R(s) and Z decay in $\mathcal{O}(\alpha_s^4)$: complete results

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in collaboration with

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RADCOR 2011

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Final results of a Large Karlsruhe based project:

- 10 years of work of the Karlsruhe multiloop group:

+ many talks on RADCORS's and LOOPS& LEGS, starting from "Five-loop vacuum polarization in pQCD: $O(\alpha_s^4 N_f^2)$ results" (Talk presented at RADCOR/Loops and Legs 2002, Kloster Banz, Germany)
+ ...
+ ...
+ recent works
  Phys.Rev.Lett.101:012002,2008; arXiv:0801.1821
  Phys.Rev.Lett.104:132004,2010; arXiv:1001.3606v1
  Nucl.Phys.B837:186-220,2010; arXiv:1004.1153 ++
  Nucl.Phys.Proc.Suppl.205-206:237-241,2010;
  P. Baikov, K. Chetyrkin, J. K. and J. Rittinger, in preparation

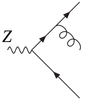
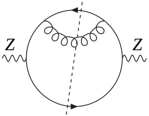
New Results to report

- **(new!) result for the singlet contribution into the (massless) $\langle VV \rangle$ corelator $\Longrightarrow$ final complete $\mathcal{O}(\alpha_s^4)$ result for the R(s)**

- **(new!) as spin-off: the QED β-function in five loops**

- **(new!) final complete result for $Z \to$ hadrons at $\mathcal{O}(\alpha_s^4)$ (the latter including in full power-not-suppressed top-mass dependence)**

$$\Gamma(Z \to \text{hadrons}) = \sum_{f_{QCD}} \int \mathrm{d}\Phi \left| \mathcal{M}(Z \to f_{QCD}) \right|^2$$

$= \int \mathrm{d}\Phi \Big|$  $\Big|^2 + \ \ldots \ \xrightarrow[Opt.Th.]{} \ \mathrm{Im}\Big[$  $\Big] + \ \ldots$

$$= \Pi^{\mu\nu} = i \int e^{iqx} \langle 0 | \; T \, j_Z^{\mu}(x) j_Z^{\nu}(0) \; | 0 \rangle \; \mathrm{d}x$$

$$= g^{\mu\nu} \Pi_1(-q^2) + q^{\mu} q^{\nu} \Pi_2(-q^2)$$

$$\Gamma(Z \to \text{hadrons}) = \Gamma_0 \cdot \frac{2\pi i}{s} \Big(\Pi_1(s - i\varepsilon) - \Pi_1(s + i\varepsilon) \Big)$$

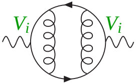
$$= \Gamma_0 \cdot R(s) \qquad \left(\Gamma_0 = \frac{G_F M_Z^3}{8\pi\sqrt{2}} \right)$$

Vector and Axial Contributions to Z-decay

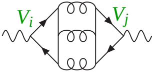
Z current to fermion i:

$$j^\mu_{Z_i} = g_i^V\, j^\mu_{V_i} + g_i^A\, j^\mu_{A_i} = g_i^V\, \overbrace{\overline{\psi}_i\, \gamma^\mu\, \psi_i}^{=V_i} + g_i^A\, \overbrace{\overline{\psi}_i\, \gamma^\mu \gamma_5\, \psi_i}^{=A_i}$$

