The Thermal Representation of Conformal Ladder Integrals

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The Big Picture

Quantum Mechanics

$\downarrow$

Thermal QFT

$\downarrow$

Single-valued Polylogarithms

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Conformal Ladder Integrals

$\downarrow$

AdS_2/CFT_1

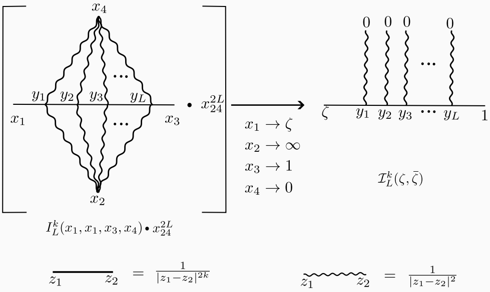
Conformal Ladder Integrals

- **Four-point functions** are fundamental objects in CFT: they encode the dynamics through the OPE
- They can be **bootstrapped** using only symmetry and unitarity
- **Perturbative calculations**:
  - Conformal symmetry simplifies conformal integrals
  - Ladder integrals: Special class with iterative structure
  - AdS/CFT correspondence renewed interest i.e. Mellin-transform
  - Applications in fishnet theories, large-N_c calculations, integrability, single-valued polylogarithms

Ladder integrals appear in:

- $\mathcal{N} = 4$ SYM (planar limit)
- Fishnet theories (Kazakov et al.)
- Large-charge expansions
- Integrability studies
- Basso-Dixon integrals
- Thermal bootstrap calculations

Conformal Ladder Integrals



L-loop ladder integral in $D = 2k+2$ dimensions

$$I_L^k(x_i) = \prod_{n=1}^{L}\left[\int \frac{d^{2k+2}y_n}{\pi^{k+1}} \frac{g^2\Gamma(k)}{|y_{n-1,n}|^{2k}|x_2-y_n|^2|x_4-y_n|^2}\right]\frac{1}{|y_L-x_3|^{2k}}$$

with $y_0 = x_1,\ y_{ij} = y_i - y_j$

Conformal Ladder Integrals

Conformal covariance:

$$I_L^k(\lambda x_i) = \lambda^{-(2k+2L)} I_L^k(x_i)$$
$$I_L^k(1/x_i) = (x_1^2 x_3^2)^k (x_2^2 x_4^2)^L I_L^k(x_i)$$

Cross-ratio representation:

$$I_L^k(x_i) = \frac{1}{x_{13}^{2k} x_{24}^{2L}} \Phi_L^k(u, v)$$

with $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

Conformal frame: Fix $x_1 = \zeta$, $x_2 = \infty$, $x_3 = 1$, $x_4 = 0$

$$\mathcal{I}_L^k(\zeta, \bar{\zeta}) = \frac{1}{|1-\zeta|^{2k}} \Phi_L^k(\zeta, \bar{\zeta})$$

Conformal Ladder Integrals

Isaev's integral representation [Isaev 2003]
For $k, L \geq 1$:

$$\mathcal{I}_L^k(\zeta,\bar{\zeta}) = -\frac{1}{(L!)^2}\int_0^\infty dt\, t^L [t - 2\ln|\zeta|]^L \partial_t \left(\frac{e^{-t}}{(1-\zeta e^{-t})(1-\bar{\zeta}e^{-t})}\right)^k$$

Using the change of variables $\zeta' = \zeta e^{-t}\,,\ \zeta = |\zeta| e^{i\theta}\,,\ \zeta' = |\zeta'| e^{i\theta}$ we obtain

$$\Phi_L^k(\zeta,\bar{\zeta}) = \frac{1}{L!(L-1)!}\int_0^{|\zeta|} \frac{d|\zeta'|}{|\zeta'|} 2\ln|\zeta'| (\ln^2|\zeta'| - \ln^2|\zeta|)^{L-1} \frac{\mathcal{D}_0^k(\zeta',\bar{\zeta}')}{\mathcal{D}_0^k(\zeta,\bar{\zeta})}$$

where

$$\mathcal{D}_0^k(\zeta,\bar{\zeta}) = \Gamma(k)\frac{(\zeta-\bar{\zeta})^k}{|1-\zeta|^{2k}}\,,\ \ k = 1, 2, ..,\ \ \mathcal{D}_0^0(\zeta,\bar{\zeta}) \equiv \ln \mathcal{Z}_0(\zeta,\bar{\zeta})$$

