One Loop Amplitudes for Multi-Jet Production

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Outline

- Numerical evaluation of one-loop amplitudes with generalised unitarity
- `NGluon` implementation and performance
- Extension to multiple fermion pairs
- Colour sums : primitive amplitudes to partial amplitudes
- Outlook

based on work with Benedikt Biederman and Peter Uwer

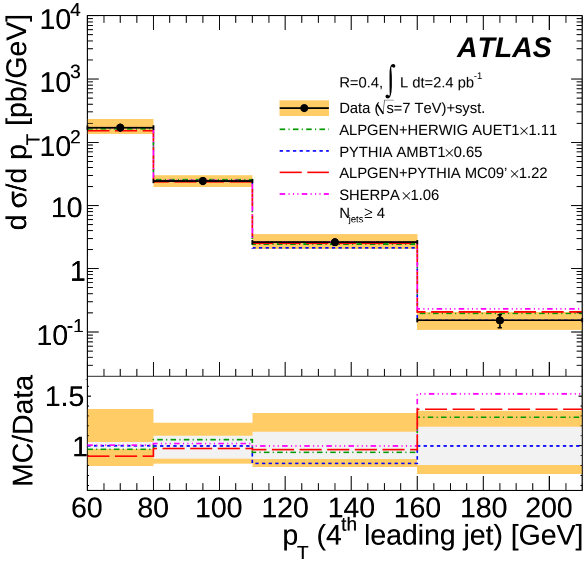
Comput.Phys.Commun. 182 (2011) [arXiv:1011.2900] + work in progress

Motivation

Motivation : Automating NLO corrections to SM backgrounds for the LHC.

see talks by: van Hameren,Becker,Englert,Shivaji,Pozzorini, Maierhoefer,Heinrich,Hirschi,Fujimoto,Riemann,Melia

$$\sigma_n^{NLO} = \sigma_n^{LO} + \sigma_n^{\text{virtual}} - \sigma_n^{\text{int.sub.}} + \sigma_{n+1}^{\text{real}} + \sigma_{n+1}^{\text{sub.}}$$



Multi-jet measurements from ATLAS

[ATLAS arXiv:1107.2092 [hep-ex]]

Computations of Virtual Corrections

- A lot of recent progress in computational methods for virtual corrections:

  Bern,Dixon,Dunbar,Kosower,Britto,Cachazo,Feng,Mastrolia,
  Ossola,Papadopoulos,Pittau,Ellis,Giele,Kunszt,Melnikov,Forde,...

- Automated numerical approaches:

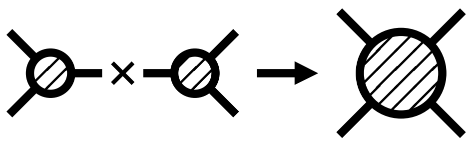
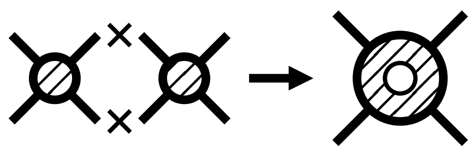
  [Grace, `BlackHat`, `Rocket`, `CutTools/Helac-1l`, Denner et al., `GOLEM/samurai`, MadLoop, ...]

  - Growing number of phenomenological studies
  - Continuing improvements: colour dressing, GPU's

    [Giele,Kunszt,Winter][Giele,Stavenga,Winter]

- Efficiency:
  - Numerical stability
  - Fast numerical evaluation
  - Complexity of processes with additional jets
  - Portability $\Rightarrow$ Public codes

On-Shell Methods

- Amplitudes without Feynman diagrams
- Physical degrees of freedom to avoid large intermediate expressions
- Use amplitudes as fundamental building blocks
  - Recursion:  [Britto,Cachazo,Feng,Witten (2004)]
  - Unitarity:  [Bern,Dixon,Dunbar,Kosower (1994)] [Bern,Dixon,Kosower (1997)] [Britto,Cachazo,Feng (2004)]
- Make extensive use of complex momenta

Colour Ordering and Primitive Amplitudes

- Order kinematic dependence w.r.t. group structure :
  Colour Ordered / Partial Amplitudes

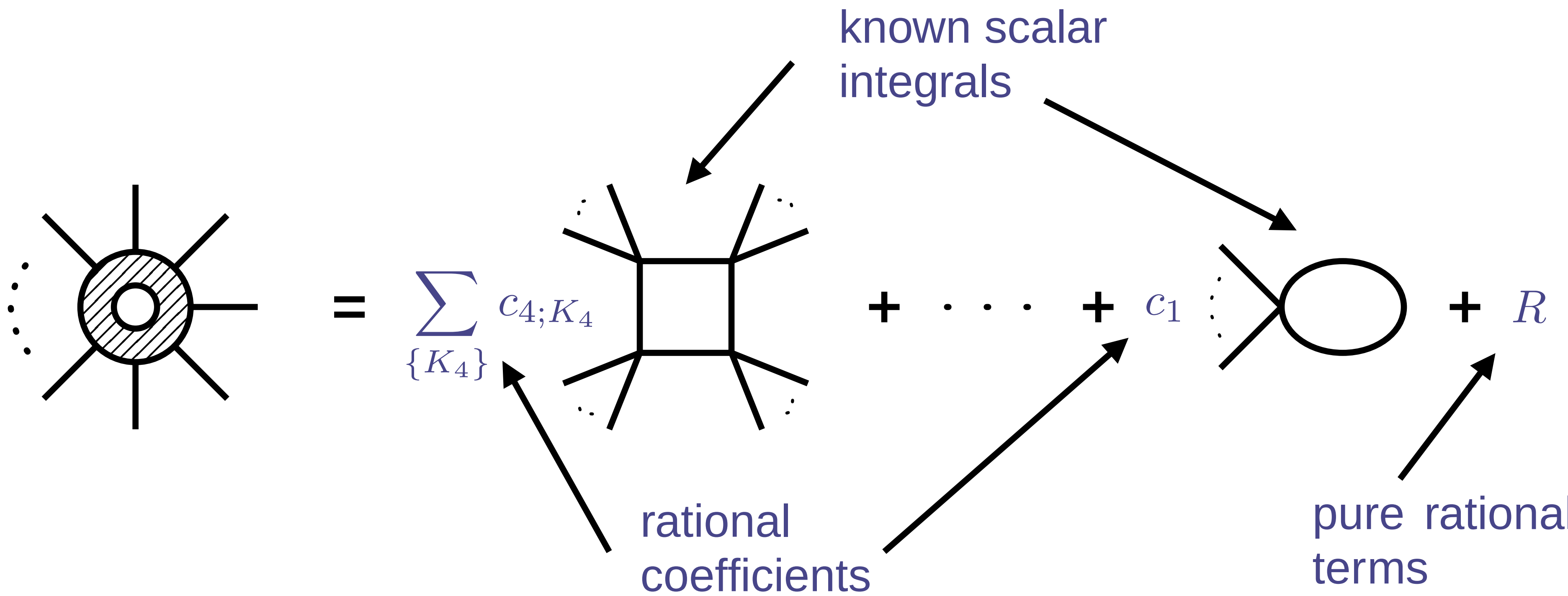
$$\mathcal{A}^{(0)}(\{a_i\},\{h_i\},\{p_i\}) = \sum_c f_c(T^{a_i}) A^{(0)}(\{h_i\},\{p_i\})$$

- At higher loops internal colour flows give additional structure :
  Primitive Amplitudes

$$A^{(l)}(N_c, N_f, \{h_i\},\{p_i\}) = \sum_p g_p(N_c, N_f) A^{(l),[p]}(\{h_i\},\{p_i\})$$

