Multiparton NLO corrections by numerical methods

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26.09.-31.09. RADCOR 2011

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Introduction

- In this talk we present an algorithm for the numerical calculation of one-loop QCD amplitudes.
- The algorithm consists of subtraction terms, approximating the soft, collinear and ultraviolet divergences of QCD one-loop amplitudes.
- The algorithm consists of a method to deform the integration contour for the loop integration into the complex plane.
- The algorithm is formulated at the amplitude level and does not rely on Feynman graphs.
- All ingredients of the algorithm can be calculated efficiently using recurrence relations.

The subtraction method

- The contributions of an infrared observable at next-to-leading order with n final state particles can be written as

$$\langle O\rangle^{NLO} \quad = \quad \int\limits_{n+1} O_{n+1} d\sigma^R + \int\limits_{n} O_n d\sigma^V + \int\limits_{n} O_n d\sigma^C .$$

  - $d\sigma^R$ denotes the real emission contribution, whose matrix elements are given by the square of the Born amplitudes with $(n+3)$ partons $|A^{(0)}_{n+3}|^2$.
  - $d\sigma^V$ denotes the virtual contribution, whose matrix elements are given by the interference term of the one-loop and Born amplitude $2\,\mathrm{Re}(A^{(0)^*}_{n+2} A^{(1)}_{n+2})$.
  - $d\sigma^C$ denotes a collinear subtraction term, which subtracts the initial state collinear singularities.
- One adds and subtracts a suitably chosen piece to be able to perform the phase space integrations by Monte Carlo methods.

$$\langle O\rangle^{NLO} \quad = \quad \int\limits_{n+1} \left(O_{n+1} d\sigma^R - O_n d\sigma^A \right) + \int\limits_{n} \left(O_n d\sigma^V + O_n d\sigma^C + O_n \int d\sigma^A \right) .$$

- On the next slide we extend this subtraction method to the virtual part.

The subtraction method for the virtual part

- The renormalised one-loop amplitude is related to the bare amplitude by

$$\mathcal{A}^{(1)} = \mathcal{A}^{(1)}_{bare} + \mathcal{A}^{(1)}_{CT},$$

where $\mathcal{A}^{(1)}_{CT}$ denotes the ultraviolet counterterm from renormalisation.

- The bare amplitude involves the loop integration

$$\mathcal{A}^{(1)}_{bare} = \int \frac{d^D k}{(2\pi)^D} \mathcal{G}^{(1)}_{bare}.$$

- Introducing subtraction terms which match locally the singular behaviour of the bare integrand.

$$\mathcal{A}^{(1)}_{bare} + \mathcal{A}^{(1)}_{CT} = \int \frac{d^D k}{(2\pi)^D} \left(\mathcal{G}^{(1)}_{bare} - \mathcal{G}^{(1)}_{soft} - \mathcal{G}^{(1)}_{coll} - \mathcal{G}^{(1)}_{UV} \right) + \left(\mathcal{A}^{(1)}_{CT} + \mathcal{A}^{(1)}_{soft} + \mathcal{A}^{(1)}_{coll} + \mathcal{A}^{(1)}_{UV} \right)$$

- The expression in the first bracket is finite and can therefore be integrated numerically in four dimensions.
- The integrated subtractions terms in the second bracket can be easily calculated analytically in D dimensions.
- Their poles in the dimensional regularisation parameter are cancelled by the corresponding poles from the ultraviolet counterterms, initial state collinear subtraction terms and dipole subtraction terms.

