Two–Loop QED Operator Matrix Elements with a Massive External Fermion Line

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Outline

1. Introduction.
2. The renormalization group technique.
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4. The $O(\alpha)$ operator matrix elements.
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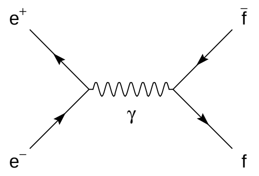
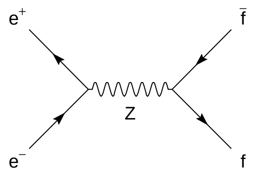
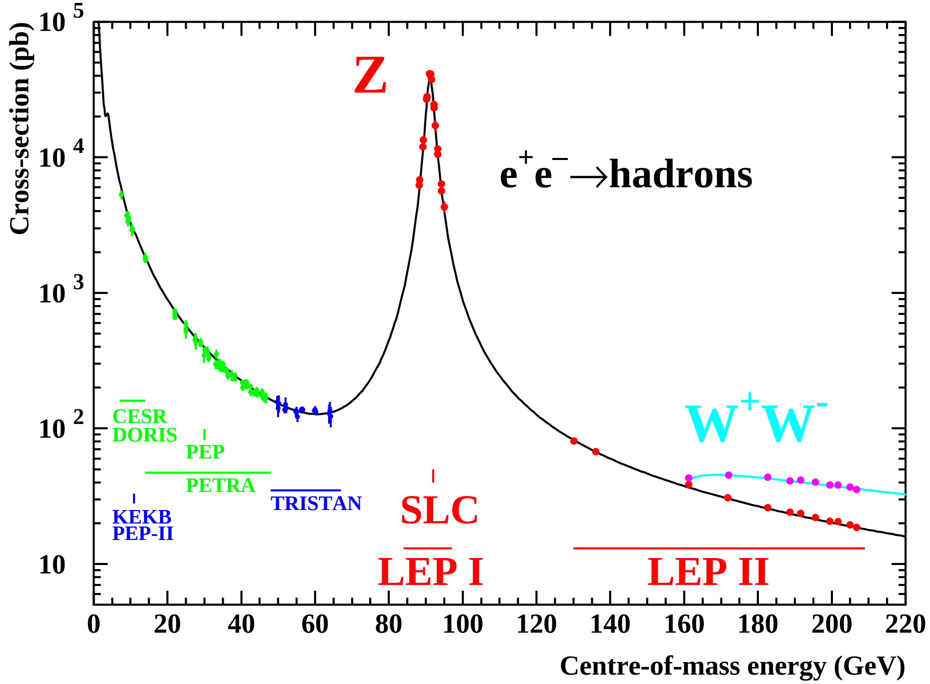
We want to revisit the calculation of the two-loop order initial state radiative corrections to electron-positron annihilation into heavy fermions:







Some time ago, Berends, van Neerven and Burgers (Nucl. Phys. **B297** 1988) 429; E: B304 (1988) 921.) calculated the corrections due to initial state radiation directly (for massive electrons), including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production.

- The process is of central importance at LEP, Giga-Z, and the ILC.
- Recalculate these corrections by a method based on the renormalization group.

The Born Cross Section : $e^+e^- \rightarrow f,\overline{f}$ $f \neq e$

$$\frac{d\sigma^{(0)}(s)}{d\Omega} = \frac{\alpha^2}{4s} N_{C,f}\sqrt{1-\frac{4m_f}{s}} \times \left[\left(1+\cos^2\theta+\frac{4m_f^2}{s}\sin^2\theta\right)G_1(s)-\frac{8m_f^2}{s}G_2(s)+2\sqrt{1-\frac{4m_f^2}{s}}\cos\theta G_3(s)\right]$$

$$\sigma^{(0)}(s) = \frac{4\pi\alpha^2}{3s}N_{C,f}\sqrt{1-\frac{4m_f}{s}}\left[\left(1+\frac{2m_f^2}{s}\right)G_1(s)-6\frac{m_f^2}{s}G_2(s)\right]$$

$$G_1(s) = Q_e^2Q_f^2+2Q_eQ_fv_ev_f\mathrm{Re}[\chi_Z(s)]+(v_e^2+a_e^2)(v_f^2+a_f^2)|\chi_Z(s)|^2$$
$$G_2(s) = (v_e^2+a_e^2)a_f^2|\chi_Z(s)|^2$$
$$G_3(s) = 2Q_eQ_fa_ea_f\mathrm{Re}[\chi_Z(s)]+4v_ev_fa_ea_f|\chi_Z(s)|^2.$$

$$\chi_Z(s) = \frac{s}{s-M_Z^2+iM_Z\Gamma_Z}$$

We represent the observable in Mellin space transforming $z = s'/s \; \epsilon [0,1]$:

The differential scattering cross section $\Sigma(z) = d\sigma_{ij}(z)/ds'$ is considered. This quantity reads in Mellin space

$$\mathbf{M}[\Sigma(z)](N) = \int_0^1 dz z^{N-1} \Sigma(z) \; .$$

In this representation the different Mellin convolutions to be performed in z–space simplify to ordinary products. The following representation is obtained

$$\frac{d\sigma_{ij}}{ds'}(N) = \frac{1}{s}\sigma^{(0}(N) \sum_{l,k} \Gamma_{l,i}\left(N, \frac{\mu^2}{m^2}\right) \tilde{\sigma}_{lk}\left(N, \frac{s'}{\mu^2}\right) \Gamma_{k,j}\left(N, \frac{\mu^2}{m^2}\right) \; .$$

- Here Γ_{li} denote massive operator matrix elements and $\tilde{\sigma}_{lk}$ the massless Wilson coefficients, both being calculated in the $\overline{\mathrm{MS}}$ scheme.
- μ is the factorization mass, which cancels in the physical cross section.
- The initial state fermion mass dependence is solely encoded in Γ_{li}.

Γ_{li} and $\tilde{\sigma}_{lk}$ obey the following renormalization group equations:

$$\begin{aligned}
\left[\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\delta_{al}+\gamma_{al}(N,g)\right]\Gamma_{li}\left(N,\frac{\mu^2}{m^2},g(\mu^2)\right) &= 0 \\
\left[\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\delta_{la}\delta_{kb}-\gamma_{la}(N,g)\delta_{kb}-\gamma_{kb}(N,g)\delta_{la}\right]\tilde{\sigma}_{lk}\left(\frac{s'}{m^2},g(\mu^2)\right) &= 0 \\
\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]\sigma_{ij}\left(\frac{s'}{\mu^2},g(\mu^2)\right) &= 0
\end{aligned}$$

For the process under consideration we obtain to $O(a^2)$:

$$\begin{aligned}
\left[\frac{\partial}{\partial\hat{L}}-\beta_0 a^2\frac{\partial}{\partial a}+\frac{1}{2}\gamma_{ee}(N,a)\right]\Gamma_{ee}\left(N,a,\frac{\mu^2}{m^2}\right)+\frac{1}{2}\gamma_{e\gamma}(N,a)\Gamma_{\gamma e}\left(N,a,\frac{\mu^2}{m^2}\right) &= 0 \\
\left[\frac{\partial}{\partial\hat{L}}-\beta_0 a^2\frac{\partial}{\partial a}-\gamma_{ee}(N,a)\right]\tilde{\sigma}_{ee}\left(N,a,\frac{s'}{\mu^2}\right)-\gamma_{\gamma e}(N,a)\tilde{\sigma}_{e\gamma}\left(N,a,\frac{s'}{\mu^2}\right) &= 0
\end{aligned}$$

where $\partial/\partial\mu$ has been replaced by $2\partial/\partial\hat{L}$, with $\hat{L}=\ln(\mu^2/M^2)$.

