dots$, ϵ^k, $0 < k < l$: required for order $n + l$

N^0 and N^{-1} transition functions Z to next-to-leading log (NLL) accuracy

$$Z\Big|_{a_s^n} = \frac{1}{\epsilon^n}\frac{\gamma_0^{n-1}}{n!}\left[\gamma_0 - \frac{\beta_0}{2} n(n-1)\right] + \sum_{l=1}^{n-1} \frac{1}{\epsilon^{n-l}} \sum_{k=1}^{n-l-1} \gamma_0^{n-l-k-1} \gamma_l \gamma_0^k \frac{(l+k)!}{n!\,l!}$$
$$- \frac{\beta_0}{2} \sum_{l=1}^{n-2} \frac{1}{\epsilon^{n-l}} \sum_{k=1}^{n-l-2} \gamma_0^{n-l-k-1} \gamma_l \gamma_0^k \frac{(l+k)!}{n!\,l!} (n(n-1) - l(l+k+1))$$
$$+ \text{NNLL contributions (explicit expressions)} + \dots$$

ϵ^{-n+l} off-diagonal entries: contributions up to $N^{-1} \ln^{n+l-1} N$

Diagonal cases: γ_0 only for N^0 part, second term with $l = 1$ for N^{-1} NLL

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$$Z\Big|_{a_s^n} = \frac{1}{\epsilon^n}\frac{\gamma_0^{n-1}}{n!}\left[\gamma_0 - \frac{\beta_0}{2}n(n-1)\right] + \sum_{l=1}^{n-1}\frac{1}{\epsilon^{n-l}}\sum_{k=1}^{n-l-1}\gamma_0^{n-l-k-1}\gamma_l\gamma_0^k\frac{(l+k)!}{n!\,l!}$$

$$-\frac{\beta_0}{2}\sum_{l=1}^{n-2}\frac{1}{\epsilon^{n-l}}\sum_{k=1}^{n-l-2}\gamma_0^{n-l-k-1}\gamma_l\gamma_0^k\frac{(l+k)!}{n!\,l!}(n(n-1) - l(l+k+1))$$

$+$NNLL contributions (explicit expressions) $+ \dots$

ϵ^{-n+l} off-diagonal entries: contributions up to $N^{-1} \ln^{n+l-1} N$

Diagonal cases: γ_0 only for N^0 part, second term with $l = 1$ for N^{-1} NLL

Threshold logarithms before factorisation (I)

Unfactorized partonic structure functions in $D = 4 - 2\epsilon$ dimensions

$$T_{a,j} = \tilde{C}_{a,i} Z_{ij}, \quad -\gamma \equiv P = \frac{dZ}{d \ln Q^2} Z^{-1}, \quad \frac{da_s}{d \ln Q^2} = -\epsilon a_s + \beta_{D=4}$$

a_s^n : $\epsilon^{-n} \ldots \epsilon^{-2}$: lower-order terms, ϵ^{-1}: n-loop splitting functions$+ \ldots$,

ϵ^0: n-loop coefficient fct's $+ \ldots$, ϵ^k, $0 < k < l$: required for order $n + l$

N^0 and N^{-1} transition functions Z to next-to-leading log (NLL) accuracy

$$Z\Big|_{a_s^n} = \frac{1}{\epsilon^n}\frac{\gamma_0^{n-1}}{n!}\left[\gamma_0 - \frac{\beta_0}{2} n(n-1)\right] + \sum_{l=1}^{n-1} \frac{1}{\epsilon^{n-l}} \sum_{k=1}^{n-l-1} \gamma_0^{n-l-k-1} \gamma_l \gamma_0^k \frac{(l+k)!}{n!l!}$$
$$- \frac{\beta_0}{2} \sum_{l=1}^{n-2} \frac{1}{\epsilon^{n-l}} \sum_{k=1}^{n-l-2} \gamma_0^{n-l-k-1} \gamma_l \gamma_0^k \frac{(l+k)!}{n!l!} (n(n-1) - l(l+k+1))$$
$$+\text{NNLL contributions (explicit expressions)} + \ldots$$

ϵ^{-n+l} off-diagonal entries: contributions up to $N^{-1} \ln^{n+l-1} N$

Diagonal cases: γ_0 only for N^0 part, second term with $l = 1$ for N^{-1} NLL

Threshold logarithms before factorisation (II)

D-dimensional coefficient functions $\tilde{C}_a$: finite for $\epsilon \to 0$

$$\tilde{C}^T_{a,i} = 1_{\text{diagonal case}} + \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} a_s^n \, \epsilon^l \, c_{a,i}^{T(n,l)}$$

$c_{a,i}^{(n,l)}$: l additional factors $\ln N$ relative to $c_{a,i}^{(n,0)} \equiv c_{a,i}^{(n)}$ discussed above.

Full N^{m}LO calc. of $T_{a,j}$: highest $m+1$ powers of ϵ^{-1} to all orders in α_s

$$\begin{aligned}
T^{T(1)} &= -\frac{1}{\epsilon} P^{T(0)} + c^{T(1,0)} + \epsilon c^{T(1,1)} + \epsilon^2 c^{T(1,2)} + \epsilon^3 c^{T(1,3)} \\
T^{T(2)} &= \frac{1}{2\epsilon^2} P^{T(0)}(P^{T(0)} + \beta_0) - \frac{1}{2\epsilon}\left[P^{T(1)} + 2P^{T(0)} c^{T(1,0)}\right] + c^{T(2,0)} - P^{T(0)} c^{T(1,1)} \\
&\quad + \epsilon \left[c^{T(2,1)} - P^{T(0)} c^{T(1,2)}\right] + \ldots \\
T^{T(3)} &= -\frac{1}{6\epsilon^3} P^{T(0)}(P^{T(0)} + \beta_0)(P^{T(0)} + 2\beta_0) \\
&\quad + \frac{1}{6\epsilon^2}\left[P^{T(1)}(3P^{T(0)} + 2\beta_0) + P^{T(0)}(3P^{T(0)} c^{T(1,0)} + 3\beta_0 c^{T(1,0)} + 2\beta_1)\right] \\
&\quad - \frac{1}{6\epsilon}\left[2P^{T(2)} + 3P^{T(1)} c^{T(1,0)} + P^{(0)}(6c^{T(2,0)} - 3P^{T(0)} c^{T(1,1)} - 3\beta_0 c^{T(1,1)})\right] \\
&\quad + c^{T(3,0)} - \frac{1}{2} P^{T(1)} c^{T(1,1)} - P^{T(0)} c^{T(2,1)} + \frac{1}{2} P^{T(0)}\left(P^{T(0)} + \beta_0\right) c^{T(1,2)} + \ldots
\end{aligned}$$

Extension to all powers of ϵ: all-order resummation of highest $m+1$ logs.