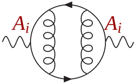
$= \sum_i (g_i^V)^2$



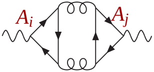
$+ \sum_{i,j} g_i^V\, g_j^V$



$+ \sum_i (g_i^A)^2$

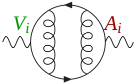


$+ \sum_{i,j} g_i^A\, g_j^A$

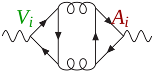


$$i, j = \{t, b, c, s, u, d\}$$

NO



and

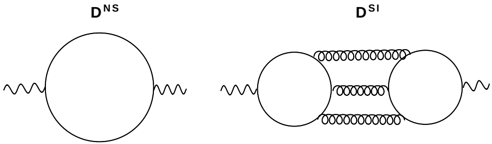


• status of theory (in the massless limit) •

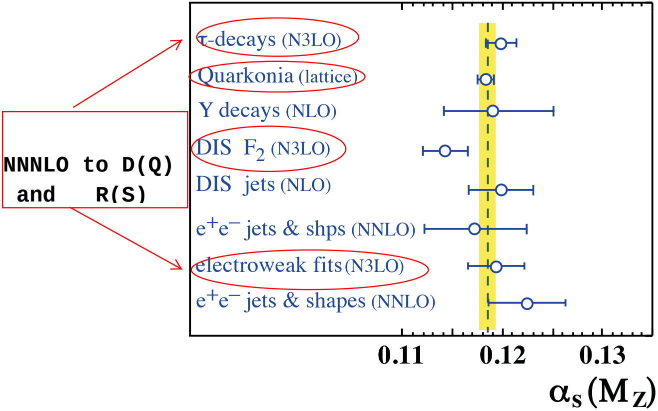
$$R^{NS} = 3\sum_i Q_i^2 \left(1 + \frac{\alpha_s}{\pi} + \# \left(\frac{\alpha_s}{\pi}\right)^2 + \# \left(\frac{\alpha_s}{\pi}\right)^3 + \# \left(\frac{\alpha_s}{\pi}\right)^4 + \cdots \right)$$

parton model	QED Källen+ Sabry 1955	Chetyrkin, Kataev, Tkachov; Dine, Sapirstein; Celmaster 1979	Gorishny, Kataev, Larin; Surguladze, Samuel 1991 Chetyrkin /gen. gauge/ 1996	Baikov, Chetyrkin, Kühn 2008; Baikov, Chetyrkin, Kühn 2010 (Feynman Gauge only)

$$R^{SI} = \left(\sum_i Q_i\right)^2 \left(\# \left(\frac{\alpha_s}{\pi}\right)^3 + \boxed{??\left(\frac{\alpha_s}{\pi}\right)^4} + \cdots \right)$$



World Summary of α_s 2009:



-> $\alpha_s(M_Z) = 0.1184 \pm 0.0007$

(Bethke, arXiv:0908.1135)

Singlet contribution to the (vector) Adler function (Last missing term!)

$$D^{SI}(Q^2) = d_R \left(\sum_{i=3}^{\infty} d_i^{SI}\, a_s^i(Q^2) \right)$$

$$d_3^{SI} = \frac{d^{abc}d^{abc}}{d_R} \left(\frac{11}{192} - \frac{1}{8}\zeta_3 \right), \; d_4^{SI} = \frac{d^{abc}d^{abc}}{d_R} \left(C_F\, d_{4,1}^{SI} + C_A\, d_{4,2}^{SI} + T\, n_f\, d_{4,3}^{SI} \right)$$

$$d_{4,1}^{SI} \;=\; = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad d_{4,3}^{SI} \;=\; \frac{-149}{576} + \frac{13}{32}\zeta_3 - \frac{5}{16}\zeta_5 + \frac{1}{8}\zeta_3^2$$

$$d_{4,2}^{SI} \;=\; \frac{3893}{4608} - \frac{169}{128}\zeta_3 + \frac{45}{64}\zeta_5 - \frac{11}{32}\zeta_3^2$$

Numerically:

$$
\begin{aligned}
R(s) \;=\;& 3\sum_f Q_f^2 \left\{1 + a_s + a_s^2\,(1.986 - 0.1153 n_f)\right. \\
&+ \left. a_s^3\,(-6.637 - 1.200 n_f - 0.00518 n_f^2)\right\} \\
&- \left(\sum_f Q_f\right)^2 \left(1.2395\, a_s^3 + \underline{(-17.8277 + 0.57489 n_f)\; a_s^4}\right)
\end{aligned}
$$

for n_f=5

$$
\frac{11}{3}\left[1 + a_s + a_s^2\,1.409 - 12.767\, a_s^3 - 79.98\, a_s^4\right] + \frac{1}{9}\left[-1.240\, a_s^3 - 14.95\, a_s^4\right]
$$