The solutions of these equations are

$$\Gamma_{ee}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{ee}^{(0)}(N)L+\Gamma_{ee}^{(0)}(N)\right]+a^2\left[\left\{\frac{1}{8}\gamma_{ee}^{(0)}(N)\left(\gamma_{ee}^{(0)}(N)-2\beta_0\right)+\frac{1}{8}\gamma_{e\gamma}^{(0)}(N)\gamma_{\gamma e}^{(0)}(N)\right\}L^2\right.$$
$$\left.+1+\frac{1}{2}\left\{-\gamma_{ee}^{(1)}(N)+2\beta_0\Gamma_{ee}^{(0)}-\gamma_{ee}^{(0)}(N)\Gamma_{ee}^{(0)}(N)-\gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}(N)\right\}L+\Gamma_{ee}^{(1)}\right]+O(a^3),$$

$$\tilde{\sigma}_{ee}\left(N,a,\frac{s'}{\mu^2}\right) = a\left[-\frac{1}{2}\gamma_{ee}^{(0)}(N)\lambda+\tilde{\sigma}_{ee}^{(0)}(N)\right]+a^2\left[\left\{\frac{1}{2}\gamma_{ee}^{(0)}(N)\left(\gamma_{ee}^{(0)}(N)+\beta_0\right)+\frac{1}{4}\gamma_{e\gamma}^{(0)}(N)\gamma_{\gamma e}^{(0)}(N)\right\}\lambda^2\right.$$
$$\left.+1+\left\{-\gamma_{ee}^{(1)}(N)-\beta_0\tilde{\sigma}_{ee}^{(0)}-\gamma_{ee}^{(0)}(N)\tilde{\sigma}_{ee}^{(0)}(N)-\gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}(N)\right\}\lambda+\tilde{\sigma}_{ee}^{(1)}\right]+O(a^3),$$

$$\Gamma_{\gamma e}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{\gamma e}^{(0)}(N)L+\Gamma_{\gamma e}^{(0)}\right]+O(a^2)$$

$$\tilde{\sigma}_{e\gamma}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{e\gamma}^{(0)}(N)\lambda+\tilde{\sigma}_{e\gamma}^{(0)}\right]+O(a^2)\ ,$$

with the logarithms $L=\ln\left(\frac{\mu^2}{m^2}\right)$ and $\lambda=\ln\left(\frac{s'}{\mu^2}\right)$

Introducing the splitting functions in N-space

$$P_{ij}^{(l)}(N) = \int_0^1 dz z^{N-1} P_{ij}^{(l)}(z) = -\gamma_{ij}^{(l)}(N)$$

one obtains

$$\begin{aligned}
\frac{d\sigma_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)\Bigg\{1 + a_0\left[P_{ee}^{(0)}\mathbf{L} + \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right] \\
&+a_0^2\Bigg\{\left[\frac{1}{2}P_{ee}^{(0)} \otimes P_{ee}^{(0)} - \frac{\beta_0}{2}P_{ee}^{(0)} + \frac{1}{4}P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}\right]\mathbf{L}^2 \\
&+\left[P_{ee}^{(1)} + P_{ee}^{(0)} \otimes \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right) - \beta_0\tilde{\sigma}_{ee}^{(0)} + P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}\right]\mathbf{L} \\
&+ \left(2\Gamma_{ee}^{(1)} + \tilde{\sigma}_{ee}^{(1)}\right) + 2\Gamma_{ee}^{0)} \otimes \tilde{\sigma}_{ee}^{(0)} + 2\tilde{\sigma}_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)} + \Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)}\Bigg\}\Bigg\}
\end{aligned}$$

with

$$\mathbf{L} = \ln\left(\frac{s'}{m^2}\right) = \ln\left(\frac{s}{m^2}\right) + \ln(z); \quad \hat{\mathbf{L}} \equiv \ln(s/m^2)\ .$$

It is convenient to represent the differential scattering cross section in terms of three contributions, the flavor non-singlet terms with a single fermion line (I), those with a closed fermion line (II), and the pure-singlet terms (III). These contributions are :

$$
\begin{aligned}
\frac{d\sigma^{\rm I}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)\Biggl\{1 + a_0\left[P_{ee}^{(0)}\mathbf{L} + \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right] \\
&\qquad + a_0^2\Biggl\{\frac{1}{2}P_{ee}^{(0)} \otimes P_{ee}^{(0)}\mathbf{L}^2 + \left[P_{ee}^{(1),\rm I} + P_{ee}^{(0)} \otimes \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right]\mathbf{L} \\
&\qquad + \left(2\Gamma_{ee}^{(1),\rm I} + \tilde{\sigma}_{ee}^{(1),\rm I}\right) + 2\Gamma_{ee}^{0)} \otimes \tilde{\sigma}_{ee}^{(0)} + \Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)}\Biggr\}\Biggr\} \\
\frac{d\sigma^{\rm II}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\Biggl\{-\frac{\beta_0}{2}P_{ee}^{(0)}\mathbf{L}^2 + \left[P_{ee}^{(1),\rm II} - \beta_0\tilde{\sigma}_{ee}^{(0)}\right]\mathbf{L} + \left(2\Gamma_{ee}^{(1),\rm II} + \tilde{\sigma}_{ee}^{(1),\rm II}\right)\Biggr\} \\
\frac{d\sigma^{\rm III}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\Biggl\{\frac{1}{4}P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}\mathbf{L}^2 + \left[P_{ee}^{(1),\rm III} + P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}\right]\mathbf{L} \\
&\qquad + \left(2\Gamma_{ee}^{(1),\rm III} + \tilde{\sigma}_{ee}^{(1),\rm III}\right) + 2\tilde{\sigma}_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)}\Biggr\}
\end{aligned}
$$

- $\tilde{\sigma}_{ij}^{(k)}$ denotes the respective contribution of the massless Drell-Yan (DY) cross section.

Different ingredients to the calculation :

- Splitting functions P_{ij} to $O(\alpha^2)$

[E. G. Floratos, D. A. Ross and C. T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545]; Nucl. Phys. B **152** (1979) 493; A. Gonzalez-Arroyo, C. Lopez and F. J. Yndurain, Nucl. Phys. B **153** (1979) 161; A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429; E. G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417; G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27; W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437; R. Hamberg and W. L. van Neerven, Nucl. Phys. B **379** (1992) 143; R. K. Ellis and W. Vogelsang, arXiv:hep-ph/9602356; S. Moch and J. A. M. Vermaseren, Nucl. Phys. B **573** (2000) 853 [arXiv:hep-ph/9912355]; and the results from the present calculation]

- massless Drell-Yan Cross Section $\tilde{\sigma}_{ij}$ to $O(\alpha^2)$

[R. Hamberg, W. L. van Neerven and T. Matsuura, Nucl. Phys. B **359** (1991) 343 [E: B **644** (2002) 403]; R. V. Harlander and W. B. Kilgore, Phys. Rev. Lett. **88** (2002) 201801.]

- massive OMEs Γ_{ij} to $O(\alpha^2)$ $\Longrightarrow$ this calculation;

Renormalization concerns the wave function, the charge renormalization, and the ultraviolet singularities of the local operators.

i) Wave function renormalization

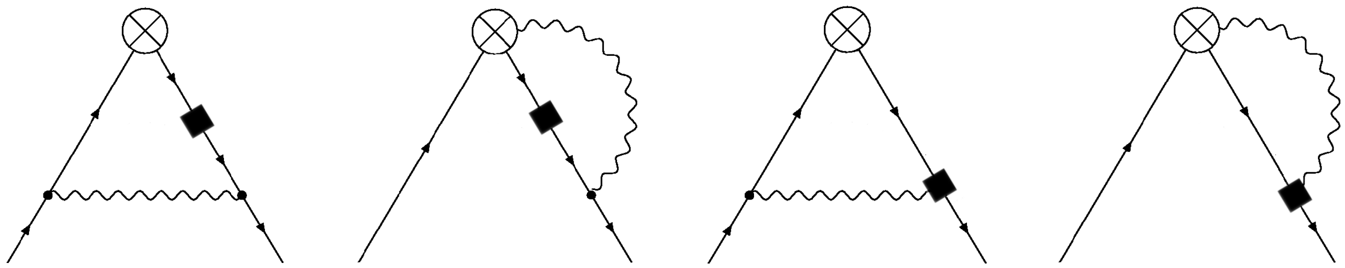
The bare wave function is renormalized by

$$\psi_0 = \sqrt{Z_2(\varepsilon)}\psi$$

$$\begin{aligned} Z_2 &= 1 + \hat{a} S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \left[\frac{6}{\varepsilon} - 4 + \left(4 + \frac{3}{4}\zeta_2\right)\varepsilon\right] \\ &\quad + \hat{a}^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{\left[18\frac{1}{\varepsilon^2} - \frac{51}{2}\frac{1}{\varepsilon} + \left(\frac{433}{8} - \frac{147}{2}\zeta_2 + 96\zeta_2\ln(2) - 24\zeta_3\right)\right]_{\mathrm{I}} \right. \\ &\quad \left. + \left[16\frac{1}{\varepsilon^2} - \frac{38}{3}\frac{1}{\varepsilon} + \left(\frac{1139}{18} - 28\zeta_2\right)\right]_{\mathrm{II}}\right\}. \end{aligned}$$

Broadhurst, et. al. Z. Phys. C 48 (1990). Melnikov and Ritbergen, Nucl. Phys. B 591 (2000)