- $d\sigma^V \propto 2\,\mathrm{Re}(\mathcal{A}_n^{(0)^*}\mathcal{A}_n^{(1)}) \quad \rightarrow \quad d\sigma^V = d\sigma^V_{bare} + d\sigma^V_{CT}$
- Putting everything together the next-to-leading order contribution reads

$$\langle O\rangle^{NLO} = \int_{n+1}\left(O_{n+1}d\sigma^R - O_n d\sigma^A\right) + \int_{n+loop}\left(O_n d\sigma^V_{bare} - O_n d\sigma^{A'}\right) + \int_n\left(O_n d\sigma^V_{CT} + O_n d\sigma^C + O_n\int d\sigma^A + O_n\int_{loop} d\sigma^{A'}\right).$$

- The complicated process-dependent one-loop integral can be performed numerically with Monte Carlo techniques.
- In practise the one-loop integral and the phase space integration in the second bracket is done with a single Monte Carlo integration.
- The integral in the second line looks rather complicated but is in practise just a born Amplitude times some prefactors and is therefore easily calculable.
- In the remaining talk I will focus on the one-loop integral only.

Colour decomposition

- Amplitudes in QCD may be decomposed into group-theoretical factors (carrying the colour structures) multiplied by kinematic factors called partial amplitudes. As an example we consider the colour decomposition of a n-gluon tree-level amplitude.

$$\mathcal{A}_n^{(0)}(g_1, g_2, \ldots, g_n) = \left(\frac{g}{\sqrt{2}}\right)^{n-2} \sum_{\sigma \in S_n \setminus Z_n} \delta_{i_{\sigma_1} j_{\sigma_2}} \delta_{i_{\sigma_2} j_{\sigma_3}} \ldots \delta_{i_{\sigma_n} j_{\sigma_1}} A_n^{(0)}(g_{\sigma_1}, g_{\sigma_2}, \ldots, g_{\sigma_n}),$$

  where the sum is over all non-cyclic permutations of the external gluon legs.
- The quantities $A_n^{(0)}(g_{\sigma_1}, g_{\sigma_2}, \ldots, g_{\sigma_n})$, called partial amplitudes, contain the kinematic information.
- At one-loop level partial amplitudes can be further decomposed into primitive amplitudes.

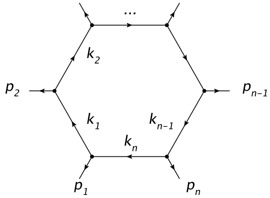
$$\mathcal{A}_n^{(1)} = \sum_{j,k} C_j A_{n,j,k}^{(1)}$$

  The colour structures are denoted by C_j, while the primitive amplitudes are denoted by $A_{n,j,k}^{(1)}$.
  - Primitive amplitudes are gauge invariant.
  - Primitive amplitudes have a fixed cyclic ordering of the external legs and a definite routing of the of the external fermion lines.
  - This ensures that the type of each loop propagator is uniquely defined, being either a quark or a gluon/ghost propagator.

Kinematics

- In a bare primitive Amplitude with n external legs, $A^{(1)}_{bare}$, only n different propagators occur in the loop integral.
- We define the kinematics as follows:

$$k_j = k - q_j,$$
$$q_j = \sum_{l=1}^{j} p_l.$$



- We define the bare one-loop integrand $G^{(1)}_{bare}$ via:

$$A^{(1)}_{bare} = \int \frac{d^D k}{(2\pi)^D} G^{(1)}_{bare}, \qquad G^{(1)}_{bare} = P(k) \prod_{j=1}^{n} \frac{1}{k_j^2 - m_j^2 + i\delta}$$

The subtraction terms

On the next slides I will present the subtraction terms for the bare one-loop amplitude. There are

- *soft* subtraction terms
- *collinear* subtraction terms
- *UV* subtraction terms

- Even if our algorithm does not depend on single Feynman diagrams, it is helpful for the derivation of the subtraction terms to define the integrand $G^{(1)}_{bare}$ by a sum of colour ordered Feynman diagrams

$$G^{(1)}_{bare} = \sum_{\mathfrak{G}} F(\mathfrak{G})$$

The soft subtraction terms for massless QCD

- Definition of the soft singularity:
  - Propagator j is soft and
  - propagator j corresponds to a gluon and
  - the external particles j and $j+1$ are on-shell.

$$k_j \to 0 \quad \text{and} \quad p_j^2 = 0 \quad \text{and} \quad p_{j+1}^2 = 0 \quad \Rightarrow \quad k_{j-1}^2 = k_j^2 = k_{j+1}^2 = 0$$