Extra suppression factor $\frac{3}{99} \approx 0.03$**!**

QED β-function in five loops

By a proper change of color factors we arrive at the full Adler function of QED in five loops $\Longrightarrow$ the QED β-function; for a QED with one charged fermion we get ($A \equiv \frac{e^2}{16\pi^2}$)

$$\beta^{QED} = \frac{4}{3} A + 4 A^2 - \frac{62}{9} A^3 - A^4 \left(\frac{5570}{243} + \frac{832}{9} \zeta_3 \right)$$

Gorishny, Kataev, Larin, Surguladze, 1991

$$-A^5 \boldsymbol{\left(\frac{195067}{486} + \frac{800}{3} \zeta_3 + \frac{416}{3} \zeta_4 - \frac{6880}{3} \zeta_5 \right)}$$

Numerically ($\mathbf{A} = \frac{\alpha}{4\pi} \approx \mathbf{5.81 \cdot 10^{-4}}$)

$$\beta^{QED} = \frac{4}{3} A \left(1 + 3 A - 5.1667 A^2 - 100.534 A^3 + 1129.51 A^4 \right)$$

No hope for a non-trivial fixed point solution $\beta(A^*) = 0$!

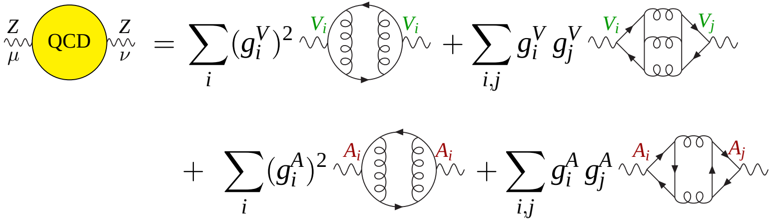
Tool-box for massless correlators at α_s^4:

- **IRR / Vladimirov, (78)/ + IR R^* -operation /Chetyrkin, Smirnov (1984)/ + resolved combinatorics /Chetyrkin, (1997)/**
- **reduction to Masters: "direct and automatic" construction of CF's through $1/D$ expansion within the Baikov's representation for Feynman integrals (Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003)**
- **algebraic /Baikov, Chetyrkin (2010); R. Lee, A. Smirnov, V. Smirnov, (2011)/ and numerical /A. Smirnov, M. Tentyukov (2010)/ evaluation of all masters (the former with two absolutely different methods!)**
- **computing: MPI-based (PARFORM) as well as thread-based (TFORM) versions of FORM**
  Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

Vector and Axial Contributions to the Z-decay

Z current to fermion i:

$$j^\mu_{Z_i} = g_i^V\, j^\mu_{V_i} + g_i^A\, j^\mu_{A_i} = g_i^V \overbrace{\bar{\psi}_i\, \gamma^\mu\, \psi_i}^{=V_i} + g_i^A \overbrace{\bar{\psi}_i\, \gamma^\mu \gamma_5\, \psi_i}^{=A_i}$$



$i, j = \{t, b, c, s, u, d\}$

- scale $\sqrt{s} = M_z \quad \Rightarrow \quad m_l = 0 \quad (l = \{b, c, s, u, d\})$
- 2 scales $\sqrt{s} = M_Z$ and m_t (but $M_Z^2/(4m_t^2) \ll 1$): Decoupling of top (factorization of top effects)

‘t Hooft-Veltman-Larin Treatment of γ_5

Problem: naive γ_5 ($[\gamma_5, \gamma_\alpha] = 0$) is not applicable for the singlet diagrams.