Counterterms:



$$Z_{CT} = \int_0^1 dx\, x^n \Bigg\{ -\frac{72}{\varepsilon^2}\delta(1-x) - \frac{24}{\varepsilon}\mathcal{D}_0(x) - 24\mathcal{D}_1(x) + 16\mathcal{D}_0(x) - (64 + 18\zeta_2)\,\delta(1-x) - 12 + N\left[\left(\frac{24}{\varepsilon} + 8\right)\delta(1-x) + 24\mathcal{D}_0(x)\right]\Bigg\}$$

ii) Charge Renormalization

Massive fermions → coupling constant first obtained in MOM-scheme, where

$$Z_g^{\mathrm{MOM}^2} = 1 + a^{\mathrm{MOM}}(\mu^2)\beta_{0,H}\left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon/2}\exp\left(\sum_{k=2}^{\infty}\frac{\zeta_k}{k}\left(\frac{\varepsilon}{2}\right)^k\right)$$
$$+O\left(a^{\mathrm{MOM}}(\mu^2)^2\right)$$

We then transform to the $\overline{\mathrm{MS}}$ scheme using

$$Z_g^{\mathrm{MOM}^2} a^{\mathrm{MOM}}(\mu^2) = Z_g^{\overline{\mathrm{MS}}^2} a^{\overline{\mathrm{MS}}}(\mu^2)$$

which implies

$$a^{\mathrm{MOM}} = a^{\overline{\mathrm{MS}}} - \beta_{0,H}\ln\left(\frac{m_e^2}{\mu^2}\right)a^{\overline{\mathrm{MS}}^2} + O\left(a^{\overline{\mathrm{MS}}^3}\right)$$

with

$$\beta_{0,H} = -\frac{4}{3}$$

iii) Renormalization of the composite operators

The inverse Z-factors are given in the MOM-scheme by

$$\begin{aligned} Z_{ij}^{-1}(a^{\mathsf{MOM}}, n_f+1, \mu) &= \delta_{ij} - a^{\mathsf{MOM}} \frac{\gamma_{ij}^{(0)}}{\varepsilon} + a^{\mathsf{MOM}^2} \Bigg[\frac{1}{\varepsilon} \Big(-\frac{1}{2} \gamma_{ij}^{(1)} - \delta a_1^{\mathsf{MOM}} \gamma_{ij}^{(0)} \Big) \\ &\quad + \frac{1}{\varepsilon^2} \Big(\frac{1}{2} \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \Big) \Bigg] + O(a^{\mathsf{MOM}^3}) \end{aligned}$$

with

$$\delta a_1^{\mathsf{MOM}} = S_\varepsilon \frac{2\beta_{0,H}}{\varepsilon} \left(\frac{m_e^2}{\mu^2} \right)^{\varepsilon/2} \exp\left[\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left(\frac{\varepsilon}{2} \right)^i \right]$$

The renormalized OMEs are given by

$$A_{ij}^{\mathsf{MOM}} = \delta_{ij} + a^{\mathsf{MOM}} \left[\hat{A}_{ij}^{(1)} + Z_{i,j}^{-1,(1)} \right] + a^{\mathsf{MOM}^2} \left[\hat{A}_{ij}^{(2)} + Z_{i,j}^{-1,(2)} + Z_{i,j}^{-1,(1)} \hat{A}_{ij}^{(1)} \right] + O(a^{\mathsf{MOM}^3})$$

The corresponding Z-factors are given by

$$\left[Z_{ee}^{\mathrm{I}}(\varepsilon,N)\right]^{-1} = 1 + a^{\mathsf{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{ee}^{(0)}(N) + a^{\mathsf{MOM}^2} S_\varepsilon^2 \left\{ \frac{1}{2\varepsilon^2} P_{ee}^{(0)^2}(N) + \frac{1}{2\varepsilon} P_{ee}^{(1),\mathrm{NS}}(N) \right\} + O(a^{\mathsf{MOM}^3})$$

$$\left[Z_{ee}^{\mathrm{II}}(\varepsilon,N)\right]^{-1} = a^{\mathsf{MOM}^2} \left\{ -\frac{1}{2\varepsilon^2} \beta_0 P_{ee}^{(0)} + \frac{2}{\varepsilon^2} \beta_{0,H} \left(\frac{m_e^2}{\mu_2}\right)^{\varepsilon/2} \exp\left[\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left(\frac{\varepsilon}{2}\right)^i\right] + \frac{1}{2\varepsilon} P_{ee}^{(1),\mathrm{II}} \right\} + O(a^{\mathsf{MOM}^3})$$

$$\left[Z_{ee}^{\mathrm{III}}(\varepsilon,N)\right]^{-1} = a^{\mathsf{MOM}^2} S_\varepsilon^2 \left\{ \frac{1}{2\varepsilon^2} P_{e\gamma}^{(0)}(N) P_{\gamma e}^{(0)}(N) + \frac{1}{2\varepsilon} \left[P_{ee}^{(1),\mathrm{III}}(N) \right] \right\} + O(a^{\mathsf{MOM}^3})$$

$$\left[Z_{e\gamma}(\varepsilon,N)\right]^{-1} = a^{\mathsf{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{e\gamma}^{(0)}(N) + O(a^{\mathsf{MOM}^2})$$

$$\left[Z_{\gamma e}^{\mathrm{NS}}(\varepsilon,N)\right]^{-1} = a^{\mathsf{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{\gamma e}^{(0)}(N) + O(a^{\mathsf{MOM}^2})$$

UV Un-renormalized Operator-Matrix Elements : after wave function and charge renormalization

$$
\begin{aligned}
\hat{A}_{ee}^{(1)} &= a^{\mathsf{MOM}} S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \left\{ \frac{1}{\varepsilon} P_{ee}^{(0)} + \Gamma_{ee}^{(0)} + \varepsilon \overline{\Gamma}_{ee}^{(0)} \right\} \\
\hat{A}_{ee}^{(2),\mathrm{I}} &= a^{\mathsf{MOM}\,2} S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{ \frac{1}{2\varepsilon^2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} - \frac{1}{2\varepsilon} \left[P_{ee}^{(1),\mathrm{I}} + 2\Gamma_{ee}^{(0)} \otimes P_{ee}^{(0)} \right] + \Gamma_{ee}^{(1),\mathrm{I}} \right\} \\
\hat{A}_{ee}^{(2),\mathrm{II}} &= a^{\mathsf{MOM}\,2} S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{ \frac{1}{\varepsilon^2} \beta_0 P_{ee}^{(0)} - \frac{1}{\varepsilon} \left[\frac{1}{2} P_{ee}^{(1),\mathrm{II}} + 2\beta_0 \Gamma_{ee}^{(0)} \right] + \Gamma_{ee}^{(1),\mathrm{II}} \right\} \\
\hat{A}_{ee}^{(2),\mathrm{III}} &= a^{\mathsf{MOM}\,2} S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{ \frac{1}{2\varepsilon^2} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)} - \frac{1}{\varepsilon} \left\{ \frac{1}{2} P_{ee}^{(1),\mathrm{III}} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} \right\} + \Gamma_{ee}^{(1),\mathrm{III}} \right\}
\end{aligned}
$$

- cf. also : [M. Buza et al., Nucl. Phys. B **472** (1996) 611; I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. B **780** (2007) 40.]

The renormalized OMEs in the $\overline{\text{MS}}$-scheme are finally given by

$$A_{ee}^{\overline{\text{MS}},\text{I}}(N) = a^{\overline{\text{MS}}}\left[-\frac{1}{2}P_{ee}^{(0)}(N)\ln\left(\frac{m_e^2}{\mu^2}\right)+\Gamma_{ee}^{(0)}(N)\right]+a^{\overline{\text{MS}}^2}\left[\frac{1}{8}P_{ee}^{(0)^2}(N)\ln^2\left(\frac{m_e^2}{\mu^2}\right)\right.$$
$$\left.-\frac{1}{2}\left[P_{ee}^{(1),\text{I}}(N)+P_{ee}^{(0)}(N)\Gamma_{ee}^{(0)}(N)\right]\ln\left(\frac{m_e^2}{\mu^2}\right)+\hat{\Gamma}_{ee}^{(1),\text{I}}(N)+P_{ee}^{(0)}(N)\overline{\Gamma}_{ee}^{(0)}(N)\right]+O(a^{\overline{\text{MS}}^3})$$

$$A_{ee}^{\overline{\text{MS}},\text{II}}(N) = a^{\overline{\text{MS}}^2}\left[\frac{\beta_{0,H}}{4}P_{ee}^{(0)}(N)\ln^2\left(\frac{m_e^2}{\mu^2}\right)-\left[\frac{1}{2}P_{ee}^{(1),\text{II}}(N)+\beta_{0,H}\Gamma_{ee}^{0}\right]\ln\left(\frac{m_e^2}{\mu^2}\right)\right.$$
$$\left.+\hat{\Gamma}_{ee}^{(1),\text{II}}(N)+2\beta_{0,H}\bar{\Gamma}_{ee}^{(0)}\right]+O(a^{\overline{\text{MS}}^3})$$

$$A_{ee}^{\overline{\text{MS}},\text{III}}(N) = a^{\overline{\text{MS}}^2}\left[\frac{1}{8}P_{e\gamma}^{(0)}(N)P_{\gamma e}^{(0)}(N)\ln^2\left(\frac{m_e^2}{\mu^2}\right)-\frac{1}{2}\left[P_{ee}^{(1),\text{III}}(N)\right.\right.$$
$$\left.\left.+P_{e\gamma}^{(0)}(N)\Gamma_{\gamma e}^{(0)}(N)\right]\ln\left(\frac{m_e^2}{\mu^2}\right)+\hat{\Gamma}_{ee}^{(1),\text{III}}(N)+P_{e\gamma}^{(0)}(N)\overline{\Gamma}_{\gamma e}^{(0)}(N)\right]+O(a^{\overline{\text{MS}}^3})$$