The Parent Quantum Mechanical System

The Hamiltonian

$$\hat{H} = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}m^2(\hat{x}_1^2 + \hat{x}_2^2) + i\mu(\hat{p}_2\hat{x}_1 - \hat{p}_1\hat{x}_2)$$

Interpretation:

- Two harmonic oscillators with frequency/mass m
- Twisted by imaginary chemical potential μ
- Equivalent to interaction with (imaginary) homogenous constant magnetic field along the third direction

The Parent Quantum Mechanical System

Interpretation:

The Hamiltonian can be thought of as a deformation of the free particle Hamiltonian $\hat{H}_0 = (\hat{p}_1^2 + \hat{p}_2^2)/2$ by the operators

$$\hat{\mathcal{O}} = \frac{1}{2}(\hat{x}_1^2 + \hat{x}_2^2)\,, \quad \hat{\mathcal{Q}} = \hat{p}_2\hat{x}_1 - \hat{p}_1\hat{x}_2$$

the deformations parameters being m and $i\mu$.

Creation/annihilation operators:

$$\hat{a}_1^\dagger = \frac{1}{2\sqrt{m}}(m(\hat{x}_1 + i\hat{x}_2) + (\hat{p}_2 - i\hat{p}_1))\,, \;\; \hat{a}_1 = \frac{1}{2\sqrt{m}}(m(\hat{x}_1 - i\hat{x}_2) + (\hat{p}_2 + i\hat{p}_1))\,,$$
$$\hat{a}_2^\dagger = \frac{1}{2\sqrt{m}}(m(\hat{x}_1 - i\hat{x}_2) - (\hat{p}_2 + i\hat{p}_1))\,, \;\; \hat{a}_2 = \frac{1}{2\sqrt{m}}(m(\hat{x}_1 + i\hat{x}_2) - (\hat{p}_2 - i\hat{p}_1))\,,$$

Diagonal form:

$$\hat{H}_0 + m^2\hat{\mathcal{O}} = m(\hat{N}_1 + \hat{N}_2 + 1)\,, \;\; \hat{\mathcal{Q}} = \hat{N}_1 - \hat{N}_2$$

with $\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i$.

The Parent Quantum Mechanical System

Grand canonical partition function

$$\mathcal{Z}_0(z,\bar{z}) = \mathrm{Tr} e^{-\beta(\hat{H}_0+m^2\hat{\mathcal{O}})-i\beta\mu\hat{\mathcal{Q}}} = \frac{|z|}{(1-z)(1-\bar{z})}$$

where $z = e^{-\beta m - i\beta\mu}$

Differential operators:

$$\hat{\mathrm{D}}_z = \frac{1}{\beta^2}\frac{\partial}{\partial m^2} = \frac{1}{2\ln|z|}(z\partial_z+\bar{z}\partial_{\bar{z}})\,, \;\; \hat{\mathrm{L}}_z = \frac{i}{\beta}\frac{\partial}{\partial\mu} = (z\partial_z-\bar{z}\partial_{\bar{z}})\,, \; [\hat{\mathrm{D}}_z,\hat{\mathrm{L}}_z] = 0$$

They commute: $[\hat{\mathrm{D}}_z,\hat{\mathrm{L}}_z] = 0$

Thermal one-point functions:

$$\langle\hat{\mathcal{O}}\rangle_0 = -\beta\hat{\mathrm{D}}_z * \ln\mathcal{Z}_0 = \frac{1}{2m}\frac{1-|z|^2}{|1-z|^2}\,, \;\; \langle\hat{\mathcal{Q}}\rangle_0 = \hat{\mathrm{L}}_z * \ln\mathcal{Z}_0 = \frac{z-\bar{z}}{|1-z|^2}$$

- These are **Poisson kernels** (harmonic functions) with constant boundary values at a single point on the unit circle/upper half-plane.
- They are **bulk-to-boundary propagators** in Euclidean AdS_2 in different coordinate patches.