Solution

‘t Hooft-Veltman ‘72: treat $\epsilon^{\mu\mu_1\mu_2\mu_3}$ as 4-dimensional object

$$\gamma_5 = \frac{i}{4!}\epsilon^{\mu_1\mu_2\mu_3\mu_4} \; \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}$$
$$(i = \{t, b, c, s, u, d\})$$

Larin ‘93: introduce a special prefactor

$$\zeta_A = 1 - \frac{4}{3}a_s + \ldots$$

and define the axial vector current as:

$$A_i = \overline{\psi}_i \, \gamma^\mu \gamma_5 \, \psi_i = \zeta_A \, \frac{i}{6} \, \epsilon^{\mu\mu_1\mu_2\mu_3} \, \overline{\psi}_i \, \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} \, \psi_i$$

(ζ_A effectively restores antisymmetry of γ_5 in *NS* diagrams; it is fixed **uniquely** from the (non-anomalous) Ward identity ,or, equivalently, by requiring scale inv. of the current $\zeta_A \, \frac{i}{6} \, \epsilon^{\mu\mu_1\mu_2\mu_3} \, [\overline{\psi}_i \, \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} \, \psi_i]_{\overline{\text{MS}}}$)

Decoupling of Top

Decoupling for QCD fields and coupling constant ($\zeta_2, \zeta_3, \bar{\zeta}_3$ and ζ_g):
6 flavour $\to$ 5 flavour

Vector current: naive decoupling (due to Ward Identity)

$$V_t^{(6)} \overset{m_t\to\infty}{=} 0 + \mathcal{O}(1/m_t^2)\,, \qquad V_b^{(6)} \overset{m_t\to\infty}{=} V_b^{(5)} + \mathcal{O}(1/m_t^2)$$

Axial vector current: **no** naive decoupling (due to potential anomaly if it it would!)

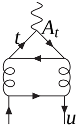
$$A_t^{(6)} \overset{m_t\to\infty}{=} C_h\, A_S^{(5)} + \mathcal{O}(1/m_t^2)\,, \qquad A_b^{(6)} \overset{m_t\to\infty}{=} A_b^{(5)} + C_\psi\, A_S^{(5)} + \mathcal{O}(1/m$$

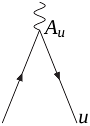
$$A_S^{(5)} = \sum_l A_l^{(5)}, \quad (A_t^{(6)} - A_b^{(6)}) \overset{m_t\to\infty}{=} (C_h - C_\psi)\, A_S^{(5)} + \mathcal{O}(1/m_t^2)$$

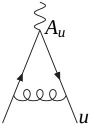
Decoupling of Top

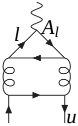
Calculation of C_h with method of projectors (Gorishny, Larin '86):

$$A_t^{(6)} \overset{m_t\to\infty}{=} C_h\left(a^{(6)}(\mu), \mu/m_t\right) A_S^{(5)}$$

$\langle T\bar{u}\, A_t\, u\rangle$: 

$\langle T\bar{u}\, A_S\, u\rangle$: 

, 

, 

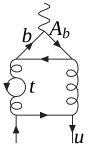
$\langle T\bar{u}\, A_i\, u\rangle\big|_{p=0}$: TAD's   Born , 0 , 0

Calculation of massive tadpoles:

- matching to topologies with EXP (Seidensticker and Steinhauser)
- reduction with Laporta alg. via CRUSHER (Marquard, Seidel)
- all master integrals up to 4 loop are known (Schröder, Vuorinen '05)

Calculation of C_ψ:

$$A_b{}^{(6)} \stackrel{m_t\to\infty}{=} A_b^{(5)} + C_\psi\big(a^{(6)}(\mu), \mu/m_t\big)\, A_S^{(5)}$$

$\langle \mathcal{T}\bar{u}\, A_b\, u\rangle$: 