- For each gluon in the loop we define the soft subtraction function

$$S_{j,soft}(\mathfrak{G}) = \frac{\lim\limits_{k_j \to 0} \left\{ k_{j-1}^2 k_j^2 k_{j+1}^2 F(\mathfrak{G}, k) \right\}}{k_{j-1}^2 k_j^2 k_{j+1}^2}$$

- The sum of the soft subtraction function over all one-loop diagrams is proportional to the tree-level amplitude $A_j^{(0)}$.
- To get the full soft subtraction term we have to sum over all gluons in the loop,

$$G_{soft}^{(1)} = i \sum_{j \in I_g} \frac{4 p_j \cdot p_{j+1}}{k_{j-1}^2 k_j^2 k_{j+1}^2} A_j^{(0)}$$

- The integrated soft subtraction term yields the expected pole-structure.

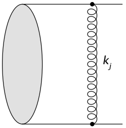
$$S_\epsilon^{-1} \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} G_{soft}^{(1)} = -\frac{1}{(4\pi)^2} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \sum_{j \in I_g} \frac{2}{\epsilon^2} \left(\frac{-2 p_j \cdot p_{j+1}}{\mu^2} \right)^{-\epsilon} A_j^{(0)} + \mathcal{O}(\epsilon).$$

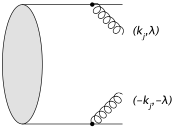
Derivation of the soft subtraction term

- In the soft limit we replace the metric tensor $g_{\mu\nu}$ of propagator j by a polarisation sum and gauge terms.

$$g_{\mu\nu} = \sum_{\lambda} \epsilon^{\mu}_{\lambda}(k_j, n)\epsilon^{\nu}_{-\lambda}(k_j, n) - 2\frac{k_j^{\mu} n^{\nu} - k_j^{\nu} n^{\mu}}{2k_j \cdot n}$$

where n^{μ} is a light like reference vector.

$\lim\limits_{k_j \to 0} \sum\limits_{\mathfrak{G}}$ 

$= \lim\limits_{k_j \to 0} \sum\limits_{\mathfrak{G}}$ 

- The terms proportional to $k_j^{\mu} n^{\nu}$ and $k_j^{\nu} n^{\mu}$ vanish due gauge invariance.
- The two "inserted" gluons lead in the soft limit to a tree-level amplitude, where these gluons are absent, times a eikonal factor $4p_j \cdot p_{j+1}$.

The collinear subtraction terms

- Definition of the collinear singularity:
  - Propagator $j-1$ is collinear to propagator j and
  - propagator j or propagator $j-1$ corresponds to a gluon and
  - the external particle j is massless and on-shell.

$$k_{j-1}||k_j \quad \text{and} \quad m_j = 0 \quad \text{and} \quad p_j^2 = 0 \quad \Rightarrow \quad k_{j-1}^2 = k_j^2 = 0$$

- For each gluon in the loop we define the collinear subtraction function

$$S_{j,coll}(\mathfrak{G}) \;=\; \frac{\lim\limits_{k_{j-1}||k_j} \left\{ k_{j-1}^2 k_j^2 F(\mathfrak{G}, k) \right\}}{k_{j-1}^2 k_j^2} \;-\; \text{soft double counting}$$

- The sum of the collinear subtraction function over all one-loop diagrams is proportional to the tree level amplitude $A_j^{(0)}$.
- We have to sum over all gluons in the loop,

$$G_{coll}^{(1)} \;=\; i \sum_{j \in I_g} (-2) \left(\frac{S_j g_{UV}(k_{j-1}^2, k_j^2)}{k_{j-1}^2 k_j^2} + \frac{S_{j+1} g_{UV}(k_j^2, k_{j+1}^2)}{k_j^2 k_{j+1}^2} \right) A_j^{(0)}.$$

$$S_q = 1,\; S_g = \frac{1}{2}, \quad \lim_{k_{j-1}||k_j} g_{UV}(k_{j-1}^2, k_j^2) \;=\; 1, \quad \lim_{k \to \infty} g_{UV}(k_{j-1}^2, k_j^2) \;=\; \mathcal{O}\left(\frac{1}{k}\right).$$