Ideal Relativistic Gas Construction

Relativistic one-particle density of states:

$$\rho_L(\omega; m; \alpha^2) = \frac{2\alpha^{2L}\beta^{2L}}{(L-1)!}\omega(\omega^2 - m^2)^{L-1}$$

with $\alpha^2 = \frac{\ell^2}{4\pi\beta^2}$ (geometric parameter)

Thermal free energy:

$$\begin{aligned}\ln \mathcal{Z}_L(z, \bar{z}; \alpha^2) &= \frac{\alpha^{2L}\beta^{2L}}{(L-1)!}\int_m^\infty d\omega\, 2\omega(\omega^2 - m^2)^{L-1} \ln \mathcal{Z}_0(z', \bar{z}') \\ &= -\frac{\alpha^{2L}}{(L-1)!}\int_0^{|z|} \frac{d|z'|}{|z'|} 2\ln|z'|(\ln^2|z'| - \ln^2|z|)^{L-1} \ln \mathcal{Z}_0(z', \bar{z}')\,,\end{aligned}$$

where $z' = e^{-\beta\omega - i\beta\mu}$ ($\ln|z'| < \ln|z|$).

$\ln \mathcal{Z}_L(z, \bar{z}; \alpha^2) = \alpha^{2L} \ln \mathcal{Z}_L(z, \bar{z})$.

Explicit Result in Terms of Polylogarithms

Result [T.P. 2021]

$$\ln \mathcal{Z}_L(z,\bar{z}) = \frac{(-1)^L L!(2\ln|z|)^{2L+1}}{2(2L+1)!} + \sum_{n=0}^{L} \frac{(2L-n)!(-2\ln|z|)^n}{(L-n)!n!} 2\Re[\mathrm{Li}_{2L+1-n}(z)]$$

Special cases:

- $L=0$: $\ln \mathcal{Z}_0 = \ln|z| - \ln(1-z) - \ln(1-\bar{z})$
- $m=\mu=0$ $(z=1)$: $\ln \mathcal{Z}_L(1,1) = \frac{(2L)!}{L!} 2\zeta(2L+1)$
- Single-valued polylogarithm representation (extra material).

Connection to Massive Complex Scalar

Euclidean action in $d = 2L + 1$:

$$S_L = \int_0^\beta d\tau \int d^{2L}\vec{x}\, |(\partial_\tau - i\mu)\phi|^2 + |\vec{\partial}\phi|^2 + m^2|\phi|^2$$

Thermal partition function:

$$\mathcal{Z}_L(\beta; m, \mu) = \frac{1}{\mathcal{Z}(\infty; 0, 0)} \int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-S_L}$$

Key result: Path integral calculation yields exactly $\ln \mathcal{Z}_L(z, \bar{z})$ with

$$\alpha^{2L} \leftrightarrow \frac{V_{2L}}{\beta^{2L}} \quad \text{(spatial volume)}$$

Recurrence Relations & Differential Equations

Recurrence relations:

$$\langle \mathcal{O} \rangle_L = -\beta \hat{\mathsf{D}}_z * \ln \mathcal{Z}_L = \beta \ln \mathcal{Z}_{L-1}$$
$$\langle \mathcal{Q} \rangle_L = \hat{\mathsf{L}}_z * \ln \mathcal{Z}_L = -\hat{\mathsf{D}}_z * \langle \mathcal{Q} \rangle_{L+1}$$

Integral operator:

$$\check{\mathsf{d}}_{z;z'} = \int_0^{|z|} \frac{d|z'|}{|z'|} 2 \ln |z'|$$

with properties:

$$\check{\mathsf{d}}_{z;z'} * \hat{\mathsf{D}}_{z'} * f(z') = \hat{\mathsf{D}}_z * \check{\mathsf{d}}_{z;z'} * f(z') = f(z) \,,$$
$$\check{\mathsf{d}}_{z;z'} * \hat{\mathsf{L}}_{z'} * f(z') = \hat{\mathsf{L}}_z * \check{\mathsf{d}}_{z;z'} * f(z') \,,$$