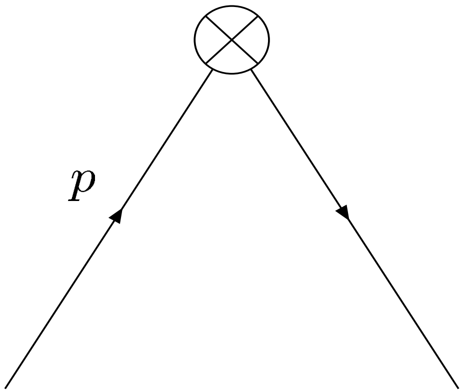
Here $\Gamma_{ij}^{(0)}$ and $\overline{\Gamma}_{ij}^{(0)}$ denote the constant and linear term in ε of the unrenormalized one-loop massive OMEs, and $\hat{\Gamma}_{ij}^{(1)}$ the corresponding constant part of the two-loop OMEs.

The unrenormalized operator matrix elements are given by

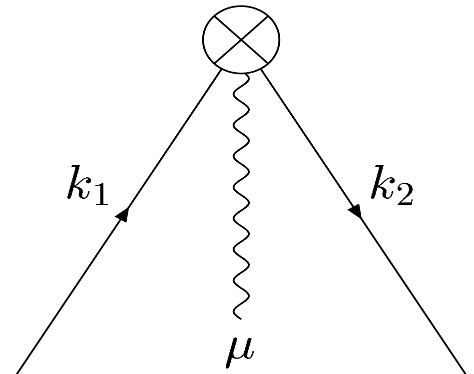
$$\hat{\hat{A}}_{ij} = \langle i|O_j|i\rangle = \sum_{k=0}^{\infty} \hat{a}^k \hat{\hat{A}}_{ij}^{(k)},$$

where in the case of QED, i and j can be either a photon or an electron $(i, j = e\gamma)$

Feynman rules for the local twist 2 operators in QED at one loop (with $\Delta \cdot \Delta = 0$):



$\not\Delta(\Delta.p)^{N-1}$



$-e\Delta_\mu \sum_{j=1}^{N-1}(\Delta.k_1)^{j-1}(\Delta.k_2)^{N-1-j}$

Notation:

- A single hat means that (only) wave function renormalization has been performed.
- A double hat means completely unrenormalized (no renormalization step has been performed.

We use dimensional regularization with $D = 4 + \varepsilon$. Then

$$\hat{A}_{ij}^{(1)} = \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} S_\varepsilon \left[-\frac{1}{\varepsilon} P_{ij}^{(0)}(x) + \Gamma_{ij}^{(0)}(x) + \varepsilon \overline{\Gamma}_{ij}^{(0)}(x) + O(\varepsilon^2)\right] ,$$

where $\Gamma_{ij}^{(0)}(x)$ and $\overline{\Gamma}_{ij}^{(0)}(x)$ represent the $O(\varepsilon^0)$ and $O(\varepsilon^1)$ terms in the ε-expansion of $\hat{A}_{ij}^{(1)}$, respectively, and the $P_{ij}^{(0)}$'s are the leading order splitting functions.

The one–loop splitting functions are factorization–scheme invariant and are given in x-space by

$$\begin{aligned} P_{ee}^{(0)}(x) &= 8\mathcal{D}_0(x) - 4(1+x) + 6\delta(1-x) = 4\left[\frac{1+x^2}{1-x}\right]_+ , \\ P_{e\gamma}^{(0)}(x) &= 4\left[x^2 + (1-x)^2\right] , \\ P_{\gamma e}^{(0)}(x) &= 4\left[\frac{1+(1-x)^2}{x}\right] , \end{aligned}$$

where we define

$$\mathcal{D}_k(x) = \left(\frac{\ln^k(1-x)}{1-x}\right)_+ .$$

The +-prescription is defined by

$$\int_0^1 dx\,[f(x)]_+\,g(x) = \int_0^1 dx f(x)\,[g(x) - g(1)]\,,$$

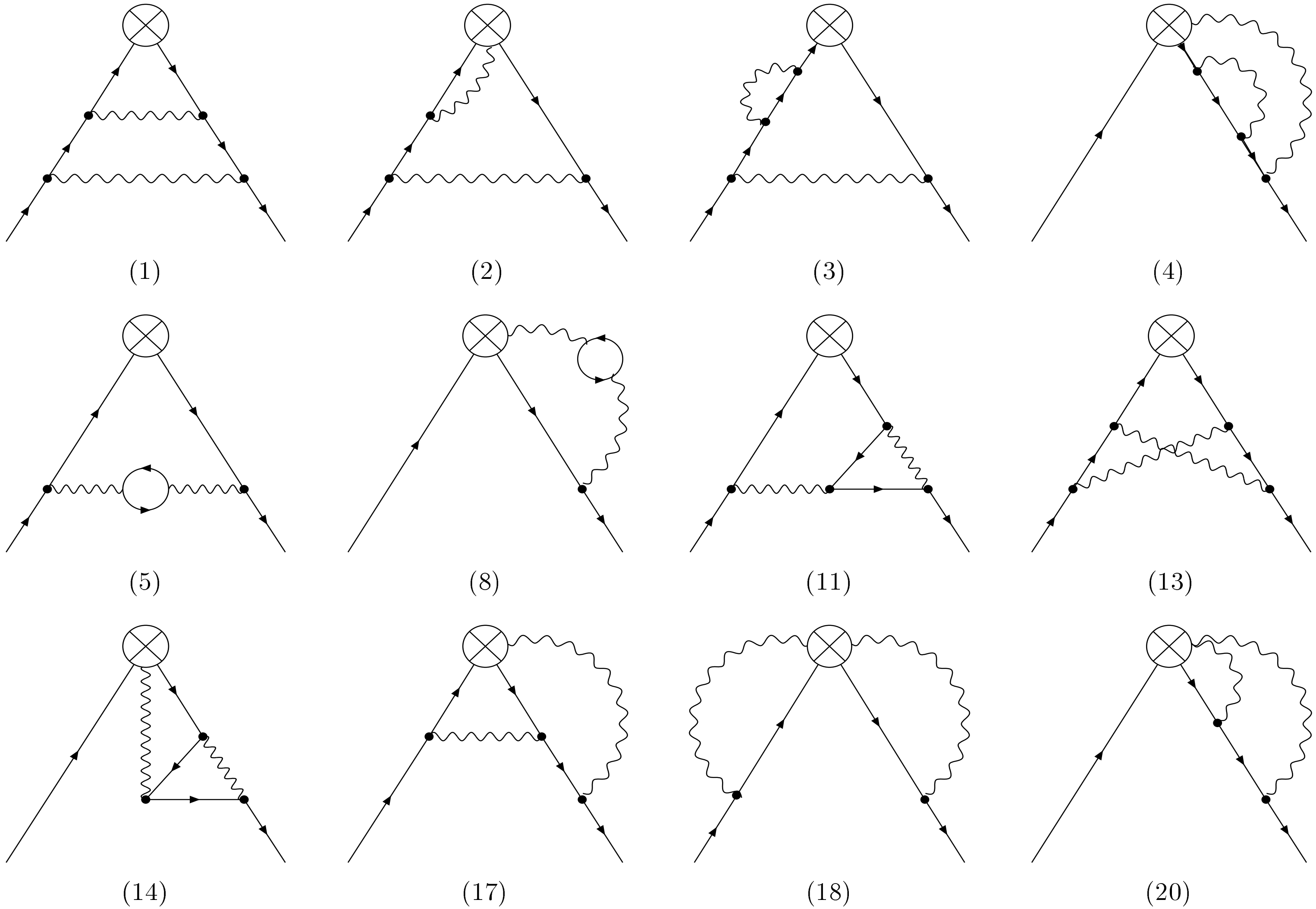
with $g(x)\,\epsilon\,\mathcal{D}[0,1]$ denoting a test function.