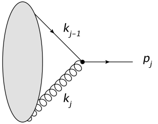
- The integrated collinear subtraction terms yields the expected pole structure:

$$S_\epsilon^{-1} \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} G_{coll}^{(1)} \;=\; -\frac{1}{(4\pi)^2} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \sum_{j \in I_g} (S_j + S_{j+1}) \left(\frac{\mu_{UV}^2}{\mu^2} \right)^{-\epsilon} \frac{2}{\epsilon} A_j^{(0)} + \mathcal{O}(\epsilon).$$

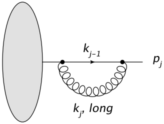
Derivation of the collinear subtraction term

- Only diagrams with collinear $q \to qg$ or $g \to gg$ splitting lead to a divergence after integration.
- As an example, the $q \to qg$ splitting.

$$\lim_{k_{j-1} \| k_j} \sum_{\mathfrak{G}}$$



$$= - \lim_{k_{j-1} \| k_j} \sum_{\tilde{\mathfrak{G}}}$$



  The sum of the left side is almost gauge invariant, only the self energies of external legs are missing.
- The self-energy insertions on the external lines introduce a spurious $1/p_j^2$-singularity. We define $p_j = k_{j-1} - k_j$ slightly off shell by introducing the Sudakov parametrisation.

$$k_{j-1} = xp + k_\perp - \frac{k_\perp^2}{x} \frac{n}{(2p \cdot n)}, \quad -k_j = (1-x)p - k_\perp - \frac{k_\perp^2}{(1-x)} \frac{n}{(2p \cdot n)}.$$

- The singular parts of the self-energies are proportional to

$$P_{q \to qg}^{long} = -\frac{2}{2k_{j-1} \cdot k_j} \left(-\frac{2}{1-x} + 2 \right) \not{p}$$

- The terms with $2/(1-x)$ correspond to the soft singularities.

The ultraviolet subtraction terms I

- We write a generic one-loop Feynman diagram

$$F_{a,n}(\mathfrak{G}, k) = P_a(\mathfrak{G}, k) \prod_{j=1}^{n} \frac{1}{k_j^2 - m_j^2}$$

where $P_a(\mathfrak{G}, k)$ is a polynomial of degree a in the loop momenta k.

- The one-loop integral of this diagram is UV-divergent, if $4 + a - 2n \geq 0$.

$$\int \frac{d^4 k}{(2\pi)^4} F_{a,n}(\mathfrak{G}, k) \rightarrow \infty \quad : \quad 4 + a - 2n \geq 0$$

- In QCD only vertex and propagator corrections are UV-divergent.
- The subtraction term has to match the UV behaviour of the one-loop integrand and has to be infrared finite. Therefore we expand the propagators $k_i^2 - m_i^2$ around the "UV-propagator" $\bar{k}^2 - \mu_{UV}^2$, with $\bar{k} = k - Q$.

$$F_{a,n}(\mathfrak{G}, k) \approx \frac{P_a(\mathfrak{G}, k)}{(\bar{k}^2 - \mu_{UV}^2)^n} \left(1 + \sum_{j=1}^{l} \frac{X_j(\bar{k})}{(\bar{k}^2 - \mu_{UV}^2)^j} \right)$$

where $X_j(\bar{k})$ is a polynomial of degree j in $\bar{k}$ and $l = 0$ for logarithmic, $l = 1$ for linear and $l = 2$ for quadratic UV-divergent diagrams.