- Primitive amplitudes are minimal gauge invariant sub-sets

Structure of One-Loop Amplitudes

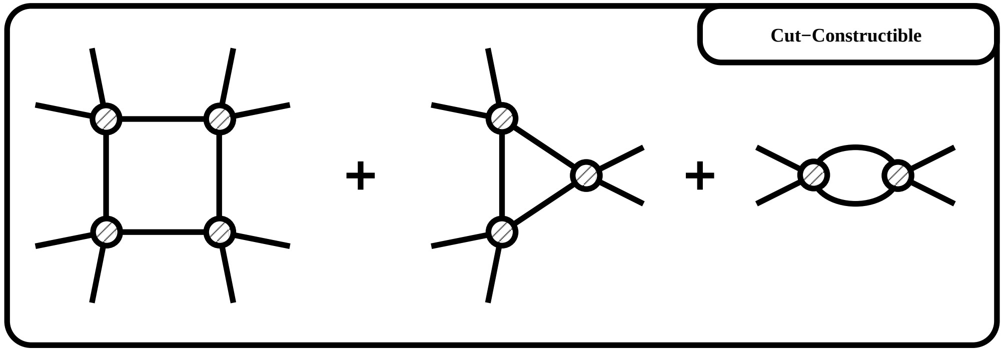


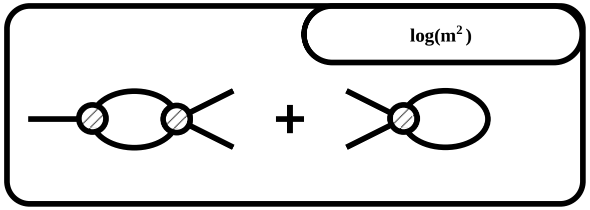
- General gauge theory amplitudes reduced to box topologies or simpler

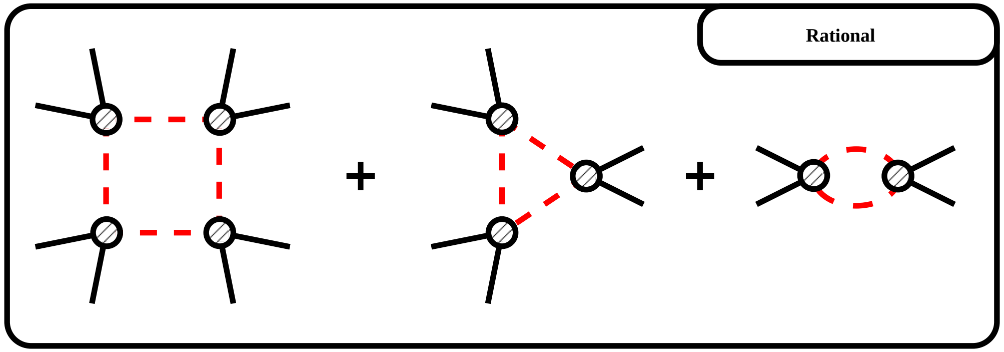
  [Passarino,Veltman;Melrose]

- Isolate logarithms with cuts $\rightarrow$ exploit on-shell simplifications
- General cutting principle: [Bern,Dixon,Kosower]
  - apply δ-functions to L and R sides
  - generate and solve the linear system for the coefficients

Generalised Unitarity for One-Loop Amplitudes







trees

Britto,Cachazo,Feng,Witten

Berends,Giele

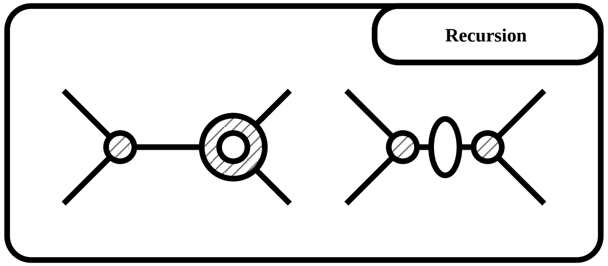
trees $\Rightarrow$ loops

Bern,Dixon,Dunbar,Kosower,Berger,Forde

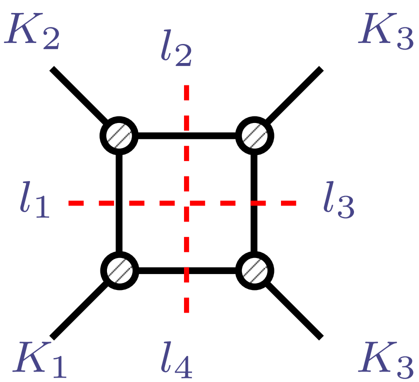
Britto,Cachazo,Feng,Mastrolia,Yang

Ossola,Papadopoulos,Pittau

Ellis,Giele,Kunszt,Melnikov



Quadruple Cuts



- Quadruple cut$\rightarrow$ 4 on-shell δ-functions
- $C_4 = \frac{1}{2}\sum_{\sigma=\pm} A_1 A_2 A_3 A_4(l_1^\sigma)$

  [Britto,Cachazo,Feng (2004)]

- Two complex solutions

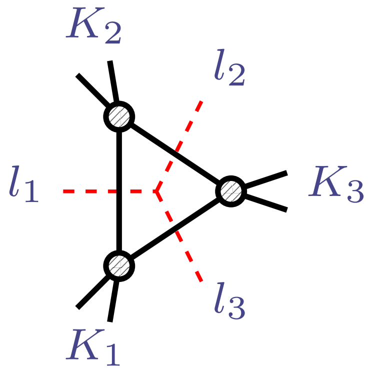
$$l_\pm^\mu = aK_1^{\flat,\mu} + bK_2^{\flat,\mu} + \frac{c}{2}\langle K_1^\flat|\gamma^\mu|K_2^\flat] + \frac{d}{2}\langle K_2^\flat|\gamma^\mu|K_1^\flat]$$

- Complete amplitude in $\mathcal{N}=4$ SYM

Triple Cuts

- Triple cut$\rightarrow$ 3 on-shell δ-functions
- Parametrise free integration

[Ossola,Papadopoulos,Pittau (2007)][Forde (2007)]

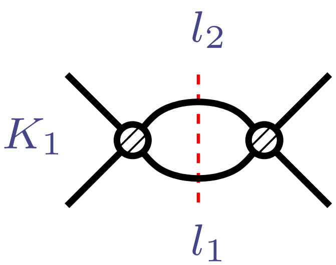


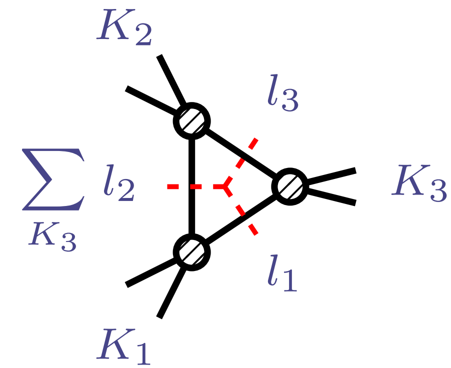
$$\oint J_t dt A_1 A_2 A_3 = \oint J_t dt \mathrm{Inf}_t[A_1 A_2 A_3(t)] + \sum_k \frac{\mathrm{Res}_{\mathrm{t}=\mathrm{t_k}}(A_1 A_2 A_3)}{\xi_k (t - t_k)}$$

- $C_3 = \frac{1}{2} \sum_{\sigma=\pm} \mathrm{Inf}_t[A_1 A_2 A_3(l_1^\sigma(t))]|_{t^0}$