Threshold logarithms before factorisation (II)

D-dimensional coefficient functions $\tilde{C}_a$: finite for $\epsilon \to 0$

$$\tilde{C}^T_{a,i} = 1_{\text{diagonal case}} + \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} a_s^n \, \epsilon^l \, c^{T(n,l)}_{a,i}$$

$c^{(n,l)}_{a,i}$: l additional factors $\ln N$ relative to $c^{(n,0)}_{a,i} \equiv c^{(n)}_{a,i}$ discussed above.

Full N^{m}LO calc. of $T_{a,j}$: highest $m+1$ powers of ϵ^{-1} to all orders in α_s

$$
\begin{aligned}
T^{T(1)} &= -\frac{1}{\epsilon} P^{T(0)} + c^{T(1,0)} + \epsilon c^{T(1,1)} + \epsilon^2 c^{T(1,2)} + \epsilon^3 c^{T(1,3)} \\
T^{T(2)} &= \frac{1}{2\epsilon^2} P^{T(0)}(P^{T(0)} + \beta_0) - \frac{1}{2\epsilon}\left[P^{T(1)} + 2P^{T(0)} c^{T(1,0)}\right] + c^{T(2,0)} - P^{T(0)} c^{T(1,1)} \\
&\quad + \epsilon \left[c^{T(2,1)} - P^{T(0)} c^{T(1,2)}\right] + \ldots \\
T^{T(3)} &= -\frac{1}{6\epsilon^3} P^{T(0)}(P^{T(0)} + \beta_0)(P^{T(0)} + 2\beta_0) \\
&\quad + \frac{1}{6\epsilon^2}\left[P^{T(1)}(3P^{T(0)} + 2\beta_0) + P^{T(0)}(3P^{T(0)} c^{T(1,0)} + 3\beta_0 c^{T(1,0)} + 2\beta_1)\right] \\
&\quad - \frac{1}{6\epsilon}\left[2P^{T(2)} + 3P^{T(1)} c^{T(1,0)} + P^{(0)}(6c^{T(2,0)} - 3P^{T(0)} c^{T(1,1)} - 3\beta_0 c^{T(1,1)})\right] \\
&\quad + c^{T(3,0)} - \frac{1}{2} P^{T(1)} c^{T(1,1)} - P^{T(0)} c^{T(2,1)} + \frac{1}{2} P^{T(0)} \left(P^{T(0)} + \beta_0\right) c^{T(1,2)} + \ldots
\end{aligned}
$$

Extension to all powers of ϵ: all-order resummation of highest $m+1$ logs.

Threshold logarithms before factorisation (II)

D-dimensional coefficient functions $\tilde{C}_a$: finite for $\epsilon \to 0$

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$c_{a,i}^{(n,l)}$: l additional factors $\ln N$ relative to $c_{a,i}^{(n,0)} \equiv c_{a,i}^{(n)}$ discussed above.

Full N^{m}LO calc. of $T_{a,j}$: highest $m+1$ powers of ϵ^{-1} to all orders in α_s

$$\begin{aligned}
T^{T(1)} &= -\frac{1}{\epsilon} P^{T(0)} + c^{T(1,0)} + \epsilon c^{T(1,1)} + \epsilon^2 c^{T(1,2)} + \epsilon^3 c^{T(1,3)} \\
T^{T(2)} &= \frac{1}{2\epsilon^2} P^{T(0)}(P^{T(0)} + \beta_0) - \frac{1}{2\epsilon}\left[P^{T(1)} + 2P^{T(0)} c^{T(1,0)}\right] + c^{T(2,0)} - P^{T(0)} c^{T(1,1)} \\
&\quad + \epsilon \left[c^{T(2,1)} - P^{T(0)} c^{T(1,2)}\right] + \ldots \\
T^{T(3)} &= -\frac{1}{6\epsilon^3} P^{T(0)}(P^{T(0)} + \beta_0)(P^{T(0)} + 2\beta_0) \\
&\quad + \frac{1}{6\epsilon^2}\left[P^{T(1)}(3P^{T(0)} + 2\beta_0) + P^{T(0)}(3P^{T(0)} c^{T(1,0)} + 3\beta_0 c^{T(1,0)} + 2\beta_1)\right] \\
&\quad - \frac{1}{6\epsilon}\left[2P^{T(2)} + 3P^{T(1)} c^{T(1,0)} + P^{(0)}(6c^{T(2,0)} - 3P^{T(0)} c^{T(1,1)} - 3\beta_0 c^{T(1,1)})\right] \\
&\quad + c^{T(3,0)} - \frac{1}{2} P^{T(1)} c^{T(1,1)} - P^{T(0)} c^{T(2,1)} + \frac{1}{2} P^{T(0)}\left(P^{T(0)} + \beta_0\right) c^{T(1,2)} + \ldots
\end{aligned}$$

Extension to all powers of ϵ: all-order resummation of highest $m+1$ logs.