The ultraviolet subtraction terms II

- We demand that the integrated subtraction term is proportional to a common pole part times the corresponding born term.

$$\int \frac{d^D k}{(2\pi)^D} S_{UV}(\mathfrak{G}) \;\propto\; \left(\frac{1}{\epsilon} - \ln\frac{\mu_{uv}^2}{\mu^2}\right) A^{(0)} + \mathcal{O}(\epsilon)$$

- The subtraction term for a UV divergent one loop diagram is

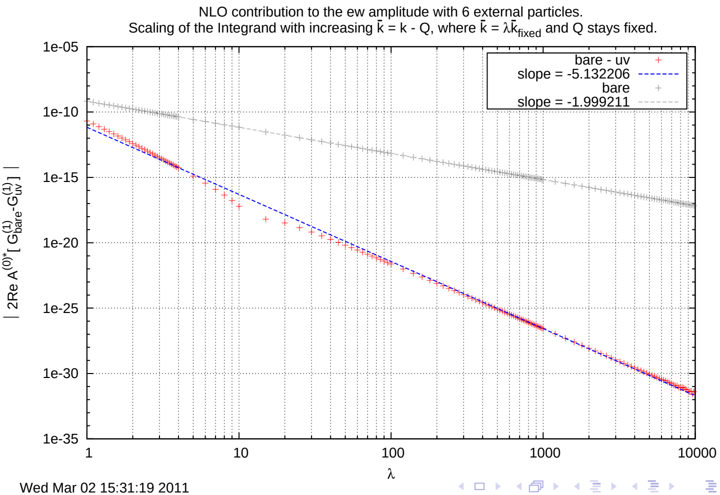
$$S_{UV}(\mathfrak{G}) \;=\; \frac{P_a(\mathfrak{G},k)}{(\bar{k}^2-\mu_{UV}^2)^n}\left(1+\sum_{j=1}^{l}\frac{X_j(\bar{k})}{(\bar{k}^2-\mu_{UV}^2)^n}\right) - \frac{-2\mu_{UV}^2}{(\bar{k}^2-\mu_{UV}^2)^3}\;R(\mathfrak{G})$$

  where $R(\mathfrak{G})$ is a finite term which ensures that the integrated subtraction term has the demanded form.

- After construction of the subtraction terms for all QCD vertex- and propagator-corrections, the unintegrated total ultraviolet subtraction term $G_{UV}^{(1)}$ can be constructed efficiently via Berends-Giele type recurrence relations.

Consistency check of the UV subtraction

- The plot shows $|2\,\mathrm{Re}(A^{(0)} G^{(1)}_{bare})|$ and $|2\,\mathrm{Re}(A^{(0)}(G^{(1)}_{bare} - G^{(1)}_{UV}))|$ over the UV scaling parameter λ for the process $e^+e^- \to 4jets$.
- The *bare* Amplitude decrease like $1/k^2$ and is therefore quadratic divergent.
- The $(bare - UV)$ Amplitude decrease like $1/k^5$ and is therefore UV-safe.



Summary of the subtraction terms

- We show that the total UV-subtraction term matches the *bare* integrand locally in the UV-limit.
- The UV-subtraction terms are constructed efficiently via Berends-Giele type recurrence relations.
- The infrared subtraction terms are formulated on amplitude level and therefore are also constructed efficiently via Berends-Giele type recurrence relations.
- All integrated subtraction terms are proportional to tree-level amplitudes

- After subtraction it is possible that one or more propagators go on-shell. Therefor we need a suitable deformation of the integration contour into the complex plane to avoid these poles.

Overview of the contour deformation

- Again the one loop integrand

$$\int \frac{d^4k}{(2\pi)^4} G^{(1)}_{bare} \quad = \quad \int \frac{d^4k}{(2\pi)^4} P(k) \prod_{j=1}^{n} \frac{1}{k_j^2 - m_j^2 + i\delta}$$

- We deform the integration contour into the complex plane to match Feynman's $+i\delta$ rule.
- Use direct deformation of the loop momenta

$$k \;\rightarrow \tilde{k} \;=\; k + i\kappa(k).$$

- After the deformation the integral reads

$$= \quad \int \frac{d^4k}{(2\pi)^4} \left| \frac{\partial \tilde{k}}{\partial k} \right| P(\tilde{k}(k)) \prod_{j=1}^{n} \frac{1}{k_j^2 - m_j^2 - \kappa^2 + 2ik_j \cdot \kappa}$$

- We have to construct the deformation vector κ such

$$k_j^2 - m_j^2 \;=\; 0 \quad \rightarrow \quad k_j \cdot \kappa \geq 0.$$

- The numeric stability of the Monte Carlo integration depends strongly on the definition of the deformation vector κ.
- At the moment we use the a algorithm by W. Gong, Z. Nagy and D. Soper to construct the deformation vector.