The $O(\varepsilon^0)$ terms are

$$\begin{aligned}
\Gamma_{ee}^{(0)}(x) &= -8\mathcal{D}_1(x) - 4\mathcal{D}_0(x) + 4\delta(1-x) + 2(1+x)\left[2\ln(1-x)+1\right] \\
&= -4\left[\frac{1+x^2}{1-x}\left\{\ln(1-x)+\frac{1}{2}\right\}\right]_+ \\
\Gamma_{e\gamma}^{(0)}(x) &= 0 \\
\Gamma_{\gamma e}^{(0)}(x) &= -2\frac{1+(1-x)^2}{x}\left[2\ln(x)+1\right] ,
\end{aligned}$$

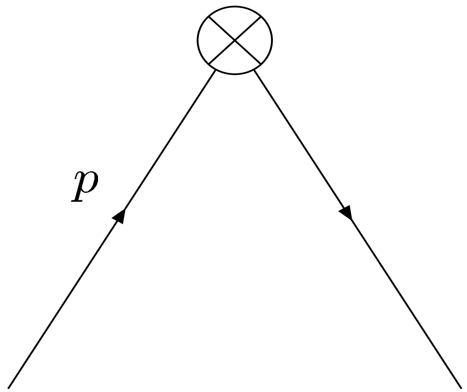
The linear term in $\varepsilon\, \overline{\Gamma}_{ee}^{(0)}(x)$ reads

$$\begin{aligned}
\overline{\Gamma}_{ee}^{(0)}(x) &= -4\mathcal{D}_2(x) - 4\mathcal{D}_1(x) - \zeta_2\mathcal{D}_0(x) - \left(4+\frac{3}{4}\zeta_2\right)\delta(1-x) \\
&\quad +2(1+x)\left[\ln^2(1-x)+\ln(1-x)+\frac{1}{4}\zeta_2\right] \\
&= -2\left[\frac{1+x^2}{1-x}\left\{\ln^2(1-x)+\ln(1-x)+\frac{1}{4}\zeta_2\right\}\right]_+ .
\end{aligned}$$

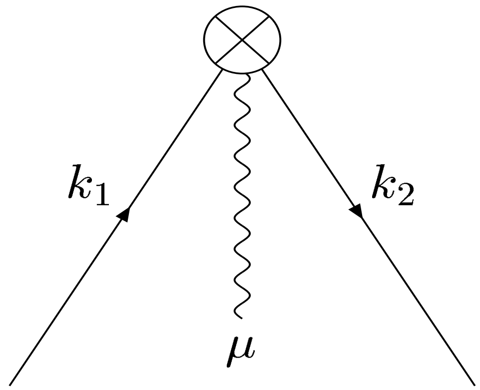


Two-loop diagrams contributing to the massive operator matrix element $A_{ee}(N, \alpha)$.
The antisymmetric diagrams count twice.

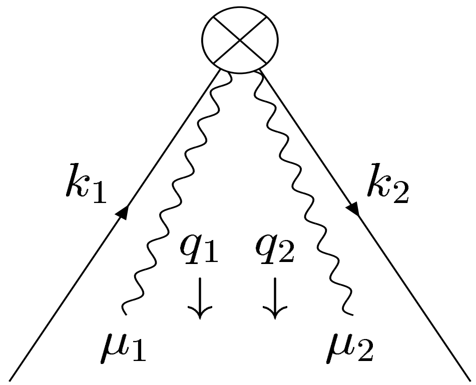
Feynman rules for the local twist 2 operators in QED (with $\Delta \cdot \Delta = 0$):



$\not{\Delta}(\Delta.p)^{N-1}$



$-e\Delta_\mu \sum_{j=1}^{N-1}(\Delta.k_1)^{j-1}(\Delta.k_2)^{N-1-j}$



$$e^2 \not{\Delta}\Delta_{\mu_1}\Delta_{\mu_2}\sum_{j=1}^{N-2}\sum_{i=1}^{j}(\Delta.k_1)^{i-1}(\Delta.k_2)^{N-2-j}\left\{[\Delta.\,(k_1-q_2)]^{j-i}+[\Delta.\,(k_2+q_2)]^{j-i}\right\}$$

All the diagrams can be written in terms of integrals of these type:

$$A^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

$$B^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{k_2 \cdot p (\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

$$E^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot k_1)^j (\Delta \cdot k_2)^{n-1-j}$$

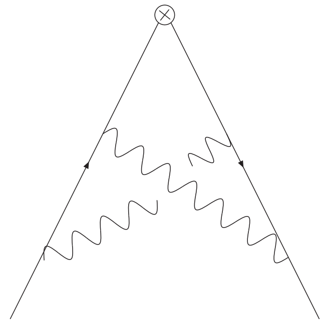
$$F^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot p)^j (\Delta \cdot k_1)^{n-1-j}$$

where,

$$D_1 = k_1^2 - m^2 \; ; \quad D_2 = k_2^2 - m^2 \; ; \quad D_3 = (k_1 - p)^2 \; ;$$

$$D_4 = (k_1 - k_2)^2 \; ; \quad D_5 = (k_2 - k_1 + p)^2 - m^2 \; ; \quad D_6 = (k_2 - p)^2$$

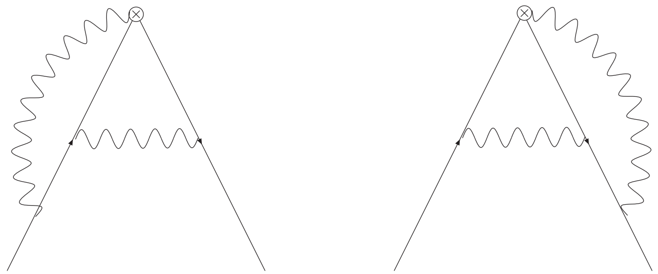
For example, in terms of these integrals, the crossed box gives



$$= \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\mathrm{Tr}[\gamma^\mu(\not p - \not k_1 + \not k_2 + m)\gamma^\nu(\not k_2 + m)\not\Delta(\not k_2 + m)\gamma_\mu(\not k_1 + m)\gamma_\nu(\not p + m)]}{D_1D_2^2D_3D_4D_5}(\Delta\cdot k_2)^n =$$

$$= (D-4)(D-2)\left[2A^{0,n+1}_{12001} - 2A^{0,n+1}_{02110} - A^{1,n}_{01111} + A^{1,n}_{11011} - A^{1,n}_{11101} + A^{1,n}_{11110} + (\Delta\cdot p)A^{0,n}_{01111}\right]$$
$$+(D-8)(D-2)\left[A^{0,n+1}_{01111} - A^{0,n+1}_{11011} - (\Delta\cdot p)A^{0,n}_{10111} + (\Delta\cdot p)A^{0,n}_{11101}\right] + 16(D-3)m^2\left[A^{0,n+1}_{12011} + A^{0,n+1}_{12101}\right]$$
$$+4(D-4)m^2\left[(\Delta\cdot p)A^{0,n}_{11111} - A^{0,n+1}_{11111}\right] + 8m^2A^{0,n+1}_{02111} + 8m^2A^{0,n+1}_{12110} + 32m^4A^{0,n+1}_{12111}$$
$$4(D-2)\left[A^{0,n+1}_{11101} - A^{0,n+1}_{11110} - 2B^{0,n+1}_{12011} - 2B^{0,n+1}_{12101} + (\Delta\cdot p)A^{0,n}_{11011}\right] \;,$$

Another example:



$$= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\mathrm{Tr}[\gamma^\mu (\not{k}_1 + m) \not{\Delta} (\not{k}_2 + m)(\not{p} - \not{k}_1 + \not{k}_2 + m) \not{\Delta} (\not{p} + m)]}{D_1 D_2 D_3 D_4 D_5} \sum_{j=0}^{n-1} (\Delta\cdot k_1)^j (\Delta\cdot k_2)^{n-1-j} =$$

$$= 2(D-4)\left[E^{2,0}_{01111} - E^{2,0}_{11011} - E^{2,0}_{11101} + E^{2,0}_{11110} + (\Delta\cdot p)E^{0,0}_{10111}\right] - 4E^{0,2}_{01111} + 4E^{0,2}_{11011} + 4(\Delta\cdot p)E^{0,1}_{10111}$$
$$+2(D-2)\left[E^{0,1}_{11011} - E^{1,0}_{11011} - E^{1,0}_{11110} - (\Delta\cdot p)E^{0,0}_{01111} + (\Delta\cdot p)E^{0,0}_{10111} - (\Delta\cdot p)E^{0,0}_{11101}\right]$$
$$+2(D-6)m^2\left[E^{1,1}_{11011} - E^{1,1}_{01111}\right] + 4(D-3)(\Delta\cdot p)\left[E^{1,0}_{11101} + E^{1,0}_{01111}\right] + 4E^{1,1}_{11101} + 4E^{1,1}_{11110}$$
$$+8m^2 E^{1,1}_{11111} - 8m^2 E^{2,0}_{11111} - 4(\Delta\cdot p)E^{0,1}_{11101} + 8m^2(\Delta\cdot p)E^{0,1}_{11111} - 2D(\Delta\cdot p)E^{1,0}_{10111} + 8m^2(\Delta\cdot p)E^{1,0}_{11111}$$