Threshold logarithms before factorisation (III)

The main part of our calculations is performed in Mellin-N space

$$f(N) = \int_0^1 dx\, x^{N-1}\, f(x) \qquad or \qquad f(N) = \int_0^1 dx\, \left(x^{N-1} - 1\right)\, f(x)_+$$

Large-x logarithms correspond to large-N logs after Mellin transform

$$\left(\frac{\ln^n(1-x)}{1-x}\right)_+ \stackrel{M}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \ldots \qquad (1-x) \ln^n(1-x) \stackrel{M}{=} \frac{(-1)^n}{N^2} \ln^n N + \ldots$$

Large-N logarithmic behaviour of coeff's and transition functions

$$c^{(n,l)}_{T,q},\ c^{T(n,l)}_{\phi,g} \sim \ln^{2n+l} N + \ldots \qquad c^{(n,l)}_{T,g},\ c^{T(n,l)}_{\phi,q} \sim \frac{1}{N} \ln^{2n-1+l} N + \ldots$$

$$c^{T(n,l)}_{L,q} \sim \frac{1}{N} \ln^{2n-2+l} N + \ldots \qquad c^{T(n,l)}_{L,g} \sim \frac{1}{N^2} \ln^{2n-2+l} N + \ldots$$

$$Z_{kk}\big|_{a_s^n} \sim \epsilon^{-n} \ln^n N + \ldots \qquad Z_{i\neq j}\big|_{a_s^n} \sim \epsilon^{-n} \frac{1}{N} \ln^{n-1} N + \ldots$$

Structure of the unfactorised amplitudes

$$T_{T,q} \simeq c_{T,q}\, Z_{qq} \sim \mathcal{O}(1) \quad \to \quad T_{T,g} = c_{T,q}\, Z_{qg} + c_{T,g}\, Z_{gg} \quad \sim \quad \mathcal{O}(1/N)$$

$$T^T_{\phi,g} \simeq c_{\phi,g}\, Z_{gg} \sim \mathcal{O}(1) \quad \to \quad T^T_{\phi,q} = c^T_{\phi,g}\, Z_{gq} + c^T_{\phi,q}\, Z_{qq} \quad \sim \quad \mathcal{O}(1/N)$$

$$T^T_{L,q} \simeq c^T_{L,q}\, Z_{qq} \sim \mathcal{O}(1/N) \quad \to \quad T^T_{L,g} = c^T_{L,q}\, Z_{qg} + c^T_{L,g}\, Z_{gg} \quad \sim \quad \mathcal{O}(1/N^2)$$

Threshold logarithms before factorisation (III)

The main part of our calculations is performed in Mellin-N space

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$$\left(\frac{\ln^n(1-x)}{1-x}\right)_+ \overset{M}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \ldots \qquad (1-x) \ln^n(1-x) \overset{M}{=} \frac{(-1)^n}{N^2} \ln^n N + \ldots$$

Large-N logarithmic behaviour of coeff's and transition functions

$$c_{T,q}^{(n,l)},\ c_{\phi,g}^{T(n,l)} \sim \ln^{2n+l} N + \ldots \qquad c_{T,g}^{(n,l)},\ c_{\phi,q}^{T(n,l)} \sim \frac{1}{N} \ln^{2n-1+l} N + \ldots$$

$$c_{L,q}^{T(n,l)} \sim \frac{1}{N} \ln^{2n-2+l} N + \ldots \qquad c_{L,g}^{T(n,l)} \sim \frac{1}{N^2} \ln^{2n-2+l} N + \ldots$$

$$Z_{kk}\big|_{a_s^n} \sim \epsilon^{-n} \ln^n N + \ldots \qquad Z_{i\neq j}\big|_{a_s^n} \sim \epsilon^{-n} \frac{1}{N} \ln^{n-1} N + \ldots$$

Structure of the unfactorised amplitudes

$$T_{T,q} \simeq c_{T,q}\, Z_{qq} \sim \mathcal{O}(1) \quad \rightarrow \quad T_{T,g} = c_{T,q}\, Z_{qg} + c_{T,g}\, Z_{gg} \sim \mathcal{O}(1/N)$$

$$T_{\phi,g}^T \simeq c_{\phi,g}\, Z_{gg} \sim \mathcal{O}(1) \quad \rightarrow \quad T_{\phi,q}^T = c_{\phi,g}^T\, Z_{gq} + c_{\phi,q}^T\, Z_{qq} \sim \mathcal{O}(1/N)$$

$$T_{L,q}^T \simeq c_{L,q}^T\, Z_{qq} \sim \mathcal{O}(1/N) \quad \rightarrow \quad T_{L,g}^T = c_{L,q}^T\, Z_{qg} + c_{L,g}^T\, Z_{gg} \sim \mathcal{O}(1/N^2)$$

Threshold logarithms before factorisation (III)

The main part of our calculations is performed in Mellin-N space

$$f(N) = \int_0^1 dx\, x^{N-1}\, f(x) \qquad or \qquad f(N) = \int_0^1 dx\, \left(x^{N-1} - 1\right)\, f(x)_+$$

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Large-N logarithmic behaviour of coeff's and transition functions

$$c_{T,q}^{(n,l)},\ c_{\phi,g}^{T(n,l)} \sim \ln^{2n+l} N + \ldots \qquad c_{T,g}^{(n,l)},\ c_{\phi,q}^{T(n,l)} \sim \frac{1}{N} \ln^{2n-1+l} N + \ldots$$

$$c_{L,q}^{T(n,l)} \sim \frac{1}{N} \ln^{2n-2+l} N + \ldots \qquad c_{L,g}^{T(n,l)} \sim \frac{1}{N^2} \ln^{2n-2+l} N + \ldots$$

$$Z_{kk}\big|_{a_s^n} \sim \epsilon^{-n} \ln^n N + \ldots \qquad Z_{i\neq j}\big|_{a_s^n} \sim \epsilon^{-n} \frac{1}{N} \ln^{n-1} N + \ldots$$

Structure of the unfactorised amplitudes

$$T_{T,q} \simeq c_{T,q} Z_{qq} \sim \mathcal{O}(1) \quad \to \quad T_{T,g} = c_{T,q} Z_{qg} + c_{T,g} Z_{gg} \quad \sim \quad \mathcal{O}(1/N)$$

$$T_{\phi,g}^T \simeq c_{\phi,g} Z_{gg} \sim \mathcal{O}(1) \quad \to \quad T_{\phi,q}^T = c_{\phi,g}^T Z_{gq} + c_{\phi,q}^T Z_{qq} \quad \sim \quad \mathcal{O}(1/N)$$

$$T_{L,q}^T \simeq c_{L,q}^T Z_{qq} \sim \mathcal{O}(1/N) \quad \to \quad T_{L,g}^T = c_{L,q}^T Z_{qg} + c_{L,g}^T Z_{gg} \quad \sim \quad \mathcal{O}(1/N^2)$$

Threshold logarithms before factorisation (III)