- $\mathrm{Inf}_t[f(t)] = \lim_{t\to\infty}(f(t))\Big|_{\mathrm{pole}} = c_0 + c_1 t + c_2 t^2 + c_3 t^3$

Double Cuts





Bubble coefficients follow from a similar analysis:

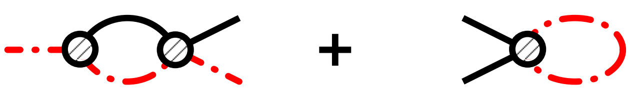
$$C_2 = \mathrm{Inf}_t \mathrm{Inf}_y [A_1 A_2(t,y)] - \frac{1}{2} \sum_{\sigma=\pm} \mathrm{Inf}_t [A_1 A_2' A_3' (t, y_\pm)]$$

3-cut: Cauchy's Theorem

[Dunbar,Perkins,Warwick (2008)]

2-cut: Stokes' Theorem

[Mastrolia (2009)]



- Wave-function bubbles $\propto \log(m^2)$
- Double cut diverges
- Solution: explicitly remove poles
  [Ellis,Giele,Kunszt,Melnikov (2008)]
- Recent analytical development
  [Britto,Mirabella (2011)]

D-dimensional Cuts

- Rational terms detected by cuts in higher dimensions [Bern,Morgan (1995)]

  Bern,Dixon,Kosower,Anastasiou, Britto,Feng,Kunszt,Mastrolia

- Higher dimensional integral coefficients determine rational terms

  [Giele,Kunszt,Melnikov (2008)]

  [Ossola,Papadopoulos,Pittau (2008)]

$$R_n = -\frac{1}{6}\sum_{K_4} C^{[4]}_{4;K_4} - \frac{1}{2}\sum_{K_3} C^{[2]}_{3;K_3} - \frac{1}{6}\sum_{K_2}\left(K_2^2 - 3(m_1^2 + m_2^2)\right) C^{[2]}_{2;K_2}$$

- Massive cuts with additional series expansion [OPP (2008)]

  [SB (2008)]

$$l^2_{[D]} = l^2_{[4]} - \mu^2, \qquad R \sim \lim_{\mu^2\to\infty} A^{(1)}(\mu^2)$$

D-dimensional Cuts via Internal Masses

- All tree level amplitudes in four dimensions
- Take $\epsilon \to 0$ limit at integrand level: [SB (2008)]

$$\oint |\mu|^{-2\epsilon} d|\mu| A_1 A_2 A_3 A_4$$

$$= \oint |\mu|^{-2\epsilon} d|\mu| \mathrm{Inf}_{\mu^2}[A_1 A_2 A_3 A_4] + \sum_k \frac{\mathrm{Res}_{\mu=\mu_k}(A_1 A_2 A_3 A_4)}{\xi_k(\mu^2 - \mu_k^2)}$$

$$\stackrel{\epsilon\to 0}{\to} \mathrm{Inf}_{\mu^2}[A_1 A_2 A_3 A_4]|_{\mu^4} = C_4^{[4]}$$

  - Easy to automate analytically: all-helicities for $gg \to 4g$
  - Numerical version used in `BlackHat` for $W/Z + 3/4j$

[Berger et. al (2009-2010)][Ita et al. (2011)]

- Numerically pentagons are required for stability (or for expansions beyond $\mathcal{O}(\epsilon^0)$)

Recent Analytic Computations

New compact analytic expressions have been useful in some cases
Made possible with new on-shell methods.

- $pp \to H + 2j$ [SB,Glover (2006)][Berger,Del Duca,Dixon (2006)] [SB,Glover,Risager (2007)] [Glover,Mastrolia,Williams (2008)] [Dixon,Sofianatos (2009)] [SB,Glover,Mastrolia,Williams (2009)] [SB,Campbell,Ellis,Williams (2009)] [Campbell,Ellis,Williams (2010)]
- $pp \to Wb\bar{b}$, massive b's with decays [SB,Campbell,Ellis (2010)]
- $pp \to VV$ (updated with interference effects) [Campbell,Ellis,Williams (2011)]
- $pp \to t\bar{t}$ helicity amplitudes [SB,Sattler,Yundin (2011)]

Available now in MCFM v6.0 `http://mcfm.fnal.gov`

Multi-Gluon One-Loop Amplitudes

- Benchmark process for automated virtual corrections
- First test for numerical generalised unitarity programs

  `Blackhat` Berger et al. (2008), Lazopoulos (2008,2009),
  `Rocket` Giele, Kunszt Melnikov (2008), Giele, Zanderighi (2008),
  Giele,Kunszt,Winter (2009) [colour-dressed]
- Numerical stability
  - Subtraction terms for non-maximal cuts, e.g.

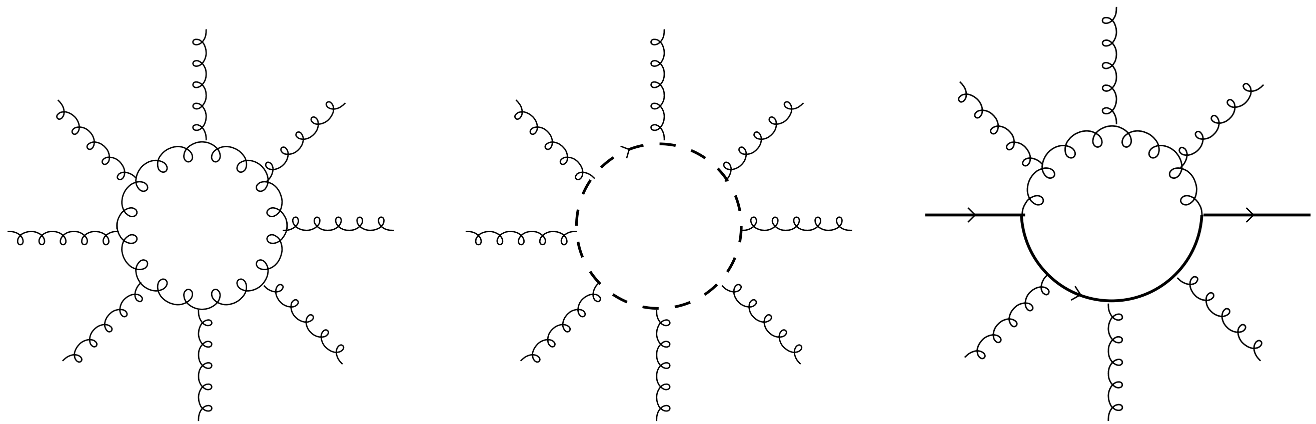
$$\overline{C_3}(K_1, K_2, K_3, t) = \tilde{C}_3(K_1, K_2, K_3; t) - \sum_{K_4} \frac{\overline{C_4}(K_1, K_2, K_3, K_4; t)}{D(K_4; t))}$$

  - Discrete Fourier projections to extract coeffcients

$$C_3(K_1, K_2, K_3) = \frac{1}{7} \sum_{p=-3}^{3} \overline{C}_3 \left(K_1, K_2, K_3; t = t_0 \exp\left(\frac{2\pi i\, p}{7}\right)\right)$$

- `NGluon` : Public c++ library [http://www.physik.hu-berlin.de/pep/tools]