Recurrence Relations & Differential Equations

Iterated integral representation:

$$\ln \mathcal{Z}_L = (-1)^L \left\{ \text{ord} \prod_{i=1}^{L} \right\} \check{\mathsf{d}}_{z_{i+1};z_i} * \ln \mathcal{Z}_0$$

where $0 \leq |z_1| \leq \cdots \leq |z_L| \leq |z|$

The "depth" k

$$(z-\bar{z})^k \left(\frac{1}{z-\bar{z}} \hat{\mathsf{L}}_z \right)^k * \ln \mathcal{Z}_0(z,\bar{z}) = \mathcal{D}_0^k(z,\bar{z}) = \Gamma(k) \frac{(\zeta - \bar{\zeta})^k}{|1-\zeta|^{2k}}$$

- This is related to the massless propagator of a scalar field $\phi(x)$ with scaling dimension $\Delta_\phi = D/2 - 1 = k$ in D-dimensions.
- It is also related to a bulk-to-boundary AdS_2 propagator for a scalar with mass $M = k(k-1)$.
- Massless two-point functions in D-dimensions can be viewed as a thermal cumulants in our parent quantum mechanical system.

Recurrence Relations & Differential Equations

A recurrence relation - weight shifting operator in AdS_2

$$\left(\hat{\mathsf{L}}_z - k\frac{z+\bar{z}}{z-\bar{z}}\right) * \mathcal{D}_0^k(z,\bar{z}) = \mathcal{D}_0^{k+1}(z,\bar{z})\,, \quad k = 0,1,2,3,\ldots\,.$$

In other words the operator

$$\hat{\mathsf{L}}_z^{(k)} \equiv \hat{\mathsf{L}}_z - k\frac{z+\bar{z}}{z-\bar{z}}$$

raises the value of the depth parameter $k \mapsto k+1$ or equivalently raises the dimension $D \mapsto D+2$, and commutes with $\check{\mathsf{d}}_{z;z'}$ as

$$\check{\mathsf{d}}_{z;z'} * \hat{\mathsf{L}}_{z'}^{(k)} * f(z') = \hat{\mathsf{L}}_z^{(k)} * \check{\mathsf{d}}_{z;z'} * f(z')$$

The above motivate us to write

$$\left\{\mathrm{ord}\prod_{n=0}^{k-1}\right\}\left[\hat{\mathsf{L}}_z - n\frac{z+\bar{z}}{z-\bar{z}}\right] * \ln\mathcal{Z}_0(z,\bar{z}) \equiv [\,\hat{\mathsf{L}}_z]^k * \ln\mathcal{Z}_0(z,\bar{z}) = \mathcal{D}_0^k(z,\bar{z})$$

where

$$[\,\hat{\mathsf{L}}_z^{(k)}, \hat{\mathsf{D}}_z] = 0\,.$$

Summary of Definitions

$$\ln \mathcal{Z}_0(z,\bar{z}) \equiv \mathcal{D}_0^0(z,\bar{z}) = \ln|z| - \ln(1-z) - \ln(1-\bar{z})$$

$$\ln \mathcal{Z}_L(z,\bar{z}) \equiv \mathcal{D}_L^0(z,\bar{z})$$

$$\mathcal{D}_0^k(z,\bar{z}) = \frac{\Gamma(k)(z-\bar{z})^k}{|1-z|^{2k}}, \quad k \geq 1$$

$$\hat{\mathsf{D}}_z = \frac{1}{\beta^2}\frac{\partial}{\partial m^2} = \frac{1}{2\ln|z|}(z\partial_z + \bar{z}\partial_{\bar{z}})$$

$$\hat{\mathsf{L}}_z = \frac{i}{\beta}\frac{\partial}{\partial \mu} = (z\partial_z - \bar{z}\partial_{\bar{z}})$$

$$\hat{\mathsf{L}}_z^{(k)} = \hat{\mathsf{L}}_z - k\frac{z+\bar{z}}{z-\bar{z}}$$

$$[\hat{\mathsf{L}}_z]^k = (z-\bar{z})^k\left(\frac{1}{z-\bar{z}}\hat{\mathsf{L}}_z\right)^k = \left\{\mathrm{ord}\prod_{n=0}^{k-1}\right\}\left[\hat{\mathsf{L}}_z - n\frac{z+\bar{z}}{z-\bar{z}}\right]$$

$$[\hat{\mathsf{D}}_z,\ \hat{\mathsf{L}}_z] = 0, \qquad [\hat{\mathsf{L}}_z^{(k)},\ \hat{\mathsf{D}}_z] = 0$$

$$\check{\mathsf{d}}_{z;z'} = \int_0^{|z|}\frac{d|z'|}{|z'|}\,2\ln|z'|, \quad [\check{\mathsf{d}}_{z;z'},\ \hat{\mathsf{L}}_{z'}] = 0$$