The main part of our calculations is performed in Mellin-N space

$$f(N) = \int_0^1 dx\, x^{N-1} f(x) \qquad or \qquad f(N) = \int_0^1 dx\, \left(x^{N-1} - 1\right)\, f(x)_+$$

Large-x logarithms correspond to large-N logs after Mellin transform

$$\left(\frac{\ln^n(1-x)}{1-x}\right)_+ \stackrel{M}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \ldots \qquad (1-x) \ln^n(1-x) \stackrel{M}{=} \frac{(-1)^n}{N^2} \ln^n N + \ldots$$

Large-N logarithmic behaviour of coeff's and transition functions

$$c^{(n,l)}_{T,q}\,,\ c^{T(n,l)}_{\phi,g} \sim \ln^{2n+l} N + \ldots \qquad c^{(n,l)}_{T,g}\,,\ c^{T(n,l)}_{\phi,q} \sim \frac{1}{N} \ln^{2n-1+l} N + \ldots$$

$$c^{T(n,l)}_{L,q} \sim \frac{1}{N} \ln^{2n-2+l} N + \ldots \qquad c^{T(n,l)}_{L,g} \sim \frac{1}{N^2} \ln^{2n-2+l} N + \ldots$$

$$Z_{kk}\Big|_{a_s^n} \sim \epsilon^{-n} \ln^n N + \ldots \qquad Z_{i\neq j}\Big|_{a_s^n} \sim \epsilon^{-n} \frac{1}{N} \ln^{n-1} N + \ldots$$

Structure of the unfactorised amplitudes

$$T_{T,q} \simeq c_{T,q} Z_{qq} \sim \mathcal{O}(1) \quad \to \quad T_{T,g} = c_{T,q} Z_{qg} + c_{T,g} Z_{gg} \sim \mathcal{O}(1/N)$$

$$T^T_{\phi,g} \simeq c_{\phi,g} Z_{gg} \sim \mathcal{O}(1) \quad \to \quad T^T_{\phi,q} = c^T_{\phi,g} Z_{gq} + c^T_{\phi,q} Z_{qq} \sim \mathcal{O}(1/N)$$

$$T^T_{L,q} \simeq c^T_{L,q} Z_{qq} \sim \mathcal{O}(1/N) \quad \to \quad T^T_{L,g} = c^T_{L,q} Z_{qg} + c^T_{L,g} Z_{gg} \sim \mathcal{O}(1/N^2)$$

Threshold logarithms before factorisation (III)

The main part of our calculations is performed in Mellin-N space

$$f(N) = \int_0^1 dx\, x^{N-1}\, f(x) \qquad or \qquad f(N) = \int_0^1 dx\, \left(x^{N-1} - 1\right)\, f(x)_+$$

Large-x logarithms correspond to large-N logs after Mellin transform

$$\left(\frac{\ln^n(1-x)}{1-x}\right)_+ \overset{M}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \dots \qquad (1-x)\ln^n(1-x) \overset{M}{=} \frac{(-1)^n}{N^2} \ln^n N + \dots$$

Large-N logarithmic behaviour of coeff's and transition functions

$$c^{(n,l)}_{T,q},\ c^{T(n,l)}_{\phi,g} \sim \ln^{2n+l} N + \dots \qquad c^{(n,l)}_{T,g},\ c^{T(n,l)}_{\phi,q} \sim \frac{1}{N}\ln^{2n-1+l} N + \dots$$

$$c^{T(n,l)}_{L,q} \sim \frac{1}{N}\ln^{2n-2+l} N + \dots \qquad c^{T(n,l)}_{L,g} \sim \frac{1}{N^2}\ln^{2n-2+l} N + \dots$$

$$Z_{kk}\big|_{a_s^n} \sim \epsilon^{-n} \ln^n N + \dots \qquad Z_{i\neq j}\big|_{a_s^n} \sim \epsilon^{-n}\frac{1}{N}\ln^{n-1} N + \dots$$

Structure of the unfactorised amplitudes

$$T_{T,q} \simeq c_{T,q}\, Z_{qq} \sim \mathcal{O}(1) \quad \to \quad T_{T,g} = c_{T,q}\, Z_{qg} + c_{T,g}\, Z_{gg} \quad \sim \quad \mathcal{O}(1/N)$$

$$T^T_{\phi,g} \simeq c_{\phi,g}\, Z_{gg} \sim \mathcal{O}(1) \quad \to \quad T^T_{\phi,q} = c^T_{\phi,g}\, Z_{gq} + c^T_{\phi,q}\, Z_{qq} \quad \sim \quad \mathcal{O}(1/N)$$

$$T^T_{L,q} \simeq c^T_{L,q}\, Z_{qq} \sim \mathcal{O}(1/N) \quad \to \quad T^T_{L,g} = c^T_{L,q}\, Z_{qg} + c^T_{L,g}\, Z_{gg} \quad \sim \quad \mathcal{O}(1/N^2)$$

D-dim. structure of unfactorised observables

Maximal phase space SIA:

$$\text{NLO}: 1 \to 2+1 \qquad (1-x)^{-1-\epsilon} x^{\cdots} \int_0^1 \text{one other variable}$$

$$\text{N}^2\text{LO}: 1 \to 2+2 \qquad (1-x)^{-1-2\epsilon} x^{\cdots} \int_0^1 \text{four other variables}$$

$$\text{N}^3\text{LO}: 1 \to 2+3 \qquad (1-x)^{-1-3\epsilon} x^{\cdots} \int_0^1 \text{seven other variables}$$

$\ldots$

Purely real contributions: no additional factors $(1-x)^{-\epsilon}$ from integral

$$T_{a,j}^{(n)R} = (1-x)^{-1-n\epsilon} \sum_{\xi=0} (1-x)^{\xi} \frac{1}{\epsilon^{2n-1}} \left\{ R_{a,j,\xi}^{(n)\text{LL}} + \epsilon R_{a,j,\xi}^{(n)\text{NLL}} + \ldots \right\}$$

Mixed contributions $(1 \to r+2)$: $n-r$ additional factors $(1-x)^{-\epsilon}$

$$T_{a,j}^{(n)M} = \sum_{l=r}^{n} (1-x)^{-1-l\epsilon} \sum_{\xi=0} (1-x)^{\xi} \frac{1}{\epsilon^{2n-1}} \left\{ M_{a,j,\xi}^{(n)\text{LL}} + \epsilon M_{a,j,\xi}^{(n)\text{NLL}} + \ldots \right\}$$