NGluon : Method



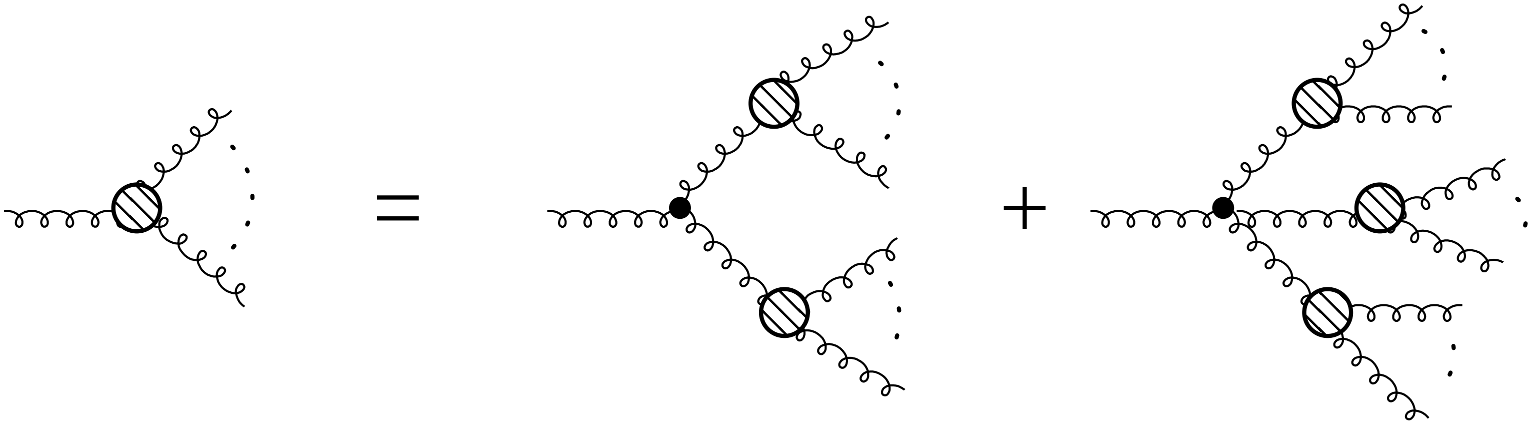
- Van-Neerven Vermaseren basis for loop momentum [Ellis,Giele,Kunszt,Melnikov]
- Subtraction terms with “spurious vectors" also computed via Fourier projections.
- Rational terms from massive scalar loop

  longditudinal polarisation of D-dimensional gluon $\equiv$ massive scalar

- Everything in 4 dimensions

Tree-level : Berends-Giele Recursion

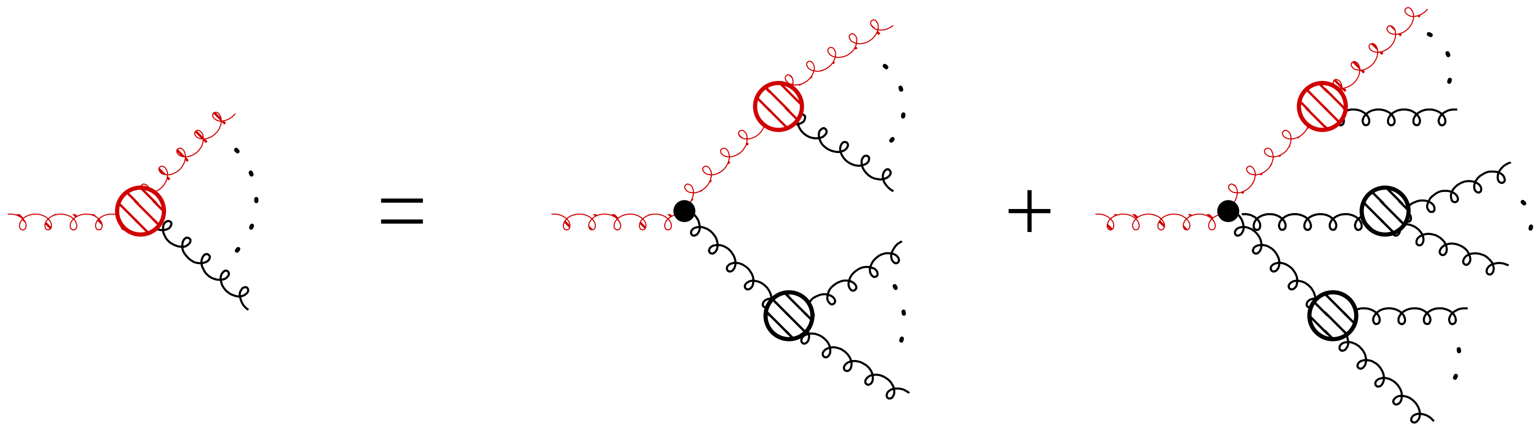
- Simple and efficient algorithm for trees : scales as $\sim n^4$
- General implementation for ordered helicity amplitudes including:
  - massless gluons and fermions
  - massive fermions and scalars (for the rational terms)



- Bottom-up approach in `NGluon`

Tree-level Currents Inside the Cuts

- Generalised unitarity computes many trees with complex momenta
- Loop momenta are adjacent $\Rightarrow$ trees scale as $\sim n^3$



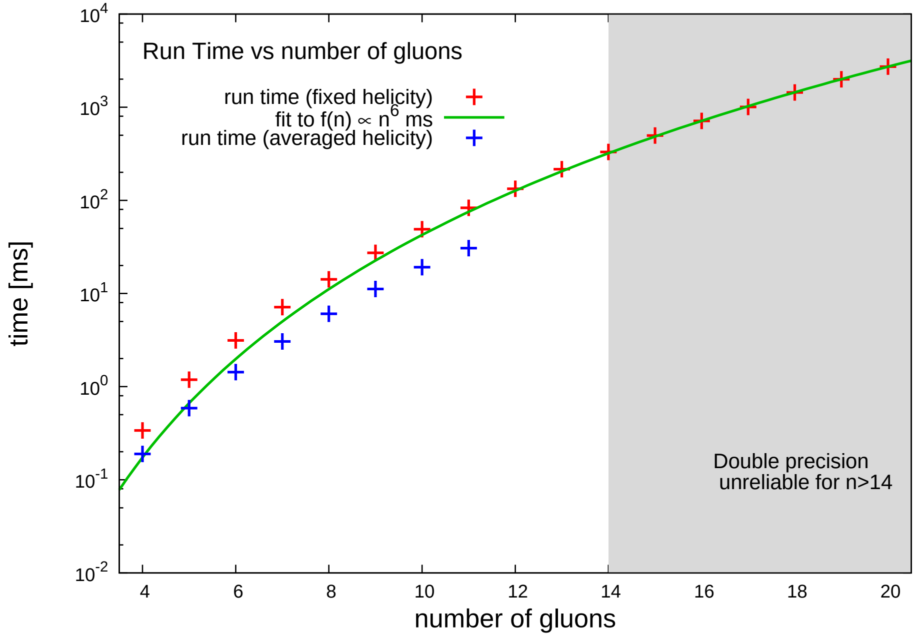
- Initialisation phase $\sim n^4$
- Coefficient computation $\sim n^3$

NGluon : Speed and Polynomial Growth

- Asymptotic growth dominated by pentagons: loops $\sim n^9$ [Giele,Zanderighi (2008)]
- NGluon asymptotic limit $n \gg 20$. Limit $\sim n^8$. Practical scaling $\sim n^6$.

n	time (ms)
4	0.19
5	0.58
6	1.42
7	3.04
8	6.02

[intel core i7 2.7GHz]



- Double precision suitable for up to 12-14 gluons
- New! Cache system for improvement of colour and helicity sums