Summary of Definitions

$\ln \mathcal{Z}_L(z, \bar{z})$	$=$	$[-\check{\mathsf{d}}]^L_{z;z'} * \ln \mathcal{Z}_0(z', \bar{z}')$
$\hat{\mathsf{D}}_z * \ln \mathcal{Z}_L(z, \bar{z})$	$=$	$\ln \mathcal{Z}_{L-1}(z, \bar{z}) = -\frac{1}{\beta}\langle \mathcal{O}(z, \bar{z})\rangle_L$
$\hat{\mathsf{L}}_z * \ln \mathcal{Z}_L(z, \bar{z})$	$=$	$\langle \mathcal{Q}(z, \bar{z})\rangle_L$
$\check{\mathsf{d}}_{z;z'} * \ln \mathcal{Z}_L(z', \bar{z}')$	$=$	$-\ln \mathcal{Z}_{L+1}(z, \bar{z})$
$\hat{\mathsf{L}}^{(k)}_z * \mathcal{D}^k_0(z, \bar{z})$	$=$	$\mathcal{D}^{k+1}_0(z, \bar{z})$
$[\hat{\mathsf{L}}_z]^k * \ln \mathcal{Z}_0(z, \bar{z})$	$=$	$\mathcal{D}^k_0(z, \bar{z})$

Main Result: The Correspondence

Conformal ladder integrals are thermal averages:

$$\mathcal{I}_L^k(\zeta,\bar{\zeta}) \longleftrightarrow \frac{1}{\Gamma(k)L!}\frac{1}{(z-\bar{z})^k}[-\check{\mathsf{d}}_{z;z'}]^L * [\hat{\mathsf{L}}_{z'}]^k * \ln \mathcal{Z}_0(z',\bar{z}')$$
$$\equiv \frac{1}{\Gamma(k)L!}\frac{1}{(z-\bar{z})^k}\mathcal{D}_L^k(z,\bar{z})$$

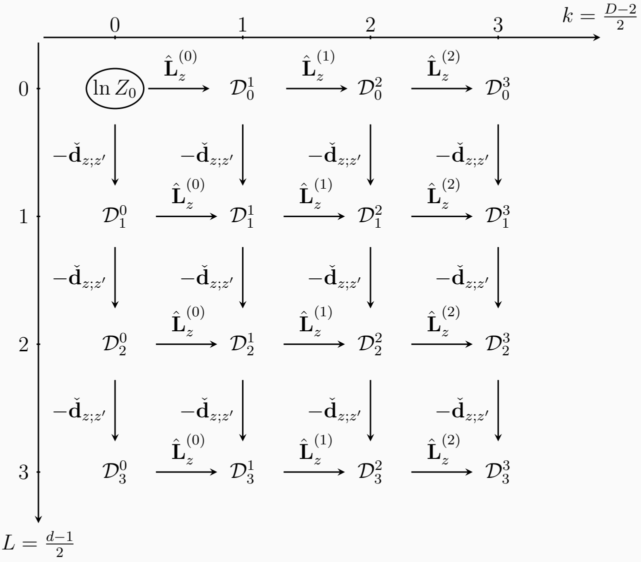
with identifications: $\zeta \leftrightarrow z$, $g^2 \leftrightarrow \alpha^2$

Equivalently:

$$\Phi_L^k(z,\bar{z}) = \frac{1}{L!}\frac{\mathcal{D}_L^k(z,\bar{z})}{\mathcal{D}_0^k(z,\bar{z})}$$

Interpretation: CFT perturbative $g^2 \ll 1 \leftrightarrow$ low temperature $\alpha^2 \ll 1$