Purely virtual part (in diagonal cases, $\xi = 0$): $\gamma^* qq$, Hgg form factors

$$T_{a,j}^{(n)V} = \delta(1-x) \frac{1}{\epsilon^{2n}} \left\{ V_{a,j,\xi}^{(n)\text{LL}} + \epsilon V_{a,j,\xi}^{(n)\text{NLL}} + \ldots \right\}$$

D-dim. structure of unfactorised observables

Maximal phase space SIA:

$$\mathrm{NLO}: \; 1 \to 2 + 1 \qquad (1-x)^{-1-\epsilon}\, x^{\cdots} \int_0^1 \text{one other variable}$$

$$\mathrm{N^2LO}: 1 \to 2 + 2 \qquad (1-x)^{-1-2\epsilon}\, x^{\cdots} \int_0^1 \text{four other variables}$$

$$\mathrm{N^3LO}: 1 \to 2 + 3 \qquad (1-x)^{-1-3\epsilon}\, x^{\cdots} \int_0^1 \text{seven other variables}$$

...

Purely real contributions: no additional factors $(1-x)^{-\epsilon}$ from integral

$$T_{a,j}^{(n)R} = (1-x)^{-1-n\epsilon} \sum_{\xi=0} (1-x)^{\xi} \frac{1}{\epsilon^{2n-1}} \left\{ R_{a,j,\xi}^{(n)\mathrm{LL}} + \epsilon R_{a,j,\xi}^{(n)\mathrm{NLL}} + \dots \right\}$$

Mixed contributions $(1 \to r+2)$: $n-r$ additional factors $(1-x)^{-\epsilon}$

$$T_{a,j}^{(n)M} = \sum_{l=r}^{n} (1-x)^{-1-l\epsilon} \sum_{\xi=0} (1-x)^{\xi} \frac{1}{\epsilon^{2n-1}} \left\{ M_{a,j,\xi}^{(n)\mathrm{LL}} + \epsilon M_{a,j,\xi}^{(n)\mathrm{NLL}} + \dots \right\}$$

Purely virtual part (in diagonal cases, $\xi = 0$): $\gamma^* qq$, Hgg form factors

$$T_{a,j}^{(n)V} = \delta(1-x) \frac{1}{\epsilon^{2n}} \left\{ V_{a,j,\xi}^{(n)\mathrm{LL}} + \epsilon V_{a,j,\xi}^{(n)\mathrm{NLL}} + \dots \right\}$$

D-dim. structure of unfactorised observables

Maximal phase space SIA:

$$\mathrm{NLO}: \ 1 \to 2+1 \qquad (1-x)^{-1-\epsilon}\, x^{\cdots} \int_0^1 \text{one other variable}$$

$$\mathrm{N^2LO}: 1 \to 2+2 \qquad (1-x)^{-1-2\epsilon}\, x^{\cdots} \int_0^1 \text{four other variables}$$

$$\mathrm{N^3LO}: 1 \to 2+3 \qquad (1-x)^{-1-3\epsilon}\, x^{\cdots} \int_0^1 \text{seven other variables}$$

...

Purely real contributions: no additional factors $(1-x)^{-\epsilon}$ from integral

$$T_{a,j}^{(n)R} = (1-x)^{-1-n\epsilon} \sum_{\xi=0} (1-x)^{\xi} \frac{1}{\epsilon^{2n-1}} \left\{ R_{a,j,\xi}^{(n)\mathrm{LL}} + \epsilon R_{a,j,\xi}^{(n)\mathrm{NLL}} + \dots \right\}$$

Mixed contributions $(1 \to r+2)$: $n-r$ additional factors $(1-x)^{-\epsilon}$

$$T_{a,j}^{(n)M} = \sum_{l=r}^{n} (1-x)^{-1-l\epsilon} \sum_{\xi=0} (1-x)^{\xi} \frac{1}{\epsilon^{2n-1}} \left\{ M_{a,j,\xi}^{(n)\mathrm{LL}} + \epsilon M_{a,j,\xi}^{(n)\mathrm{NLL}} + \dots \right\}$$

Purely virtual part (in diagonal cases, $\xi = 0$): $\gamma^* qq$, Hgg form factors

$$T_{a,j}^{(n)V} = \delta(1-x) \frac{1}{\epsilon^{2n}} \left\{ V_{a,j,\xi}^{(n)\mathrm{LL}} + \epsilon V_{a,j,\xi}^{(n)\mathrm{NLL}} + \dots \right\}$$

D-dim. structure of unfactorised observables

Maximal phase space SIA:

$$\mathrm{NLO}: \; 1 \to 2 + 1 \qquad (1-x)^{-1-\epsilon}\, x^{\cdots} \int_0^1 \text{one other variable}$$

$$\mathrm{N^2LO}: 1 \to 2 + 2 \qquad (1-x)^{-1-2\epsilon}\, x^{\cdots} \int_0^1 \text{four other variables}$$

$$\mathrm{N^3LO}: 1 \to 2 + 3 \qquad (1-x)^{-1-3\epsilon}\, x^{\cdots} \int_0^1 \text{seven other variables}$$

$\dots$

Purely real contributions: no additional factors $(1-x)^{-\epsilon}$ from integral

$$T^{(n)R}_{a,j} = (1-x)^{-1-n\epsilon} \sum_{\xi=0} (1-x)^{\xi} \frac{1}{\epsilon^{2n-1}} \left\{ R^{(n)\mathrm{LL}}_{a,j,\xi} + \epsilon R^{(n)\mathrm{NLL}}_{a,j,\xi} + \dots \right\}$$

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Resulting resummation of large-x double logs

KLN cancellation between purely real, mixed and purely virtual contributions

$$T_{a,j}^{(n)} = T_{a,j}^{(n)R} + T_{a,j}^{(n)M} + \left(T_{a,j}^{(n)V}\right)_{\rm diag} = \frac{1}{\epsilon^n}\left\{T_{a,j}^{(n)0} + \epsilon\, T_{a,j}^{(n)1} + \epsilon^2\, T_{a,j}^{(n)2} + \dots\right\}$$

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Log expansion: N^{k}LL higher-order coefficients completely fixed, if first $k+1$ powers of ϵ known to all orders - provided by N^{k}LO calculations.