NGluon : Example

- Example output: check against known analytic formula [Forde,Kosower (2005)]

```
./NGluon-demo-dd --MHVcheck --ngluon=32
# INTEGRALS: FF [1] and QCDLoop [2] are used to calculate the
# INTEGRALS: scalar one-loop integrals
# INTEGRALS: [1] van Oldenborgh: FF: A Package To Evaluate One Loop Feynman Diagrams
# INTEGRALS:     Comput.Phys.Commun.66:1-15,1991
# INTEGRALS: [2] R.Keith Ellis, Giulia Zanderighi,  Scalar one-loop integrals for QCD,
# INTEGRALS:     JHEP 0802:002,2008
# ONELOOP: Scaling test to check accuracy is used
# KINCUTS: kinematical cuts: jade
# KINCUTS:   -> set deltaS =  1.000000e-08

A32_finite       = (4.857249473081236e+48,-1.094068968087778e+48)+/-(-2.165e+33,-6.000e+33)
RAT32_num        = (4.470262889576663e+47,-3.865634861714300e+47)
RAT32_ana        = (4.470262889576666e+47,-3.865634861714277e+47)
Relative accuracy via internal scale check      = 1.1e-14
Relative accuracy via external analytic check   = 3.8e-15
# Time used for this run: 3.6486e+03
```

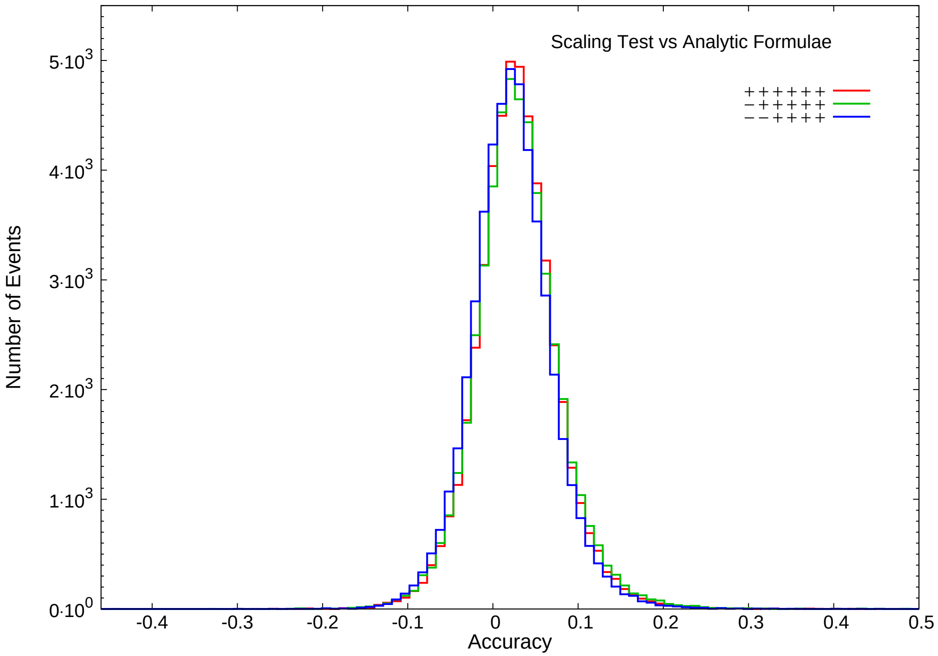
- quadruple and octuple precision implemented via QD [Hida,Li,Bailey]

$\mathtt{NGluon}$: Accuracy

- Scaling test to determine accuracy

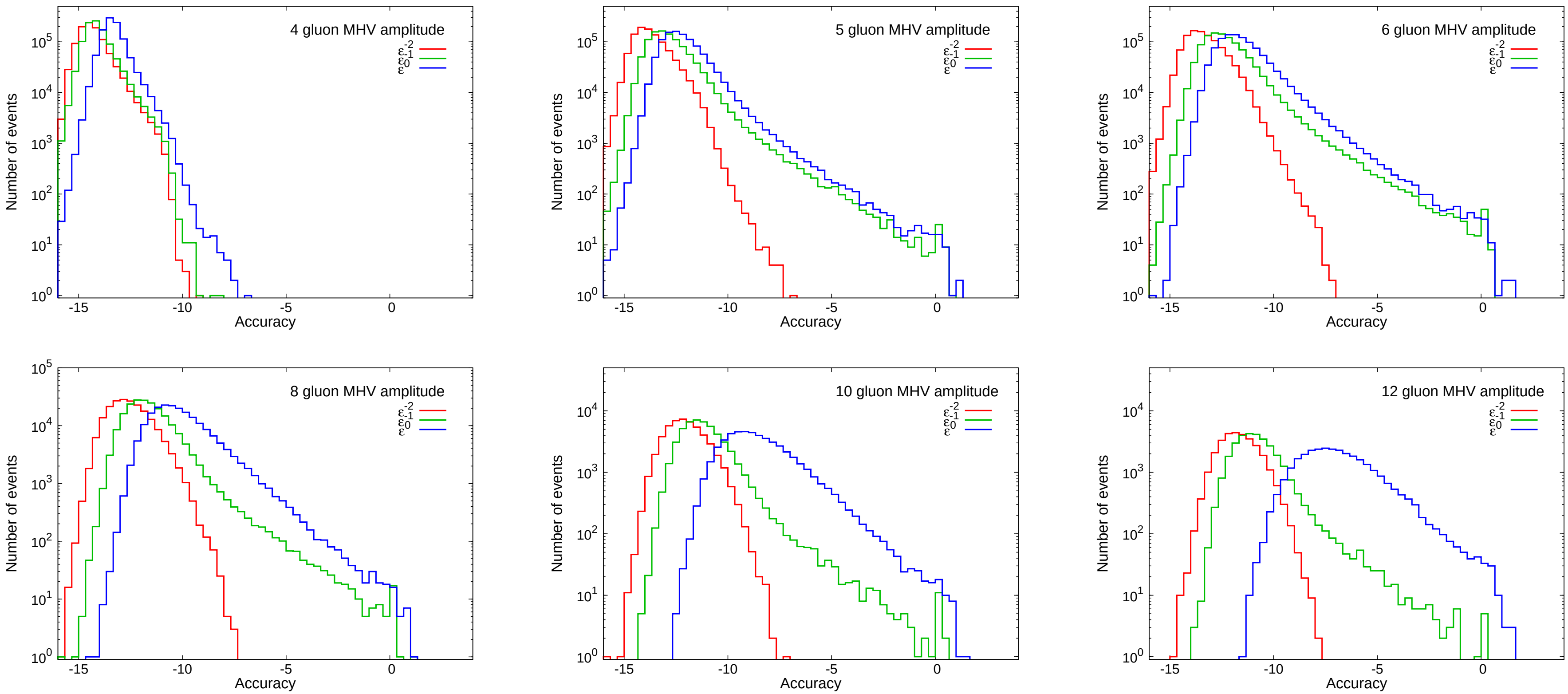
$$\log\left(\frac{A_{\mathtt{NGluon}} - A_{\text{analytic}}}{A_{\text{analytic}}}\right)$$

$$\log\left(\frac{2(A_{\mathtt{NGluon}} - A^{\text{scaled}}_{\mathtt{NGluon}})}{A_{\mathtt{NGluon}} + A^{\text{scaled}}_{\mathtt{NGluon}}}\right)$$



- Very reliable
- Alternative: use higher order terms in Fourier projection
  - offers possibility of speed improvement