Graphical Representation



The Differential Equation

Second-order differential equation

$$\left(\frac{1}{4\beta^2}\hat{\Delta}_z - L\hat{\mathrm{D}}_z + k(k-1)\frac{|z|^2}{(z-\bar{z})^2}\right) * \mathcal{D}^k_L(z,\bar{z}) = 0$$

where $\hat{\Delta}_z = \partial_m^2 + \partial_\mu^2 = 4\beta^2 z\bar{z}\partial_z\partial_{\bar{z}}$

In (m,μ) variables:

$$\left[(\partial_m^2 + \partial_\mu^2) - \frac{2L}{m}\partial_m - \beta^2\frac{k(k-1)}{\sin^2(\beta\mu)}\right]\mathcal{D}^k_L(m,\mu) = 0$$

Special case $L = 0$: Scalar field on Euclidean AdS_2 with metric

$$ds^2 = \frac{1}{\sin^2(\beta\mu)}(dm^2 + d\mu^2)$$

and mass $M^2 = \beta^2 k(k-1)$

Thermal One-Point Functions of Conserved Currents

- **Thermal two-point function** of massive complex scalar:

$$g^{(L)}(\tau,\vec{x};m,\mu) = \langle \phi^\dagger(\tau,\vec{x})\phi(0)\rangle^{(L)}$$

- **Integrable part** (annihilated by $\Box_d$):

$$g^{(L)} = \frac{1}{r^{2\Delta_\phi}}\left[1 + \sum_s a^L_{\mathcal{O}_s}(z,\bar{z})\left(\frac{r}{\beta}\right)^{\Delta_s} C_s^{\frac{d}{2}-1}(\cos\theta) + \text{shadows}\right]$$

- **Key result**:

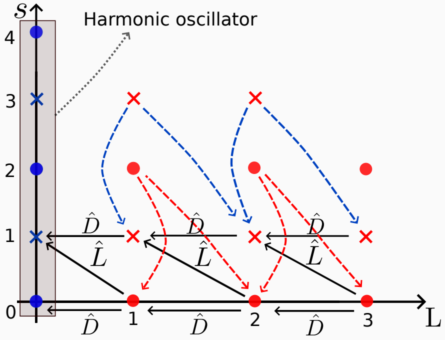
$$a^L_{\mathcal{O}_0}(z,\bar{z}) = \frac{1}{(4\pi)^L\alpha^{2L}}\mathcal{D}^0_{L-1}(z,\bar{z}) \quad \text{(spin-0)}$$
$$a^L_{\mathcal{O}_1}(z,\bar{z}) = \frac{1}{(4\pi)^L\alpha^{2L}}\frac{1}{2}\mathcal{D}^1_{L}(z,\bar{z}) \quad \text{(spin-1)}$$

Thermal One-Point Functions of Conserved Currents

Recurrence relation [Karydas et al. 2023]

$$a^{L}_{\mathcal{O}_{s+2}}(z,\bar{z}) = \frac{2\pi}{2L-1} a^{L+1}_{\mathcal{O}_s}(z,\bar{z}) + \frac{\ln^2|z|}{(2L-1+2s)(2L+1+2s)} a^{L}_{\mathcal{O}_s}(z,\bar{z})$$

valid for $s \geq 0$



Implication: All higher-spin thermal one-point functions determined by

- $\mathcal{D}^{0}_{L-1}$ (ladders in $D = 2$) for even spin
- $\mathcal{D}^{1}_{L}$ (ladders in $D = 4$) for odd spin

Hyper-Partition Function & All-Loop Summation

Hyper-partition function (sum over all dimensions):

$$\mathcal{Z}(z,\bar{z};\alpha^2) = \prod_{L=0}^{\infty} \mathcal{Z}_L(z,\bar{z};\alpha^2)$$

Differential equation:

$$(\hat{\mathsf{D}}_z + \alpha^2) * \ln \mathcal{Z}(z,\bar{z};\alpha^2) + \ln \mathcal{Z}_{-1}(z,\bar{z}) = 0$$

$$\left(\frac{\partial}{\partial m} + 2\alpha^2\beta^2 m\right) \ln \mathcal{Z}(m,\mu;\alpha^2) = -\beta \frac{\sinh \beta m}{\cosh \beta m - \cos \beta\mu} \,.$$