Type A integrals

ν_1	ν_2	ν_3	ν_4	ν_5	$(a\ ,\ b)$
1	1	1	1	0	
1	1	1	0	1	
1	1	0	1	1	
1	0	1	1	1	0,n
0	1	1	1	1	1,n
2	1	1	1	0	n,0
1	2	1	1	0	n,1
1	2	1	0	1	
2	1	0	1	1	
0	2	1	1	1	
2	0	1	1	1	n,0
1	1	1	0	2	n,1
1	1	0	1	2	
1	1	1	2	0	0,n
0	1	1	2	1	1,n
1	0	1	1	2	
1	0	1	2	1	
2	1	1	0	2	0,n
1	2	1	0	2	n,0
3	1	1	1	0	
0	3	1	1	1	n,0
1	3	1	0	1	
2	2	1	1	0	
2	1	2	1	0	
1	1	3	1	0	
1	2	2	1	0	
0	1	3	1	1	
0	2	2	1	1	0,n
3	1	1	0	1	
1	1	2	0	2	
2	1	2	0	1	
1	0	3	1	1	
3	0	1	1	1	
2	0	2	1	1	

Type B integrals

ν_1	ν_2	ν_3	ν_4	ν_5	$(a\ ,\ b)$
2	1	1	1	0	
2	1	1	0	1	n,0
2	1	0	1	1	
1	2	1	1	0	
1	2	1	0	1	0,n
1	2	0	1	1	

Type E and F integrals

1	1	1	1	0	
1	1	1	0	1	1,0
1	1	0	1	1	0,1
0	1	1	1	1	1,1
1	2	1	0	1	2,0
1	1	1	0	2	
0	2	1	1	1	
					0,0
1	0	1	1	1	1,0
					0,1
0	1	1	1	1	0,0
1	1	0	1	1	0,2
2	1	1	1	0	1,1

In total there are 155+ integrals

We do the 5-propagator integrals using the following IBP identities.

$$A^{n,0}_{11111} = \frac{1}{\epsilon}\left(A^{n,0}_{12101} - A^{n,0}_{02111} + A^{n,0}_{11102}\right)$$

$$A^{0,n}_{11111} = \frac{1}{\epsilon}\left(A^{0,n}_{21011} + A^{0,n}_{11120} + A^{0,n}_{10121} - A^{0,n}_{10121} + A^{0,n}_{11012} - A^{0,n}_{10112}\right)$$

$$A^{n,0}_{21111} = -\frac{1}{\epsilon}\left(A^{n,0}_{22101} + A^{n,0}_{21102}\right) + \frac{1}{(1-\epsilon)}\left(2A^{n,0}_{03111} - 2A^{n,0}_{13101} - A^{n,0}_{12102}\right)$$

$$\begin{aligned} A^{0,n}_{12111} = -\frac{1}{\epsilon}\Bigg[& - A^{0,n}_{12210} - A^{0,n}_{02211} - A^{0,n}_{12102} - A^{0,n}_{22101} \\ & + \frac{1}{1-\epsilon}\Big(-2A^{0,n}_{31101} + 2A^{0,n}_{30111} - A^{0,n}_{21210} + 2A^{0,n}_{20211} - A^{0,n}_{21102} \\ & -A^{0,n}_{21201} - 2A^{0,n}_{11310} - 2A^{0,n}_{10311} - A^{0,n}_{01311} - A^{0,n}_{11202}\Big) \Bigg] \end{aligned}$$

The 4-propagator integrals can be represented in terms of up to three Feynman parameter integrals over the unit cube. In some cases, a direct calculation will give integrals with the following structure

$$I(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \; x^n f(x, y, z; \varepsilon),$$

while in other cases they will be of the form

$$I(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \; x^n y^n f(x, y, z; \varepsilon).$$

In the first case, the integrals turn out to be expressed directly as Mellin transforms after integrating in y and z. In the second case, the Feynman parameters can be rearranged into the first form using the following change of variables

$$x' = xy, \quad y' = \frac{x(1-y)}{1-xy},$$

$$
\begin{aligned}
A_{12111}^{0,n} \quad = \quad \int_0^1 dx\; x^n \Biggl\{ & \frac{2}{3}\mathrm{Li}_2(1-x) + \frac{1}{3}\ln^2(x) + \frac{2}{3(1-x)}\ln(x) + \frac{2}{3}\zeta_2 \\
& - \frac{1}{3}(1-x)^{-3+2\epsilon}\ln^2(x) - \frac{2}{3}(1-x)^{-2+2\epsilon}\ln(x) \\
& + \frac{2}{3}(1-x)^{-1+2\epsilon} - \epsilon(1-x)^{-1+2\epsilon}(2\zeta_2 - 1) \\
& + (-1)^n \Biggl[\frac{4}{3}\mathrm{Li}_2(-x) - \frac{2}{3}\left(1 - \frac{2}{(1+x)^3}\right)\mathrm{Li}_2(1-x) \\
& \qquad - \left(1 - \frac{1}{(1+x)^3}\right)\ln^2(x) + \frac{4}{3}\ln(x)\ln(1+x) \\
& \qquad + \frac{2}{3(1+x)}\left(\frac{2}{(1+x)^2} - \frac{1}{(1+x)} - 1\right)\ln(x) \\
& \qquad - \frac{2}{3(1+x)} + \frac{4}{3(1+x)^2} + \frac{2}{3}\zeta_2\Biggr]\Biggr\}
\end{aligned}
$$

There are also integration by parts identities for the G integrals with 5 propagators:

$$E^{0,1}_{11111} = \frac{1}{1-\epsilon}\Big(A^{n,0}_{11111} + (1+n)A^{0,n}_{11111} + E^{0,1}_{12101} + E^{0,1}_{11102} + E^{0,1}_{02111}\Big)$$

$$E^{1,0}_{11111} = \frac{1}{1-\epsilon}\Big(nA^{1,n-1}_{11111} + +E^{1,0}_{12101} + E^{1,0}_{11102} + E^{1,0}_{02111}\Big)$$

$$E^{1,1}_{11111} = \frac{1}{1-\epsilon}\Big(A^{n+1,0}_{11111} + (1+n)A^{1,n}_{11111} + +E^{1,1}_{12101} + E^{1,1}_{11102} + E^{1,1}_{02111}\Big)$$

$$E^{2,0}_{11111} = \frac{1}{1-\epsilon}\Big(nA^{2,n-1}_{11111} + +E^{1,1}_{12101} + E^{1,1}_{11102} + E^{1,1}_{02111}\Big)$$

There are similar relations for the F integrals. The factors of n multiplying the A integrals in these relations can be reabsorbed into x^n by integration by parts.

We checked the integrals by several means:

- Use of the `Mathematica` package `Tarcer` (R. Mertig and R. Scharf, hep-ph/9801383), which allows to check the integrals for a few low moments.

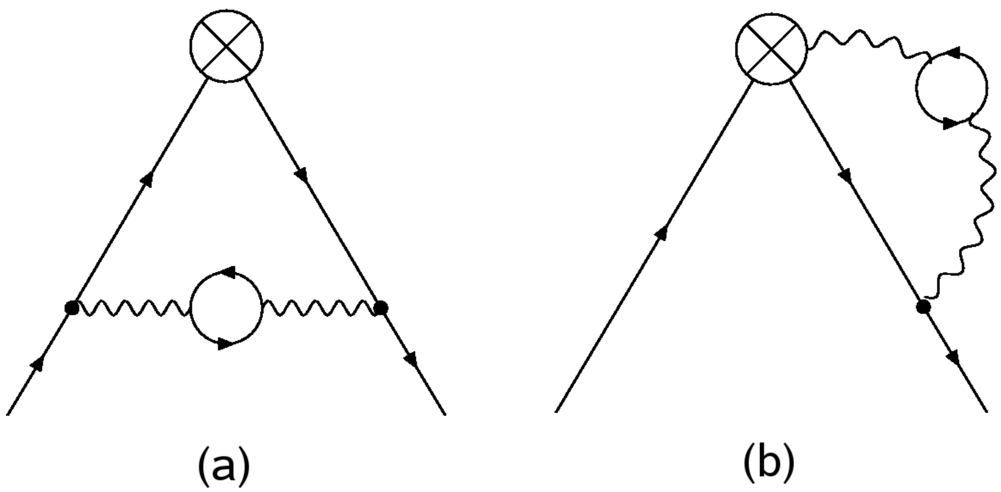
- Use of Mellin-Barnes integrals (V.A. Smirnov, hep-ph/9905323), and then use of the MB package (M. Czakon, hep-ph/0511200) to integrate numerically.
  - Advantage: can check very high moments at some numerical accuracy.
  - Disadvantage: couldn't find contour for some of our integrals.

- Additional: integration by parts identities.