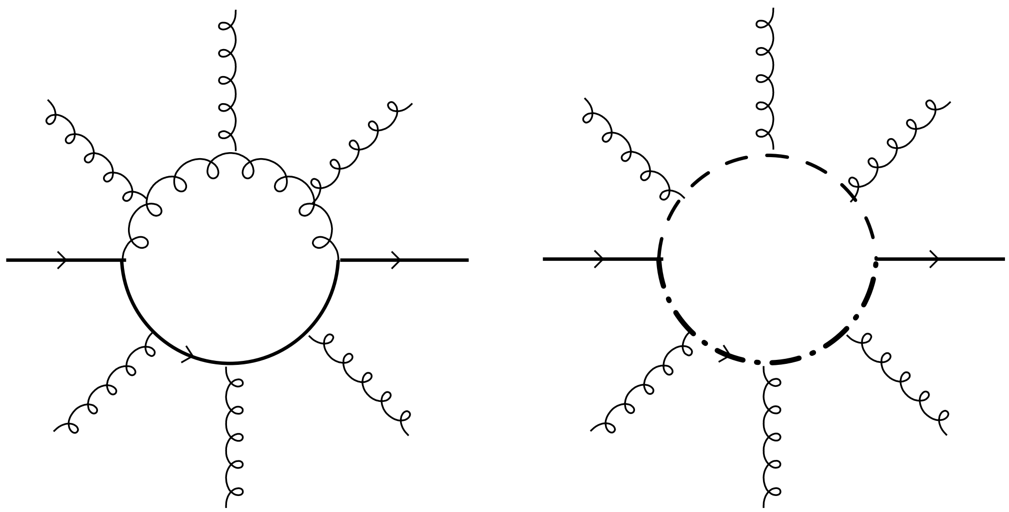
NGluon : Stability



NB: $\log$ scale for #points: Accuracy $= \log\left(\frac{2(A_{\mathtt{NGluon}} - A^{\mathrm{scaled}}_{\mathtt{NGluon}})}{A_{\mathtt{NGluon}} + A^{\mathrm{scaled}}_{\mathtt{NGluon}}}\right)$

$p_T > \frac{\sqrt{s}}{2} 10^{-2}, \eta < 3, \Delta R > 0.4$

Multiple Fermion Primitive Amplitudes



- Massive fermions and scalars used to compute rational terms.
- Cross checked against known amplitudes: [all finite amps. Bern,Dixon,Kosower (2005)]

[all $\bar{q}q3g$ amp. Bern,Dixon,Kosower (1995)]

[all $2 \to 2$ amps. Ellis,Sexton(1986) : Kunszt,Soper(1992) : `POWHEG BOX`]

[fermion loops in $n(g)$ primitive amps. for $n \leq 10$ van Hameren (2009)]

Colour Sums

- Turning primitive amplitudes into partial amplitudes
- Colour sums performed using Kleiss-Kuijf basis of $(n-2)!$ trees

$$A_n^{(0)}(1,\{a\}_m,2,\{b\}_{n-m-2}) = \sum_{\sigma\in OP\{a\}^T\cup\{b\}} (-1)^m A_n^{(0)}(1,2,\{\sigma\})$$

- Primitive amplitudes form a basis of $(n-1)!/2$

[c.f. ‘F’-basis Dixon,Del-Duca,Maltoni (1999)]

- Sub-leading colour amplitudes from leading colour primitives

$$\mathcal{A}_n^{(1)} = \sum_{\sigma\in S_{n-1}} \mathrm{tr}(\{\sigma\}) A_{n;1}(\{\sigma\}) \sum_{c=3}^{[n/2]+1} \sum_{\sigma\in S_n/S_{n;c}} \mathrm{tr}(\{\sigma\}_c)\,\mathrm{tr}(\{\sigma\}_{n-c}) A_{n;c}(\{\sigma\})$$

where

$$A_{n;c}(1,\{a\}_{c-2},\{b\}_{n-c+1}) = \sum_{\sigma\in COP\{a\}^T\cup\{b\}} (-1)^{c+1} A_{n;1}(1,\{\sigma\})$$

Colour Sums II

- Amplitudes with a single fermion pair follow similiar pattern
- General partial amplitude decomposition for multiple fermion?
- Full colour amplitudes checked against colour correlated Born amplitudes:

$$Re(\mathcal{A}^{(1)}.\mathcal{A}^{(0)}) = |\mathcal{A}^{(0)}|^2 \sum_i \left(\frac{C_i}{\epsilon^2} + \frac{\gamma_i}{\epsilon} \right) + \frac{1}{\epsilon} \sum_{i,j} |\mathcal{A}^{(0)}_{ij}|^2 \log\left(\frac{\mu_R^2}{-s_{ij}} \right) + \text{finite}$$

where

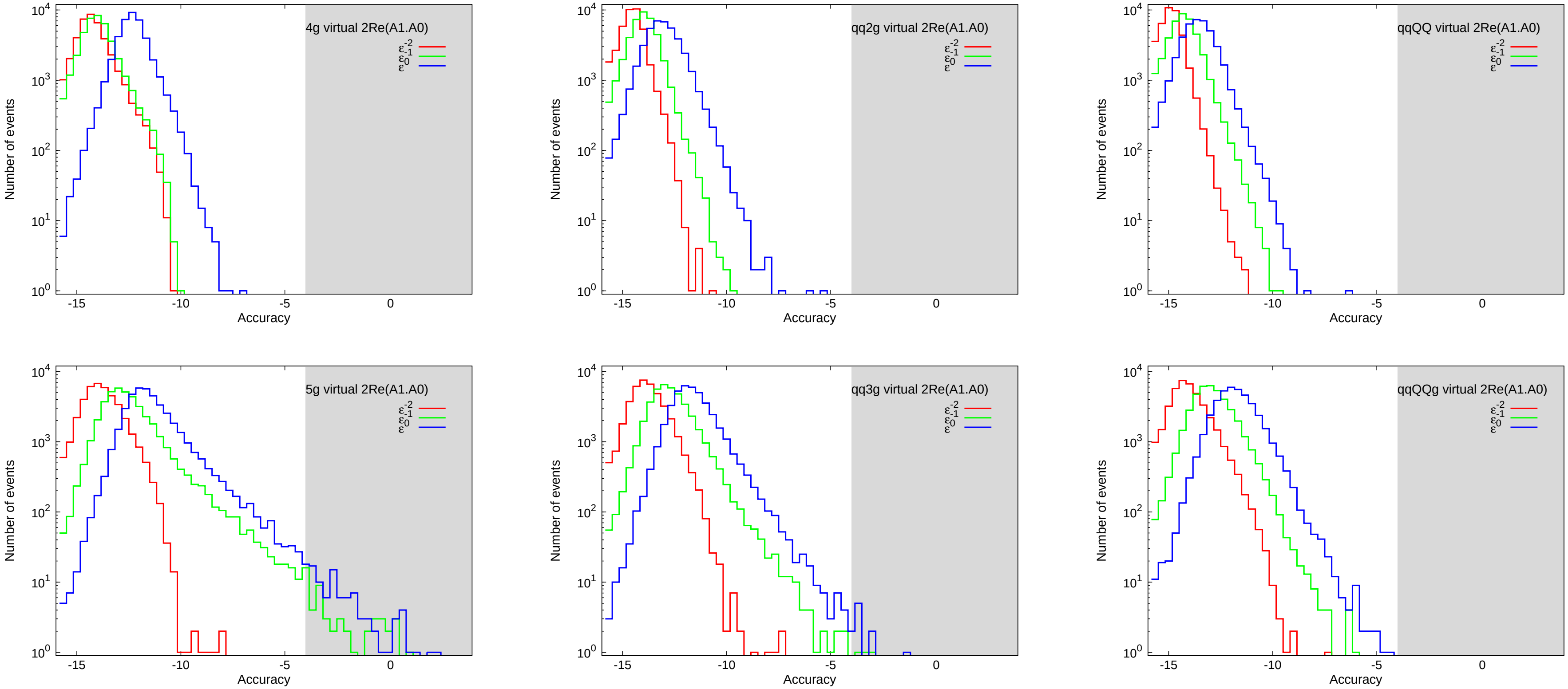
$$|\mathcal{A}^{(0)}_{ij}|^2 = \sum_{c,h} \mathcal{A}^{(0)}.T_i.T_j.\mathcal{A}^{(0),\dagger}$$