Decoupling of Top

$$A_t^{(6)} \overset{m_t \to \infty}{=} C_h \, A_S^{(5)} \,, \qquad A_b^{(6)} \overset{m_t \to \infty}{=} A_b^{(5)} + C_\psi \, A_S^{(5)} \,.$$

$$\begin{aligned} C_h = &+ \left(\frac{\alpha^{(6)}(\mu)}{\pi}\right)^2 \left[+0.125 - 0.5 \ln\left(\frac{\mu^2}{m_t^2}\right)\right] \\ &+ \left(\frac{\alpha^{(6)}(\mu)}{\pi}\right)^3 \left[-0.515 - 0.417 \ln\left(\frac{\mu^2}{m_t^2}\right) - 0.875 \ln^2\left(\frac{\mu^2}{m_t^2}\right)\right] \\ &+ \left(\frac{\alpha^{(6)}(\mu)}{\pi}\right)^4 \left[-18.335 + 15.563 \ln\left(\frac{\mu^2}{m_t^2}\right) - 0.767 \ln^2\left(\frac{\mu^2}{m_t^2}\right) - 1.531 \ln^3\left(\frac{\mu^2}{m_t^2}\right)\right] \end{aligned}$$

$$\begin{aligned} C_\psi = &+ \left(\frac{\alpha^{(6)}(\mu)}{\pi}\right)^3 \left[-0.265 - 0.014 \ln\left(\frac{\mu^2}{m_t^2}\right) - 0.084 \ln^2\left(\frac{\mu^2}{m_t^2}\right)\right] \\ &+ \left(\frac{\alpha^{(6)}(\mu)}{\pi}\right)^4 \left[-1.282 + 0.393 \ln\left(\frac{\mu^2}{m_t^2}\right) - 0.406 \ln^2\left(\frac{\mu^2}{m_t^2}\right) - 0.132 \ln^3\left(\frac{\mu^2}{m_t^2}\right)\right] \end{aligned}$$

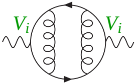
- 2 loop Collins, Wilczek and Zee '78
- 3 loop log enhanced terms Chetyrkin, Kühn '93
- 3 loop Larin, Ritbergen, Vermaseren '93; Chetyrkin, Tarasov '94
- 4 loop is new

Axial Vector Correlator

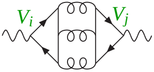
$$\Gamma(Z \to \text{hadrons}) = \Gamma_0 \cdot R(s) = \Gamma_0 \cdot \left(R^V(s) + R^A(s)\right)$$

Optical Theorem:

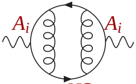
$= \sum_i (g_i^V)^2$



$+ \sum_{i,j} g_i^V g_j^V$

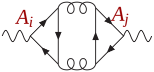


$+ \sum_i (g_i^A)^2$



R^{NS}

$+ \sum_{i,j} g_i^A g_j^A$



$R^{A,S}_{i,j}$

Decoupling: $A_t^{(6)} \overset{m_t \to \infty}{=} C_h \, A_S^{(5)}$, $\quad A_b^{(6)} \overset{m_t \to \infty}{=} A_b^{(5)} + C_\psi \, A_S^{(5)}$.

$$R^A \overset{\mathcal{O}(\alpha^4)}{=} \left(5\, R^{NS} + C_h^2 - 2(C_h - C_\psi)(R^{NS} + R^{A,S}_{b,S}) + R^{A,S}_{b,b}\right)$$

Master Formula for Γ_Z^h

$$\Gamma_Z^h = \Gamma_0 \times$$

$$\left(\sum_i (g_i^V)^2 + \sum_i (g_i^V)^2 \right) \left\{ 1 + a_s + 1.40923\, a_s^2 - 12.7671\, a_s^3 - 79.9806\, a_s^4 \right\}$$

$$+ \left(\sum_i g_i^V \right)^2 \left(-0.41318\, a_s^3 - 4.9841\, a_s^4 \right)$$