The result for the the matrix element $\hat{\Gamma}_{ee}^{(1),\mathrm{I}}$ is

$$
\begin{aligned}
&\frac{1+3x^2}{1-x}\left[6\zeta_2\ln(x)-8\ln(x)\mathrm{Li}_2(1-x)-4\ln^2(x)\ln(1-x)\right]+\left(\frac{122}{3}x+22+\frac{32}{1-x}\right)\zeta_2+(8-112\zeta_2)\,\mathcal{D}_1(x)\\
&+16\frac{1+x^2}{1-x}\left[2\mathrm{Li}_3(-x)-\ln(x)\mathrm{Li}_2(-x)\right]+\frac{80}{3(1-x)}+56(1+x)\zeta_2\ln(1-x)+(16-52\zeta_2+128\zeta_3)\,\mathcal{D}_0(x)\\
&+\left(\frac{22}{3}x+32+\frac{64}{3(1-x)^2}-\frac{51}{1-x}-\frac{16}{3(1-x)^3}\right)\ln^2(x)-(92+20x)\ln^2(1-x)+14(x-2)\ln(1-x)+120\mathcal{D}_2(x)\\
&+\left(\frac{178}{3}-36x+\frac{64}{3(1-x)^2}-\frac{140}{3(1-x)}-\frac{48}{1+x}\right)\ln(x)-\frac{1}{3}(1+x)\ln^3(x)+4\frac{x^2-8x-6}{1-x}\ln(x)\ln(1-x)\\
&-2\frac{1+17x^2}{1-x}\ln(x)\ln^2(1-x)-\frac{112}{3}(1+x)\ln^3(1-x)+32\frac{1+x}{1-x}\left[\ln(x)\ln(1+x)+\mathrm{Li}_2(-x)\right]-22x-\frac{62}{3}\\
&-4\frac{13x^2+9}{1-x}\mathrm{S}_{1,2}(1-x)+4\frac{5-11x^2}{1-x}\left[\ln(1-x)\mathrm{Li}_2(1-x)-\mathrm{Li}_3(1-x)-2\zeta_3\right]+\frac{4(16x^2-10x-27)}{3(1-x)}\mathrm{Li}_2(1-x)\\
&+\frac{224}{3}\mathcal{D}_3(x)+\left[\frac{433}{8}-\frac{67}{45}\pi^4+\left(\frac{37}{2}-48\ln(2)\right)\zeta_2+58\zeta_3\right]\delta(1-x)+(-1)^n\left\{\frac{2(1-x)(45x^2+74x+45)}{3(1+x)^2}\right.\\
&+\frac{2(9+12x+30x^2-20x^3-15x^4)}{3(1+x)^3}\ln(x)+\frac{4(x^2+10x-3)}{3(1+x)}\left(\zeta_2+2\mathrm{Li}_2(-x)+2\ln(x)\ln(1+x)\right)\\
&+\frac{1+x^2}{1+x}\Big[36\zeta_3-24\zeta_2\ln(1+x)+8\zeta_2\ln(x)-\frac{2}{3}\ln^3(x)+40\mathrm{Li}_3(-x)-4\ln^2(x)\ln(1+x)-24\ln(x)\ln^2(1+x)\\
&-24\ln(x)\mathrm{Li}_2(-x)-48\ln(1+x)\mathrm{Li}_2(-x)-8\ln(x)\mathrm{Li}_2(1-x)-16\mathrm{S}_{1,2}(1-x)-48\mathrm{S}_{1,2}(-x)\Big]\\
&\left.-\frac{16(x^4+12x^3+12x^2+8x+3)}{3(1+x)^3}\mathrm{Li}_2(1-x)+4x\frac{1-x-5x^2+x^3}{(1+x)^3}\ln^2(x)\right\}
\end{aligned}
$$

Diagrams for process II:



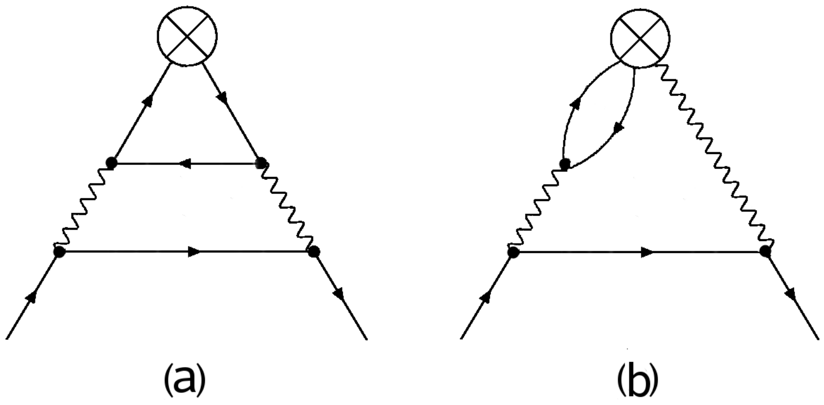
Very simple to calculate. We use the well known result for the one-loop vacuum diagram:

$$\Pi_2^{\mu\nu}(q) = -\frac{8e^2}{(4\pi)^{D/2}}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\int_0^1 dx \frac{\Gamma(2-D/2)\, x(1-x)}{(m^2 - x(1-x)q^2)^{2-D/2}}\ ,$$

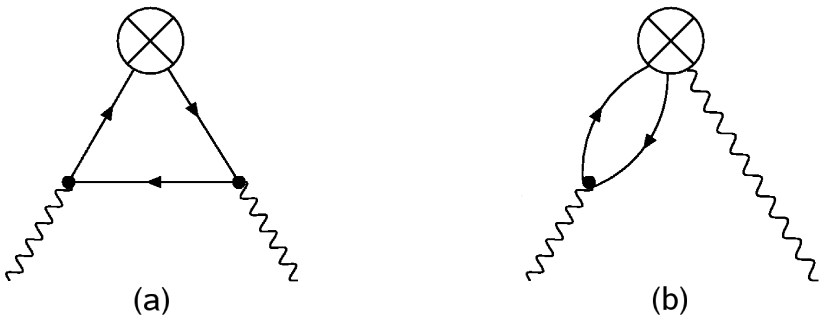
The result for $\hat{\Gamma}^{(1),\mathrm{II}}_{ee}$ is

$$\frac{76}{27}x-\frac{572}{27}-\left(12x+\frac{4}{3}+\frac{8}{1-x}\right)\ln(x)+\frac{128}{9(1-x)^2}+\frac{80}{27(1-x)}-\frac{64}{9(1-x)^3}$$
$$-\frac{32}{9}\left(\frac{1}{(1-x)^2}-\frac{5}{(1-x)^3}+\frac{2}{(1-x)^4}\right)\ln(x)+\frac{16}{3}(1+x)\left(\ln(1-x)+\ln^2(1-x)\right)$$
$$-\frac{2(1+x^2)}{3(1-x)}\ln^2(x)+\left(\frac{224}{27}-\frac{8}{3}\zeta_2\right)\mathcal{D}_0(x)+\frac{4}{3}(1+x)\zeta_2-\frac{32}{3}\left(\mathcal{D}_1(x)+\mathcal{D}_2(x)\right)$$
$$+\left(\frac{8}{3}\zeta_3+10\zeta_2-\frac{1411}{162}\right)\delta(1-x)$$

Diagrams for process III:



They can be calculated using the one-loop insertions



which can be found in Blümlein, Bierenbaum and Klein, Nucl. Phys. B 780 (2007) 40

For diagram (a) one obtains

$$
\begin{aligned}
I^{(a)}_{\mu\nu}(k) \;=\;& 4\frac{(\Delta\cdot k)^{n-1}}{(4\pi)^{D/2}}\Gamma(3-D/2)\int_0^1 dx\; x^{n+D/2-2}(1-x)^{-2+D/2}\Bigg[\\
& -2x(1-x)\frac{g_{\mu\nu}k^2-2k_\mu k_\nu}{(-k^2+M^2)^{3-D/2}}(\Delta\cdot k)^2-\frac{2m^2 g_{\mu\nu}}{(-k^2+M^2)^{3-D/2}}(\Delta\cdot k)^2\\
& +(1-x)\frac{2(n+1)x-n}{2-D/2}\,\frac{k_\mu\Delta_\nu+k_\nu\Delta_\mu}{(-k^2+M^2)^{2-D/2}}(\Delta\cdot k)\\
& +(1-x)\frac{n(1-2x)-Dx}{2-D/2}\,\frac{g_{\mu\nu}}{(-k^2+M^2)^{2-D/2}}(\Delta\cdot k)^2\\
& -(1-x)n\frac{(n+1)(1-x)-1}{(1-D/2)(2-D/2)}\,\frac{\Delta_\mu\Delta_\nu}{(-k^2+M^2)^{1-D/2}}\Bigg]\;,
\end{aligned}
$$

while diagram (b) gives

$$
I^{(b)}_{\mu\nu}(k)=8\frac{(\Delta\cdot k)^{n-1}}{(4\pi)^{D/2}}\Gamma(2-D/2)\int_0^1 dx\;(x-x^2)^{D/2-1}(x^n+(1-x)^n)\left[\frac{(\Delta\cdot k)\Delta_\mu k_\nu-k^2\Delta_\mu\Delta_\nu}{(-k^2+M^2)^{2-D/2}}\right]\;,
$$