- Possible improvements using BCJ relations [Bern,Carrasco,Johansson (2008)]

$$A_n^{(0)}(1,2,\{a\}_m,3,\{b\}_{n-m-3}) = \sum_{\sigma \in POP\{a\} \cup \{b\}} \prod_{k=4}^{m} \frac{\mathcal{F}(3,\{\sigma\},1|k)}{s_{2,4,\ldots,k}} A_n^{(0)}(1,2,3,\{\sigma\})$$

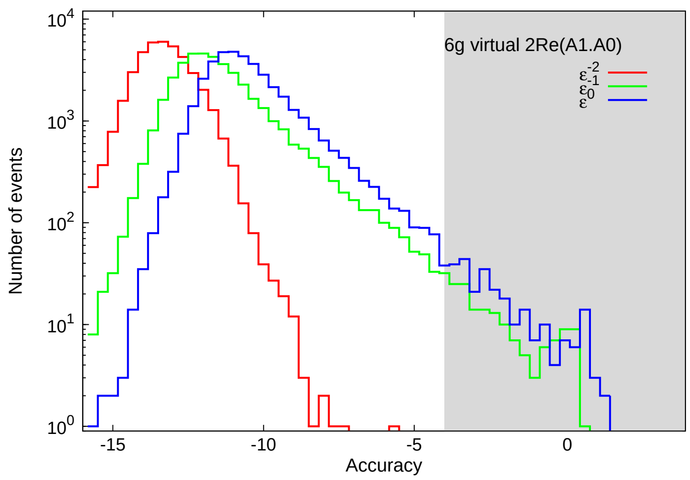
Stability of colour summed amplitudes

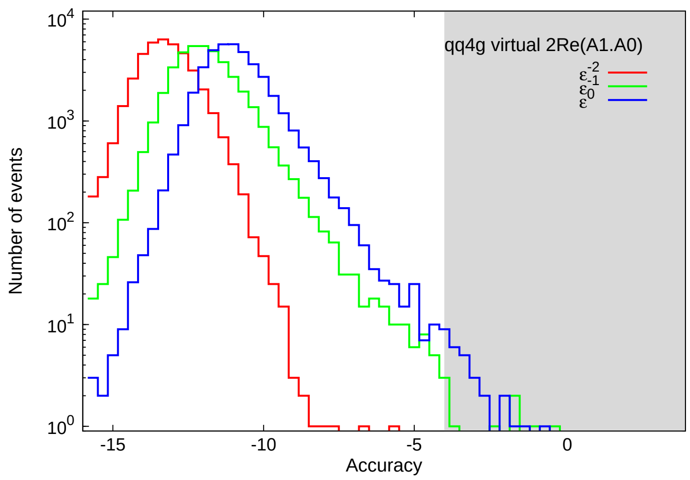
- First look : no current caching



- Accuracy improves with addition of more fermion pairs

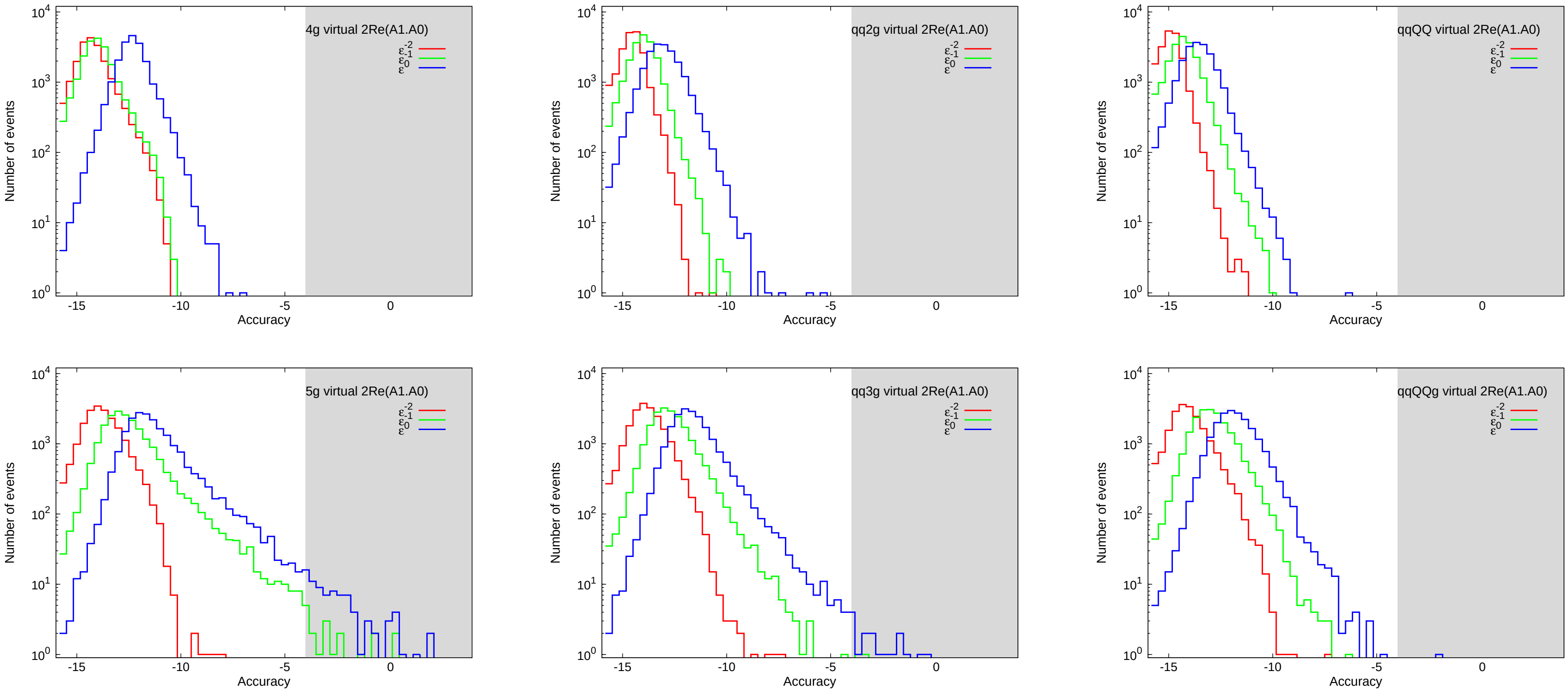
Stability of colour summed amplitudes





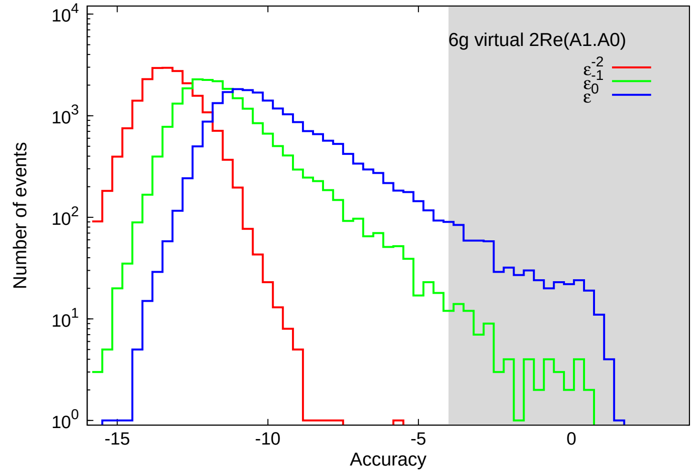
- $2 \to 4$ results under control
- Never more than 1% need re-evaluation
- Main bottleneck expected in the subtraction terms