Present situation: (a) N^{2}LO for $F^T_{a\neq L}$ (b) NLO for F^T_L

$\Rightarrow$ resummation of the (a) three and (b) two highest $N^{-1}\ln^k N$ terms to all orders in α_s: consistent with, and extending, MV Physical Kernel results

In Mellin-N space one can rewrite the (off-diag) unfactorised amplitudes like

$$T_{a,k}^{T(n)}(N) = \frac{1}{N\,\epsilon^{2n-1}}\sum_{i=0}^{n-1}\left(A_{a,k}^{T(n,i)} + \epsilon B_{a,k}^{T(n,i)} + \epsilon^2 C_{a,k}^{T(n,i)} + \dots\right)\exp(\epsilon\,(n-i)\ln N)$$

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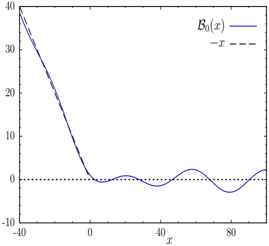
All order expressions: new functions involving Bernoulli numbers [LL:Vogt('10)].

Relation between even-n Bernoulli numbers and the Riemann ζ-function

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{[(2n)!]^2} |B_{2n}| x^{2n} = 1 - \frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \zeta_{2n} \left(\frac{x}{2\pi}\right)^{2n}$$

$\mathcal{B}_0$ to appear for the LL result; further $\mathcal{B}$-functions for NLL and NNLL results.



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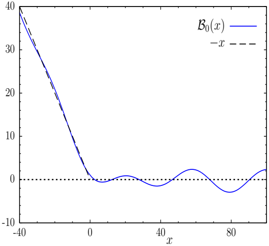

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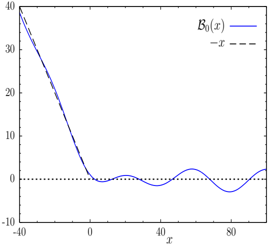

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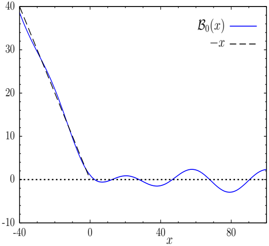

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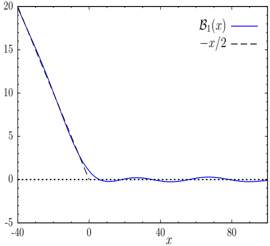

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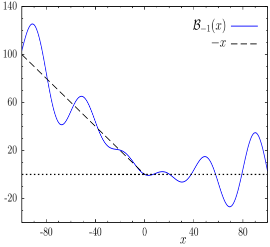

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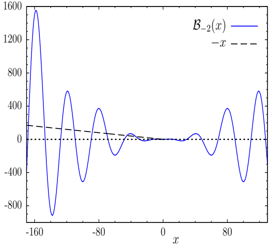

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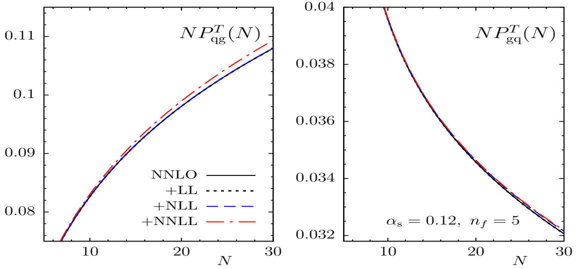
NNLL resummation of the off-diagonal splitting functions

$$\tilde{N} \equiv N e^{\gamma_e} \quad \text{and} \quad \tilde{a}_s \equiv 4 a_s (C_A - C_F) \ln^2 \tilde{N}$$

$$\begin{aligned} N\, P^T_{qg}(N, \alpha_s) \;=\;& 2 a_s n_f \mathcal{B}_0(\tilde{a}_s) \\ & + a_s^2 \ln \tilde{N}\, n_f \left[(-12 C_F + 6\beta_0) \frac{1}{\tilde{a}_s} \mathcal{B}_{-1}(\tilde{a}_s) + \frac{\beta_0}{\tilde{a}_s} \mathcal{B}_{-2}(\tilde{a}_s) + (6 C_F - \beta_0) \mathcal{B}_1(\tilde{a}_s) \right] \\ & + \text{known NNLL contributions (tables)} + \ldots \end{aligned}$$

$$\begin{aligned} N\, P^T_{gq}(N, \alpha_s) \;=\;& 2 a_s C_F \mathcal{B}_0(-\tilde{a}_s) \\ & + a_s^2 \ln \tilde{N}\, C_F \left[(-12 C_F + 2\beta_0) \frac{1}{\tilde{a}_s} \mathcal{B}_{-1}(-\tilde{a}_s) + \frac{\beta_0}{\tilde{a}_s} \mathcal{B}_{-2}(-\tilde{a}_s) + (8 C_F - 2 C_A - \beta_0) \mathcal{B}_1(-\tilde{a}_s) \right] \\ & + \text{known NNLL contributions (tables)} + \ldots \end{aligned}$$



NNLL resummation of the off-diagonal splitting functions

$$\tilde{N} \equiv N e^{\gamma_e} \quad \text{and} \quad \tilde{a}_s \equiv 4 a_s (C_A - C_F) \ln^2 \tilde{N}$$

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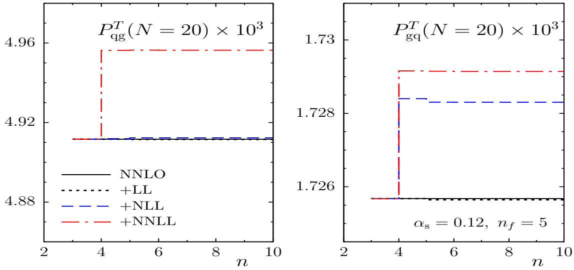
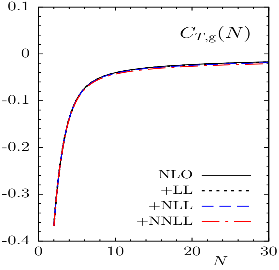
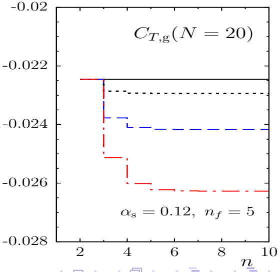