Proof of principle - $e^+e^- \to jets$

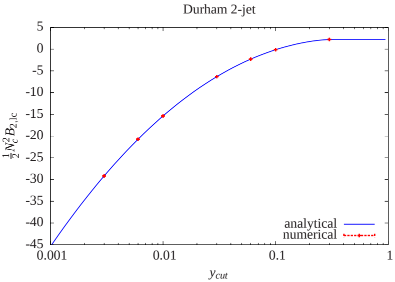
- The cross section for n jets normalised to the LO cross section for $e^+e^- \to$ hadrons.

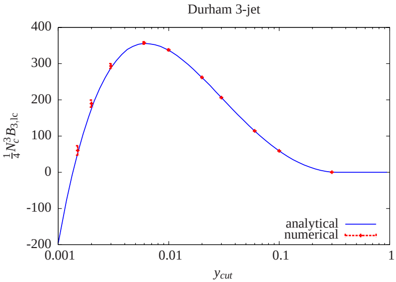
$$\frac{\sigma_{n-jet}}{\sigma_0} = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-2} A_n(\mu) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-1} B_n(\mu) + \mathcal{O}(\alpha_s^n).$$

- We expand the NLO perturbative coefficient B_n in $1/N_c$.

$$B_n = N_c \left(\frac{N_c}{2}\right)^{n-1} \left[B_{n,lc} + \mathcal{O}\left(\frac{1}{N_c}\right)\right]$$

- We calculate the NLO coefficient in leading colour up to $n = 5$ i.e. up to six-point functions.
- We plot $N_c(N_c/2)^{n-1} B_{n,lc}$ over the resolution parameter y_{cut} in the Durham algorithm.





- The cross section for $n - jets$ normalised to the LO cross section for $e^+e^- \to$ hadrons.

$$\frac{\sigma_{n-jet}}{\sigma_0} = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-2} A_n(\mu) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-1} B_n(\mu) + \mathcal{O}(\alpha_s^n).$$

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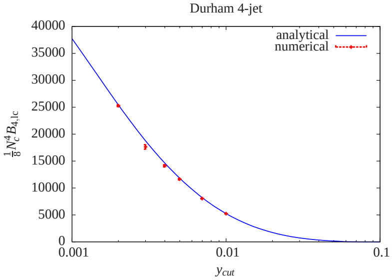
- We calculate the NLO coefficient in leading colour up to $n = 5$ i.e. up to six-point functions.
- We plot $N_c(N_c/2)^{n-1} B_{n,lc}$ over the resolution parameter y_{cut} which is corresponded to the Durham algorithm.



Durham 5-jet

y_{cut}	$\frac{N_c^5}{16} B_{5,lc}$
0.002	$(4.275 \pm 0.006) \cdot 10^5$
0.001	$(1.050 \pm 0.026) \cdot 10^6$
0.0006	$(1.84 \pm 0.15) \cdot 10^6$

Summary and outlook

Summary

- In this talk the extension of the subtraction method to the virtual corrections was presented.
- The major ingredients of this subtraction method, the subtraction terms, were also presented.
- All required ingredients can be calculated efficiently using recurrence relations and a suitable contour deformation is provided.
- We demonstrated the functionality of the algorithm on the process $e^+e^- \to jets$.

Outlook

- Improving the efficiency of the Monte Carlo.
- Extend the contour deformation to massive QCD.
- Z-Production for the LHC.
- Full colour calculations.

Thank you for your attention!

ltr: Daniel Götz, Sebastian Becker, Stefan Weinzierl, Christopher Schwan, Christian Reuschle.