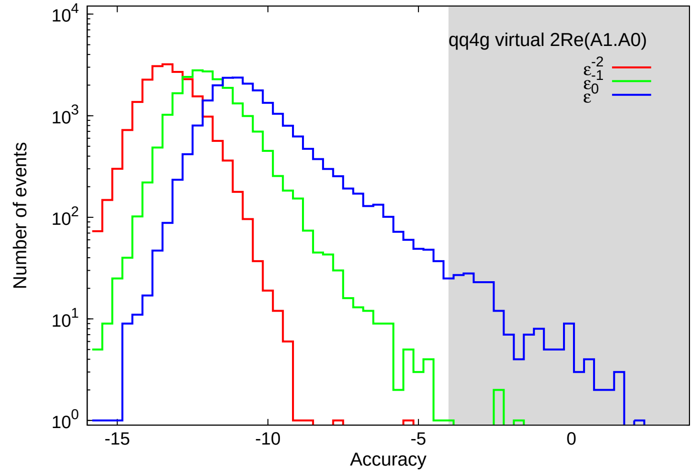
Stability of colour summed amplitudes



- Using cache system switched on improvement of order 2-3
- Evaluation times $\sim 1s$ for $2 \to 3$ processes

Stability of colour summed amplitudes





- Caching with permutations gives a few more unstable points
- Evaluation times : $\sim 15s\ 6g$, $\sim 20s\ \bar{q}q + 4g$
- Suitable for evaluation by a re-weighting procedure

Outlook

- On-shell techniques opening up multi-particle final states @ NLO
- Numerical computations with `N(Gluon)`
  - `NGluon` package for multi-gluon amplitudes [public c++ library]
  - multi-gluon and fermion primitve amplitudes [release for this year]
- Future directions
  - Phenomenological applications (combination with real radiation)
  - Matching to parton showers [`POWHEG BOX`,`aMC@NLO`]
  - Massive amplitudes (top, EW corrections etc.)

Massive Fermions and Rational Terms

- Mass cuts vs. D-dimensional cuts. recall: $\int d^D l = \int d^{-2\epsilon}\mu \int d^4\bar{l}$
- Longditudinal D-d gluon polarisation is equivalent to a massive scalar
- loop momentum in D-dimensions : $l^\mu = \bar{l} + \tilde{l}$ where $\tilde{l}^2 = -\mu^2$
- Consider:

$$A^D(\mu) = \dots \frac{\not{l} + m}{l^2 - m^2} \dots = \dots \frac{\bar{\not{l}} + \tilde{\not{l}} + m}{l^2 - \mu^2 - m^2} \dots$$

$$A^{\tilde{m}}(\mu) = \dots \frac{\not{l} + \tilde{m}}{l^2 - \tilde{m}^2} \dots$$

- $\tilde{m} = \sqrt{m^2 + \mu^2}$ and $\{\tilde{\not{l}}, \not{p}\} = 0$. However $[\tilde{\not{l}}, \gamma_5] = 0$. [Bern,Morgan (1995)]

$$A^D(\mu) = a_0 + a_1 f_1(\tilde{l}) + a_2\mu^2 + \dots$$

$$A^{\tilde{m}}(\mu) = a_0 - a_1'\mu + a_2\mu^2 + \dots$$

but: $\int d^{-2\epsilon}\mu \frac{\mu^{2n-1}}{\prod_{p=0}^{n} l_p^2 - m_p^2} = 0$

Van Neerven-Vermaseren Basis

- When computing numerically we must worry subtract higher cut terms
- Integrand level expression involves both coefficient and spurious terms
- V-NV basis gives a convenient parameterisation:

$$\overline{C}_4(l) = c_4 + \tilde{c}_4 l \cdot n_4$$

- e.g. 4-cut has 3d real space $\vec{K} = \mathrm{span}\{K_1, K_2, K_3\}$
- Spurious space (1d in this case) given by n_4 perp. to $\mathbf{K}$
  - compute an extended Gram matrix
  - then compute determinants to find n_4 s.t.

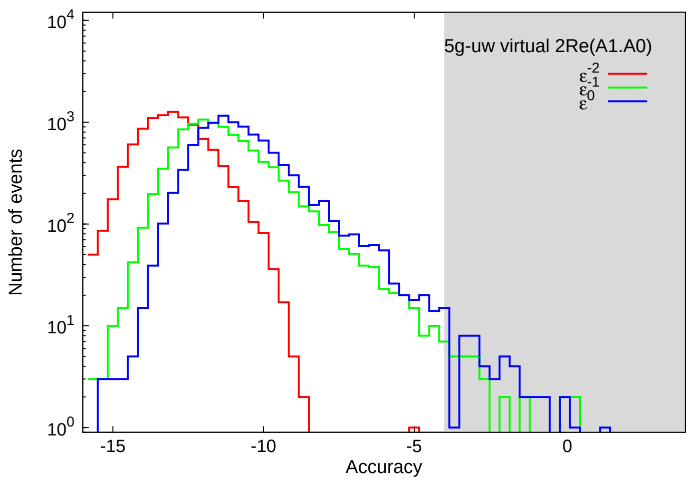
$$\Delta(m) = \begin{pmatrix} (\vec{K}_i \cdot \vec{K}_j) & (\vec{K} \cdot m) \\ (\vec{K} \cdot m) & m \cdot m \end{pmatrix} \quad \longrightarrow \quad \Delta(n_4) = \begin{pmatrix} (\vec{K}_i \cdot \vec{K}_j) & (0) \\ (0) & 1 \end{pmatrix}$$

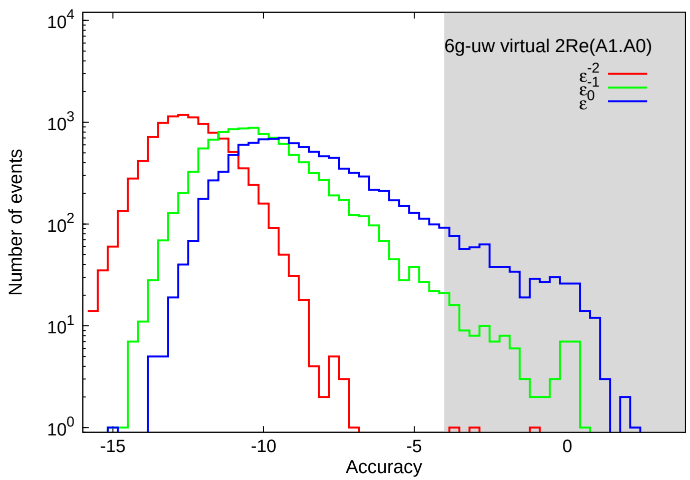
- Analagous method for lower multiplicity cuts (larger spurious spaces)

Evaluation on unweighted phase-space

- Unweighted events of 5 and 6 gluons generated with Madgraph v5

[Alwall et al. (2011)]





- $10,000$ events with caching ($p_T > 20$GeV, $\eta < 5$, $\Delta R > 0.4$)
- Accuracy slightly improved compared to flat space results