NNLL resummation of the coefficient functions

$$
\begin{aligned}
N\,C_{T,g}(N,\alpha_s) = &\ \frac{1}{2\ln\tilde N}\frac{C_F}{C_A-C_F}\Big[\exp(2a_sC_F\ln^2\tilde N)\mathcal{B}_0(\tilde a_s)-\exp(2a_sC_A\ln^2\tilde N)\Big]\\
&-\frac{1}{8\ln^2\tilde N}\frac{C_F(3C_F-b_0)}{(C_A-C_F)^2}\Big[\exp(2a_sC_F\ln^2\tilde N)\mathcal{B}_0(\tilde a_s)-\exp(2a_sC_A\ln^2\tilde N)\Big]\\
&-\frac{a_s}{4}\frac{C_F}{C_A-C_F}\exp(2a_sC_F\ln^2\tilde N)(8C_A+4C_F-\beta_0)\\
&-\frac{a_s}{4}\frac{n_f}{C_A-C_F}\exp(2a_sC_F\ln^2\tilde N)\Big[-6C_F\mathcal{B}_0(\tilde a_s)-(8C_A-2C_F-\beta_0)\mathcal{B}_1(\tilde a_s)\\
&\quad-(12C_F-4\beta_0)\frac{1}{\tilde a_s}\mathcal{B}_{-1}(\tilde a_s)-\frac{\beta_0}{\tilde a_s}\mathcal{B}_{-2}(\tilde a_s)\Big]\\
&-\frac{a_s^2}{3}\beta_0\ln^2\tilde N\frac{C_F}{C_A-C_F}\Big[C_A\exp(2a_sC_A\ln^2\tilde N)-C_F\exp(2a_sC_F\ln^2\tilde N)\mathcal{B}_0(\tilde a_s)\Big]\\
&+\text{known NNLL contributions (tables)}+\ldots
\end{aligned}
$$





NNLL resummation of the coefficient functions

$$
\begin{aligned}
N\, C^{T}_{\phi,q}(N,\alpha_s) \quad = \quad & \frac{1}{2\ \ln \tilde{N}} \frac{n_f}{C_F - C_A} \Big[\exp(2a_s C_A \ln^2 \tilde{N}) \mathcal{B}_0(-\tilde{a}_s) - \exp(2a_s C_F \ln^2 \tilde{N})\Big] \\
& + \frac{1}{8\ \ln^2 \tilde{N}} \frac{n_f(3C_F - b_0)}{(C_F - C_A)^2} \Big[\exp(2a_s C_A \ln^2 \tilde{N}) \mathcal{B}_0(-\tilde{a}_s) - \exp(2a_s C_F \ln^2 \tilde{N})\Big] \\
& + \frac{a_s}{4} \frac{n_f}{C_F - C_A} \exp(2a_s C_F \ln^2 \tilde{N})(12C_A - 18C_F - \beta_0) \\
& + \frac{a_s}{4} \frac{n_f}{C_F - C_A} \exp(2a_s C_A \ln^2 \tilde{N}) \Big[2\beta_0 \mathcal{B}_0(-\tilde{a}_s) - (\beta_0 - 6C_F)\mathcal{B}_1(-\tilde{a}_s) \\
& \quad -(4\beta_0 - 12C_F)\frac{1}{\tilde{a}_s}\mathcal{B}_{-1}(-\tilde{a}_s) - \frac{\beta_0}{\tilde{a}_s}\mathcal{B}_{-2}(-\tilde{a}_s)\Big] \\
& + \frac{a_s^2}{3}\beta_0 \ln^2 \tilde{N} \frac{n_f}{C_F - C_A} \Big[C_A \exp(2a_s C_A \ln^2 \tilde{N}) \mathcal{B}_0(-\tilde{a}_s) - C_F \exp(2a_s C_F \ln^2 \tilde{N})\Big] \\
& + \text{known NNLL contributions (tables)} + \ldots
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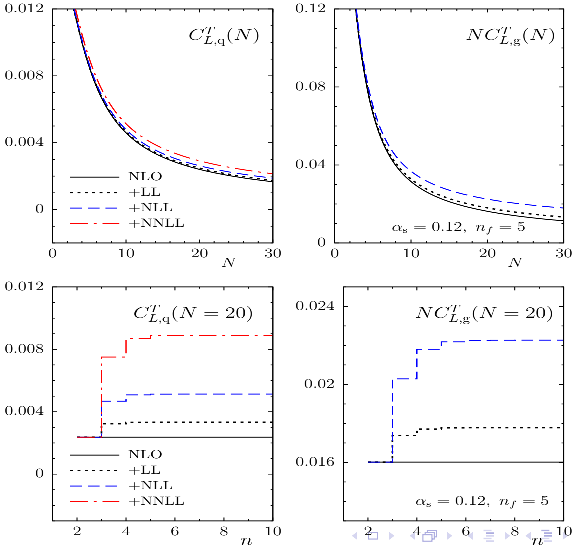
$$

$(C_A - C_F)$ denominators are cancelled by corresponding numerator factors.

Unlike the splitting functions, coefficient functions do not vanish for $C_A = C_F$.

NNLL resummation of the coefficient functions

$$N^2 C^T_{L,g}(N,\alpha_s) = 8a_s C_F \exp(2a_s C_A \ln^2 \tilde{N}) + 2a_s C_F N C^{LL}_{T,g}(N,\alpha_s)$$
$$+16 a_s^2 \ln \tilde{N} n_f \exp(2a_s C_A \ln^2 \tilde{N}) \left[\left(4C_A - \frac{5}{4}C_F\right) + \frac{1}{3} a_s \ln^2 \tilde{N} C_A \beta_0 \right]$$



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- As in DIS: double logs fixed ('inherited from lower orders') by D-dim structure and mass-factorisation.
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Back-up slides

Diagonal amplitudes $T_{T,q}$ and $T^T_{\phi,g}$ (SGE)

Our goal is the resummation of the off-diagonal amplitudes $T_{T,g}$ and $T^T_{\phi,q}$, suppressed by N^{-1}.