$$+ \Big[\left(-3.08333 - \ln_{M_t} \right) a_s^2$$

$$+ \left(-15.9877 - 3.72222\, \ln_{M_t} + 1.91667\, \ln^2_{M_t} \right) a_s^3$$

$$+ \left(49.162 + 17.6822\, \ln_{M_t} + 14.7153\, \ln^2_{M_t} - 3.67361\, \ln^3_{M_t} \right) a_s^4 \Big] \Big\}$$

$$\Gamma_0 = \frac{G_F\, M_Z^3}{8\, \pi \sqrt{2}}\,, \quad a_s = \frac{\alpha_s^{(5)}}{\pi}\,, \quad \ln_{M_t} = \ln \left(\frac{M_t^{\text{pole}}}{M_Z} \right)^2$$

Effective Master Formula for Γ_Z^h

Using as inputs $M_t = 172.0$, $s_W^2 = 0.232$ and $\alpha_s(M_Z) = .1190$ we arrive to an "effective" Master Formula illustrating the total effect of subleading singlet contributions in comparison to the non-singlet ones:

$$\Gamma_Z = \Gamma_0 \left(\sum_i (g_i^V)^2 + \sum_i (g_i^A)^2 \right) \Bigg\{$$

$$1 + a_s + 1.409\, a_s^2 - 12.77\, a_s^3 - \mathbf{79.99\, a_s^4} \quad \Longleftarrow \text{NS}$$

$$-0.1054\, a_s^3 - \boxed{\mathbf{1.272\, a_s^4}} \quad \Longleftarrow \text{V SI}$$

$$-0.6315\, a_s^2 - 3.0341\, a_s^3 + \boxed{\mathbf{11.452\, a_s^4}} \quad \Longleftarrow \text{A SI} \Bigg\}$$

Conclusions I

- The "10 years +" project of computing R(s) and $\Gamma(Z \to \text{hadrons})$ at order α_s^4 is finished!
- The last missing ingredients — singlet contributions— to the (massless) VV and AA as well as (massive) $\langle A_t A_t \rangle$ and $\langle A_t A_b \rangle$ correlators at $\mathcal{O}(\alpha_s^4)$ are now available
- singlet $\mathcal{O}(\alpha_s^4)$ contributions are numerically tiny
- the net effects of $\mathcal{O}(\alpha_s^4)$ term in Γ_Z^h are: an increase of $\delta\alpha_s(M_Z) = \mathbf{0.0005}$

$$\mathcal{O}(\alpha_s^3):\ \ \alpha_s(M_Z)^{NNLO} = \mathbf{0.1185} \pm \mathbf{0.0026}^{\mathbf{exp}} \pm \mathbf{0.002}^{\mathbf{th}}$$

$$\mathcal{O}(\alpha_s^4):\ \ \alpha_s(M_Z)^{NNNLO} = \mathbf{0.1190} \pm \mathbf{0.0026}^{\mathbf{exp}} \pm \mathbf{0.0005}^{\mathbf{th}}$$

and *four-fold* decrease of the theory error!
/ K.Ch, Baikov and Kühn, PLR 101 (2008) 012002/

Conclusions II

- the **5-loop** QED β-function is computed $\Longleftarrow$ the first example of 5-loop RG function in a (normal) 4-D gauge theory: almost exactly thirty yeas after similar result in a ϕ^4 model /K.Ch, Gorishnii, Larin, and Tkachov, Phys.Lett. B132 (1983) 351/; Kleinert, Neu, Schulte-Frohlinde , K.Ch., and Larin, Phys.Lett. B272 (1991) 39-44/
- All our methods and tools are equally well applicable to evaluation of the **5-loop** β-functions and anomalous dimensions in general **non-Abelian** gauge theories. A couple of interesting examples could be:

1. QCD β-function (important for a better understanding of the τ-lepton decay rate within the so-called contour-improved method)
2. the anom. dim. of the Konishi operator in N=4 supersymmetrical YM theory at **5 loops** (important for for better understanding of Ads/CFT correspondence, integrability, ABA and TBA . . .)