The result for $\hat{\Gamma}_{ee}^{(1),\mathrm{III}}$ is

$$\frac{2}{x}(1-x)(4x^2+13x+4)\zeta_2+\frac{1}{3x}(8x^3+135x^2+75x+32)\ln^2(x)$$
$$+\left[\frac{304}{9x}-\frac{80}{9}x^2-\frac{32}{3}x+108-\frac{32}{1+x}-\frac{64(1+2x)}{3(1+x)^3}\right]\ln(x)-\frac{224}{27}x^2$$
$$+16\frac{1-x}{3x}(x^2+4x+1)\left[2\ln(x)\ln(1+x)-\mathrm{Li}_2(1-x)+2\mathrm{Li}_2(-x)\right]$$
$$+(1+x)\Bigg[4\zeta_2\ln(x)+\frac{14}{3}\ln^3(x)-32\ln(x)\mathrm{Li}_2(-x)-16\ln(x)\mathrm{Li}_2(x)+64\mathrm{Li}_3(-x)$$
$$+32\mathrm{Li}_3(x)+16\zeta_3\Bigg]-\frac{182}{3}x+50-\frac{32}{1+x}+\frac{800}{27x}+\frac{64}{3(1+x)^2}$$

The first moment vanishes for all three contributions $\hat{\Gamma}_{ee}^{(1),\mathrm{I}}$, $\hat{\Gamma}_{ee}^{(1),\mathrm{II}}$ and $\hat{\Gamma}_{ee}^{(1),\mathrm{II}}$.

→ Fermion number conservation is satisfied.

The 2–loop corrections to the process $e^+e^- \to Z^0$ can be organized in the following form :

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s)\left\{1 + a_0\left[T_{11}\hat{\mathbf{L}} + T_{10}\right] + a_0^2\left[T_{22}\hat{\mathbf{L}}^2 + T_{21}\hat{\mathbf{L}} + T_{20}\right]\right\}$$

$$a_0 = \frac{\alpha(m_e^2)}{4\pi}$$

- Universal Corrections : $T_{ii}(z)$ $\Longrightarrow$ depend on LO splitting functions and β_0

$$T_{11} = 8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z) = 4\left[\frac{1+z^2}{1-z}\right]_+ ,$$

$$\begin{aligned} T_{22} = & \Big\{64\mathcal{D}_1(z) + 48D_0(z) + (18 - 32\zeta_2)\delta(1-z) \\ & -32\frac{\ln(z)}{1-z} - 32(1+z)\ln(1-z) + 24(1+z)\ln(z) - 8(5+z)\Big\}_{\text{I}} \\ & +\frac{2}{3}\Big\{8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z)\Big\}_{\text{II}} + 16\Big\{\frac{1}{2}(1-z)\ln(z) + \frac{1}{4}(1-z) + \frac{1}{3}\frac{1}{3z}(1-z^3)\Big\}_{\text{III}} . \end{aligned}$$

- $O(\alpha)$ Term : $T_{10}(z)$ $\Longrightarrow$ depend on LO OME + LO DY

$$T_{10} = -4\left[\frac{1+z^2}{1-z}\right]_+ + 2(4\zeta_2 - 1)\delta(1-z)$$

$$T_{11}\hat{\mathbf{L}} + T_{10} = P_{ee}^{(0)}(z)\left[\hat{\mathbf{L}} - 1\right] + 2(4\zeta_2 - 1)\delta(1-z)\ .$$

Complete 1-Loop Result.

- $O(\alpha^2\hat{\mathbf{L}})$ Terms : $T_{21}(z)$ $\Longrightarrow$ depend on LO,NLO splitting fcts., LO OME + LO DY

Contributions to the three main processes I-III :

$$T_{21}^{\rm I} = 16\left\{-8\mathcal{D}_1(z) - (7-4\zeta_2)\mathcal{D}_0(z) + \left(-\frac{45}{16} + \frac{11}{2}\zeta_2 + 3\zeta_3\right)\delta(1-z)\right.$$
$$+\left(\frac{1+z^2}{1-z}\right)\left[\ln(z)\ln(1-z) - \ln^2(z) + \frac{11}{4}\ln(z)\right]$$
$$\left. +(1+z)\left[4\ln(1-z) + \frac{1}{4}\ln^2(z) - \frac{7}{4}\ln(z) - 2\zeta_2\right] - \ln(z) + 3 + 4z\right\}$$

$$T_{21}^{\mathrm{II}} = 16\left\{\frac{4}{3}\mathcal{D}_1(z) - \frac{10}{9}\mathcal{D}_0(z) - \frac{17}{12}\delta(1-z) - \frac{2}{3}\frac{\ln(z)}{1-z} - \frac{1}{3}(1+z)\left[2\ln(1-z) - \ln(z)\right] - \frac{1}{9} + \frac{11}{9}z\right\}$$

$$T_{21}^{\mathrm{III}} = 16\left\{(1+z)\left[2\mathrm{Li}_2(1-z) - \ln^2(z) + 2\ln(z)\ln(1-z)\right] + \left(\frac{4}{3}\frac{1}{z} + 1 - z - \frac{4}{3}z^2\right)\ln(1-z) - \left(\frac{2}{3}\frac{1}{z} + 1 - \frac{1}{2}z - \frac{4}{3}z^2\right)\ln(z) - \frac{8}{9}\frac{1}{z} - \frac{8}{3} + \frac{8}{3}z + \frac{8}{9}z^2\right\}$$

Up to this point, we find agreement with Berends et al. (1988).

So, the factorization of the QED initial state corrections to e^+e^- annihilation into a virtual boson for large cms energies into massive OMEs and the massless Wilson coefficients of the Drell-Yan process (adapting the colors factors to the case of QED), works at one-loop order and at two-loop order up to linear order in $\ln(s/m_e^2)$ at present.

Our calculations show explicitly that inserting local operators with external fermion lines, this procedure does not give the correct result for the two-loop constant term.. For example, we saw that

$$
\begin{aligned}
\hat{\Gamma}_{ee}^{(1),\mathrm{II}} = \; & \frac{76}{27}x - \frac{572}{27} - \left(12x + \frac{4}{3} + \frac{8}{1-x}\right)\ln(x) + \frac{128}{9(1-x)^2} + \frac{80}{27(1-x)} - \frac{64}{9(1-x)^3} \\
& - \frac{32}{9}\left(\frac{1}{(1-x)^2} - \frac{5}{(1-x)^3}\right)\ln(x) + \frac{2}{(1-x)^4}\ln(x) + \frac{16}{3}(1+x)\left(\ln(1-x) + \ln^2(1-x)\right) \\
& - \frac{2(1+x^2)}{3(1-x)}\ln^2(x) + \left(\frac{224}{27} - \frac{8}{3}\zeta_2\right)\mathcal{D}_0(x) + \frac{4}{3}(1+x)\zeta_2 - \frac{32}{3}\left(\mathcal{D}_1(x) + \mathcal{D}_2(x)\right) \\
& + \left(\frac{8}{3}\zeta_3 + 10\zeta_2 - \frac{1411}{162}\right)\delta(1-x)
\end{aligned}
$$

The terms highlighted in red don't appear neither in the final result by BBN, nor in the massless Wilson coefficients. Further investigations are needed to clarify this.

7. Conclusions

1. We calculated the $O(\alpha^2)$ massive operator matrix elements in QED, which contribute to the 2-loop initial state corrections for $e^+e^- \to Z^*/\gamma^*$ in the limit $m_f^2/s \to 0$ using the renormalization group method.
2. We have obtained all logarithmic contributions $O((\alpha L)^2), O(\alpha^2 L), O(\alpha L)$ and the constant contributions $O(\alpha)$ correctly.
3. The literal application of the $s \gg m_f^2$ expansion, as proposed by BBN does not deliver the result obtained by conventional integration. Here, local operator matrix elements were used.
4. On the other hand, we obtained our results for the 2-loop matrix elements by two independent methods, which agree on the results. Furthermore, the complete OMEs obey Fermion number conservation, and renormalize as expected. The 2-loop anomalous dimensions are correctly obtained.
5. In case of massless external lines, massive OMEs can be calculated without any problem and the results agree in all cases investigated with that obtained in the limit $m^2/\mu^2 \to 0$. We conjecture, that the present problem is caused dealing with massive external fermions being on-shell, which might imply different Feynman rules for the operators. We will pin down the source for this further.