Expressions for the N^0 parts of Z_{kk}, $c^{(n,l)}_{T,q}$ and $c^{T(n,l)}_{\phi,g}$ are required.

These quantities can be determined from the diagonal amplitudes $T_{T,q}$ and $T^T_{\phi,g}$ in the limit governed by SGE

$$T_{a,k} = \exp\left(\hat{a}_s\,\tilde{T}^{(1)}_{a,k} + \hat{a}_s^2\,\tilde{T}^{(2)}_{a,k} + \hat{a}_s^3\,\tilde{T}^{(3)}_{a,k} + \ldots\right)$$

$$\tilde{T}^{(n)}_{a,k} = \sum_{l=-n-1}^{\infty} \epsilon^l \left(R^{(n,l)}_{a,k}\exp(n\epsilon\ln N) - V^{(n,l)}_{a,k}\right)$$

To N^{3}LL accuracy these results are converted to the renormalised coupling via

$$\hat{a}_s = a_s - \frac{\beta_0}{\epsilon}a_s^2 + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)a_s^3 + \frac{\beta_0^3}{\epsilon^3}a_s^4$$

After the transformation to the renormalised coupling $T_{\phi,g}$ needs to be multiplied by the renormalisation constant of $G^{\mu\nu}G_{\mu\nu}$

$$1 - 2\beta_0\epsilon^{-1}1, a_s^2 + 3\beta_0^2\epsilon^{-2}a_s^3 + \ldots$$

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Off-diagonal amplitudes $T_{T,g}$ and $T_{\phi,q}$

In Mellin-N space one can rewrite the unfactorised amplitudes like

$$T_{a,k}^{T(n)}(N) = \frac{1}{N\,\epsilon^{2n-1}} \sum_{i=0}^{n-1} \left(A_{a,k}^{T(n,i)} + \epsilon B_{a,k}^{T(n,i)} + \epsilon^2 C_{a,k}^{T(n,i)} + \dots \right) \exp(\epsilon\,(n-i)\,\ln N)$$

Mass factorisation links $A_{a,k}^{T(n,i)}$, $B_{a,k}^{T(n,i)}$, $C_{a,k}^{T(n,i)}$ to lower-order quantities.

$$\frac{1}{n_f} A_{T,g}^{(n,0)} = \frac{1}{C_F} A_{\phi,q}^{T(n,0)} = -2^{2n-1} \frac{1}{n!} \sum_{l=0}^{n-1} C_F^l C_A^{n-l-1}$$

KLN $\rightarrow$ only one(LO)/two(NLO)/three(NNLO) independent coeff's $\forall\ n,\ a,\ k$

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The general expressions for $B_{a,k}^{T(n,i)}$ and especially $C_{a,k}^{T(n,i)}$ are rather lengthy...

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Off-diagonal amplitudes $T_{T,g}$ and $T_{\phi,q}$

In Mellin-N space one can rewrite the unfactorised amplitudes like

$$T^{T(n)}_{a,k}(N) = \frac{1}{N\,\epsilon^{2n-1}} \sum_{i=0}^{n-1} \left(A^{T(n,i)}_{a,k} + \epsilon B^{T(n,i)}_{a,k} + \epsilon^2 C^{T(n,i)}_{a,k} + \dots \right) \exp(\epsilon\,(n-i)\,\ln N)$$

Mass factorisation links $A^{T(n,i)}_{a,k}$, $B^{T(n,i)}_{a,k}$, $C^{T(n,i)}_{a,k}$ to lower-order quantities.

$$\frac{1}{n_f} A^{(n,0)}_{T,g} = \frac{1}{C_F} A^{T(n,0)}_{\phi,q} = -2^{2n-1} \frac{1}{n!} \sum_{l=0}^{n-1} C_F^l C_A^{n-l-1}$$

KLN $\rightarrow$ only one(LO)/two(NLO)/three(NNLO) independent coeff's $\forall\ \ n,\ a,\ k$

$$A^{T(n,i)}_{a,k} = (-1)^i \binom{n-1}{i} A^{T(n,0)}_{a,k}$$

$$B^{T(n,i+1)}_{a,k} = (-1)^i \left[\binom{n-2}{i} B^{T(n,1)}_{a,k} + i \binom{n-1}{i+1} B^{T(n,0)}_{a,k} \right]$$

$$C^{T(n,i+2)}_{a,k} = (-1)^i \left[\binom{n-3}{i} C^{T(n,2)}_{a,k} + i \binom{n-2}{i+1} C^{T(n,1)}_{a,k} + \frac{1}{2} i(i+1) \binom{n-1}{i+2} C^{(n,0)}_{a,k} \right]$$

The general expressions for $B^{T(n,i)}_{a,k}$ and especially $C^{T(n,i)}_{a,k}$ are rather lengthy...

Resummation of the longitudinal structure function

Up to an additional power of ϵ and N^{-1} longitudinal fragmentation functions are built up in the same way

$$T^{T(n)}_{L,k}(N) = \frac{\epsilon^{-2n+2}}{N^{1+\delta_{kg}}} \sum_{i=0}^{n-1} \left(A^{T(n,i)}_{L,k} + \epsilon B^{T(n,i)}_{L,k} + \epsilon^2 C^{T(n,i)}_{L,k} + \ldots \right) \exp(\epsilon(n-i) \ln N)$$

In this case also ϵ^{-n} poles vanish at order α_s^n. The coefficients for LL resummation are

$$A^{T(n,0)}_{L,q} = \frac{2^{2n}}{(n-1)!} C_F^n \ , \qquad A^{T(n,0)}_{L,g} = \frac{2^{2n+1}}{(n-1)!} C_A^{n-1} n_f$$

Same KLN relations as for $T_{T,g}$ and $T^T_{\phi,q}$ apply.

Once the coefficients $A^{T(n,i)}_{a,k}$, $B^{T(n,i)}_{a,k}$ and $C^{T(n,i)}_{a,k}$ are obtained, the unfactorised amplitude at all orders at NNLL is known.

We extract the coefficient function at all order at such logarithmic accuracy using *mass factorisation* relations.

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