All-loop resummation of ladder integrals:

$$\mathcal{I}^k(z,\bar{z};g^2) = \sum_{L=0}^{\infty} (-g^2)^L \mathcal{I}_L^k(z,\bar{z}) = \frac{1}{\Gamma(k)(z-\bar{z})^k} [\hat{\mathsf{L}}_z]^k * \sum_{L=0}^{\infty} \frac{1}{L!} \ln \mathcal{Z}_L(z,\bar{z};-g^2)$$

Result: For $k=1$ ($D=4$), matches Broadhurst's resummation:

$$\mathcal{I}^1 = \frac{1}{2|z|} \int_m^{\infty} d\omega \, J_0(2g\beta\sqrt{\omega^2 - m^2}) \frac{\sinh(\beta\omega)}{(\cosh(\beta\omega) - \cos(\beta\mu))^2}$$

Extra Material

The General Result [Karydas et. al. (2023)]:

$$a^L_{\mathcal{O}_s} = \frac{\Gamma\left(L-\frac{1}{2}\right)}{\Gamma\left(L+s-\frac{1}{2}\right)(4\pi)^L 2^{2s}} \sum_{n=0}^{L-1+s} \frac{2^n}{n!} \frac{(\beta m)^n (2L-2+s-n)!}{(L-1+s-n)!} \times \left[Li_{2L-1+s-n}(z) + (-1)^s Li_{2L-1+s-n}(\bar{z})\right] ,$$

The Fishnet Models:

The $s = 2k$ case was a puzzle. It arises in conformal ladder integrals of the singular fishnet model for $D \to 2$, $\omega \to 1$.

$$\mathcal{L} = N_c \mathrm{Tr}\left[\phi_1^\dagger(-\partial^2)^\omega \phi_1 + \phi_2^\dagger(-\partial^2)^{\frac{D-2\omega}{2}} \phi_2 + a^2_{D,\omega} \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right] . \qquad (1)$$

$\phi_{1,2}$ belong to the adjoint of $SU(N_c)$, $\omega \in \left(0, \frac{D}{2}\right)$ and the coupling $a^2_{D,\omega}$ is classically dimensionless.

Extra Material

The Ladder Graphs in the Fishnet Models: We consider the 4pt function

$$G^{(L)}_{D,\omega}(\{x_i\}) = \langle \mathrm{Tr}\left[\phi_2^L(x_1)\phi_1(x_3)\phi_2^{\dagger L}(x_2)\phi_1^{\dagger}(x_4)\right]\rangle ,$$

whose leading N_c contribution comes from a unique L-loop conformal ladder graph.

Effective coupling: $\tilde{a}_{D,\omega} = a_{D,\omega}/\Gamma(D/2-\omega)$

Result:

$$G^{(L)}_{2,1}(z,\bar{z}) \equiv \mathcal{I}^0_L(z,\bar{z}) = \frac{2\pi}{L!} \ln \mathcal{Z}_L(z,\bar{z})$$

Summary

Conformal ladder integrals		Thermal 1pt functions
Dimension D	$D = 2k + 2$	Depth k
Loop order L	$d = 2L + 1$	Dimension d
Spacetime points		Mass m,
$x_i = (0, 1, z, \infty)$	$z = e^{-\beta m - i\beta\mu}$	chemical potential μ
Dimensionless coupling g^2	$g^2 = \alpha^2$	Geometric parameter $\alpha^2 = \frac{l^2}{4\pi\beta^2}$

Outlook

1. **AGT conjecture-like**: Spacetime cross-ratios as moduli parameters
2. **Large-charge expansions**: Resummation of ladder integrals [Caetano et. al. 2024]
3. **Thermal bootstrap**: KMS condition constraints on conformal integrals [Barrat et. al. 2025]
4. **Modular properties**: Connection to string amplitudes and integrated correlators [Dorigoni. et. al. 2024]
5. **Higher-derivative theories**: Long-range critical systems [Giombi et. al. 2023]
6. **Integrability**: Toda-like equations [Loebbert et al. 2024]
7. **Holography**: AdS_2/CFT_1 via conformal QM [Hartnoll et. al. 2025]